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Abstract

We consider a quasi-static fluid-structure interaction problem where the fluid is modeled by the Stokes equations and the structure is an active and elastic medium. More precisely, the displacement of the structure verifies the equations of elasticity with an active stress, which models the presence of internal biological motors in the structure. Under smallness assumptions on the data, we prove the existence of a unique solution for this strongly coupled system.

Keywords: Fluid-structure interaction, active structure, strong solution, existence result.

1 Introduction

Many living beings move, breathe and reproduce themselves by means of thin active structures that interact with fluids. Cilia and flagella are examples of such soft materials that deform themselves using internal biological motors and thus, induce a flow within the surrounding fluid. Understanding the underlying mechanisms of the coupling of such a fluid-structure system is of great interest for biological, medical and even engineering applications. The problem we are interested in is the interaction between an elastic medium, subjected to an internal time depending stress, and a viscous fluid, whose domain depends on the displacement of the elastic medium. In this article, we prove the existence and the uniqueness for small enough data of a regular solution to a quasi-static interaction problem involving an

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elastic structure subjected to an internal stress and a Newtonian viscous incompressible homogeneous fluid, modeled by the Stokes equations.

Existence of solutions for fluid-structure interaction problems has been the subject of numerous works in the last years. Concerning interaction problems involving a viscous incompressible homogeneous fluid and a passive solid medium, several models have been studied. In the case of a 3D linear elastic structure evolving in a 3D viscous incompressible Newtonian flow, we refer the reader to [17] and [8] where the structure is described by a finite number of eigenmodes or to [5], [6] for an artificially damped elastic structure. Both articles illustrate the use of a finite-dimensional approximation of the equation of linearized elasticity or a damped structure in order to avoid the loss of regularity inherent to this fluid-structure system.

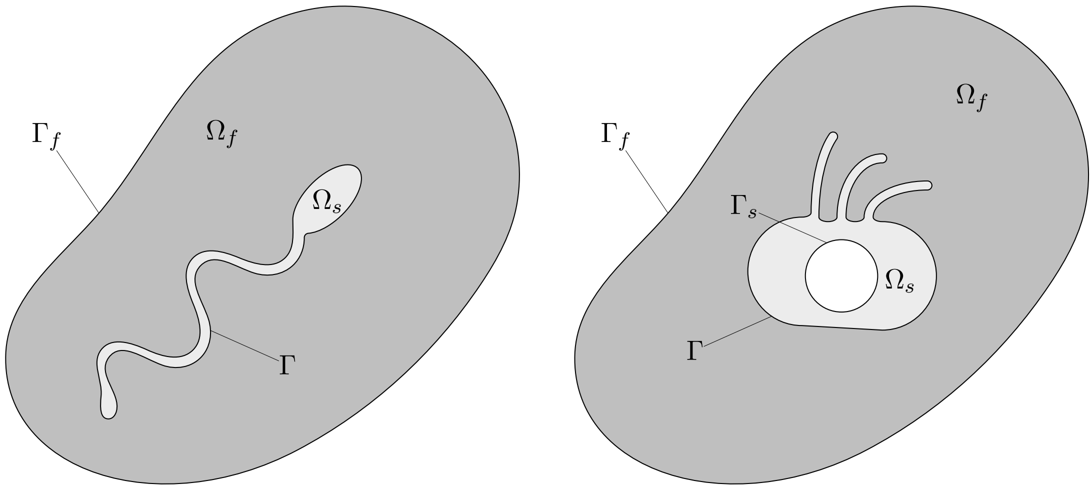
For systems coupling the incompressible Navier-Stokes equations with the equations of linearized elasticity, results have been successively obtained in [15], [21], [24] and more recently in [7], for the existence and uniqueness of strong solutions locally in time. In [15], the authors consider the existence of strong solutions for small enough data locally in time, requiring higher regularity assumptions on the initial data than the one obtained on the solution, the uniqueness being obtained under even stronger regularity assumptions. In [21], [24], the existence of local-in-time strong solutions is proven in the case where the fluid structure interface is flat and for a zero initial displacement field, once again with a gap between the regularity of the initial data and the one of the solution. One of the main difficulty is to fill the gap between the fluid (parabolic) and structure (hyperbolic) regularities. This issue has been partially solved in [7], in the case of an immersed structure with a zero initial displacement field, by obtaining hidden regularity of the structure displacement.

Finally, when the structure is modeled with the Saint Venant-Kirchhoff law, very few results on the well-posedness of the fluid-structure system are known. In [16], the Navier-Stokes equations coupled to the equations of elastodynamics, are considered and the existence and uniqueness of strong solutions are proven locally in time, under compatibility conditions on the initial data. In [19], a result of existence of strong solutions is proven in the steady-state situation, considering either the Stokes or the Navier-Stokes equations, if the data are sufficiently small.

Concerning the well-posedness of fluid-structure problems involving active structures and viscous incompressible homogeneous fluid, few results are available. In [18], the steady self-propelled motion of a constant shape solid in a non-inertial fluid, modeled by either the Stokes or steady state Navier-Stokes equations, is studied. The velocity of the solid on its boundary is divided in two parts: the first one is imposed and represents the self-propelled velocity of the solid, whereas the other one is due to the interaction with the surrounding fluid. The existence of solutions is proven and conditions under which a distribution of the self-propelled velocity on the boundary of the structure is able to propel the solid is investigated. In [25], an initial and boundary value problem for the swimming of a fish-like deformable structure is treated. In this model the movement of the structure is divided in two: the rigid part of the displacement results from the interaction of the fluid and the solid, whereas the deformation part of the displacement is imposed. The resulting coupled system between the Navier-Stokes equations for the fluid and Newton's laws for the structure is proven to be well-posed. For the same problem, the existence and uniqueness of weak solutions is studied in [22]. The same system is studied in [14] in three space dimensions and limiting the regularity of the imposed displacement of the structure (still a datum of the problem). The author proves local existence in time for any data and global existence in time under smallness assumptions on the data.

In all these works the strategy is the same: decompose the movements of the structures in two parts. The first part is imposed and describes the internal activity of the structure, while the second part is a rigid movement which satisfies Newton's law and results from the interaction with the fluid. The originality of the present work lies in the fact that the activity of the structure is modeled by a given internal active stress, such that the whole movement of the structure results from the interaction with the surrounding fluid. As we are interested in the modelling of active suspensions in viscous flows at low Reynolds number we assume that the inertial effects can be neglected so that the coupled system is a quasi-static one.

Let us now introduce more precisely the fluid-structure problem we are considering. Let n be the space dimension (2 or 3) and Ω be a regular open connected bounded subset of $\mathbb{R}^n$, whose definition



(a) (b)

Figure 1: Two-dimensional examples of the geometry of the fluid-structure problem, where $|\Gamma_s| = 0$ (a) and $|\Gamma_s| \neq 0$ (b).

will be made precise by hypotheses $(\mathbf{H_1})$-$(\mathbf{H_4})$. The domain Ω is supposed to be the union of two domains: $\Omega = \Omega_s \cup \Omega_f$, with $\Omega_s \cap \Omega_f = \emptyset$, such that Ω_s and Ω_f.The interface between Ω_s and Ω_f is denoted by Γ and we define the remaining frontiers of Ω_s and Ω_f by $\Gamma_s = \partial\Omega_s \setminus \Gamma$ and $\Gamma_f = \partial\Omega_f \setminus \Gamma$. Examples of such domains are given in Figure 1. Figure 1 (a) shows a spermcell embedded in a fluid in the case where $|\Gamma_s| = 0$. On contrary, the case where $|\Gamma_s| \neq 0$ is illustrated in Figure 1 where cilia attached to a wall are depicted.

The domain Ω_s is filed with an active elastic medium. At time t, the behavior of the structure is modeled by the non-inertial equations of elasticity, set in the reference configuration Ω_s. Denoting by d_s the displacement of the structure, these equations write

$$\begin{array}{rcll} -\mathrm{div}(\Pi_s(d_s(t),t)) &=& f_s(t) & \text{in } \Omega_s, \\ d_s(t) &=& 0 & \text{on } \Gamma_s, \end{array} \tag{1.1}$$

where f_s denotes the exterior body forces applied on the structure. The tensor Π_s represents the first Piola-Kirchhoff stress tensor of the structure which, in the present study, includes the active component and will now be defined. In the framework of continuum mechanics, two popular approaches are used to model contractility in solid materials, namely the active-stress and active-strain methods (see [2]). The former consists in adding an active component to the passive stress tensor, while in the later, the activation is considered as a pre-strain in a multiplicative decomposition of the tensor gradient of deformation. Both techniques have been studied for biological structures and particularly in a context of myocardium and arteries studies but never to model cilia and flagella, to our knowledge. For more information on models for active organs, we refer to [23, Chapter 2]. In this work, we choose the active-stress formalism to model internal motors and denote by Σ^* the active stress tensor added to the passive component. Moreover, we do not suppose Σ^* to have any particular shape, but we consider the general case where this tensor depends on both the time t and the material position x. If we consider the Saint Venant-Kirchhoff behavior law for the passive component of the elastic medium, its constitutive equations at time t become

$$\begin{array}{rcl} \Pi_s(d_s(t),t) &=& (I + \nabla d_s(t))(\Sigma_s(d_s(t)) - \Sigma^*(t)), \\ \Sigma_s(d_s) &=& 2\mu_s E(d_s(t)) + \lambda_s \mathrm{tr}(E(d_s(t)))I, \\ E(d_s) &=& \dfrac{1}{2}(\nabla d_s(t) + \nabla d_s(t)^T + \nabla d_s(t)^T \cdot \nabla d_s(t)), \end{array} \tag{1.2}$$

where μ_s and λ_s are Lamé's parameters ($\lambda_s > 0$, $\mu_s > 0$) and I stands for the identity matrix of $\mathbb{R}^n$. Thus, with the present elasticity model, the activity of the structure is completely internal and enables to fully take into account the fluid-structure interaction, whereas imposing only one part of the velocity or the deformation as in [18] or [25] does not.

At time t, the structure moves under the influence of its internal activity and the action of the surrounding fluid on its boundary. Then, the deformation of the fluid domain at time t, denoted by $\Phi(d_s(t))$, depends on the displacement of the structure on the interface Γ and satisfies

$$\Phi(d_s(t)) \quad = \quad \mathcal{I} + d_s(t) \quad \text{on } \Gamma,$$

where $\mathcal{I}$ is the identity mapping in $\mathbb{R}^n$. Moreover, let γ_Γ be the trace operator from Ω_s onto Γ and $\mathcal{R}$ be a continuous linear lifting from Γ to Ω_f (in spaces made precise later on). Then, we define the fluid domain deformation $\Phi(d_s(t))$ in the whole domain Ω_f as follows:

$$\Phi(d_s(t)) \quad = \quad \mathcal{I} + \mathcal{R}(\gamma_\Gamma(d_s(t))) \quad \text{in } \Omega_f,$$

such that the deformation is equal to the identity on the fluid boundary Γ_f. Thus the mapping $\Phi(d_s(t))$ maps the reference fluid domain Ω_f into the deformed fluid domain at time t denoted by $\Phi(d_s(t))(\Omega_f)$.

The domain Ω_f is filled with a Newtonian viscous fluid whose viscosity is denoted by μ_f. The velocity u_f and the pressure p_f of the fluid satisfy, at each time t, the Stokes equations in the deformed configuration $\Phi(d_s(t))(\Omega_f)$, which write

$$\begin{array}{rcll} -\mathrm{div}\,(\sigma_f(u_f(t), p_f(t))) & = & 0 & \text{in } \Phi(d_s(t))(\Omega_f), \\ \mathrm{div}(u_f(t)) & = & 0 & \text{in } \Phi(d_s(t))(\Omega_f), \\ u_f(t) & = & 0 & \text{on } \Phi(d_s(t))(\Gamma_f) = \Gamma_f, \end{array} \tag{1.3}$$

where

$$\begin{array}{rcl} \sigma_f(u_f(t), p_f(t)) & = & 2\mu_f D(u_f(t)) - p_f(t) I, \\ D(u_f(t)) & = & \dfrac{1}{2}\left(\nabla u_f(t) + \nabla u_f(t)^T\right). \end{array}$$

The tensor $\sigma_f(u_f(t), p_f(t))$ is the fluid stress tensor written in the deformed configuration $\Phi(d_s(t))(\Omega_f)$ and $D(u_f(t))$ is the symmetric part of the gradient of the fluid velocity. Moreover, we suppose that no external force is applied to the fluid.

To complete the set of equations (1.1), (1.2) and (1.3), we add the usual coupling conditions on the fluid-structure interface Γ, namely the continuity conditions on the velocities and on the normal component of the stress tensors that traduce the action reaction principle:

$$\frac{\partial d_s}{\partial t}(t) = w_f(t) \text{ on } \Gamma, \tag{1.4}$$

$$\Pi_s(d_s(t), t) n_f = \Pi_f(w_f(t), q_f(t)) n_f \text{ on } \Gamma, \tag{1.5}$$

where n_f is the exterior unit normal vector of $\partial\Omega_f$ and where w_f and q_f are respectively the fluid velocity and fluid pressure written in the reference configuration, and $\Pi_f(w_f(t), q_f(t))$ is the fluid stress tensor also written in the reference configuration that will be made precise below. More precosely, the fluid velocity w_f and the fluid pressure q_f are defined at time t by

$$w_f(t, \cdot) = u_f(t, \Phi(d_s(t)(\cdot)) \quad \text{and} \quad q_f(t, \cdot) = p_f(t, \Phi(d_s(t)(\cdot)) \quad \text{in } \Omega_f.$$

Even though the interaction problem we consider in the present study is non-inertial, it requires an initial condition for the displacement of the structure on the interface Γ at $t = 0$, because of the condition on the continuity of velocities in (1.4). For the sake of simplicity, as in [7] and [24] for instance, we suppose that the interface is in its reference configuration initially, i.e. that

$$d_s(0) = 0 \text{ on } \Gamma, \tag{1.6}$$

so that the kinematic boundary condition (1.4) writes

$$d_s(t) = \int_0^t w_f(s) ds \text{ on } \Gamma. \tag{1.7}$$

Conditions (1.4) and (1.7) differ in the sense that condition (1.4) can be seen as a Dirichlet boundary condition for the fluid problem whereas the condition (1.7) is a Dirichlet boundary condition for the structure problem.

Moreover, we will at first suppose that the structure is at rest initially, i.e. that $f_s(0) = \mathrm{div}\,(\Sigma^*(0))$ which, combined with the initial condition (1.6), implies that $d_s(0) = 0$ in Ω_s. The more general case where $f_s(0) \neq \mathrm{div}\,(\Sigma^*(0))$ is discussed in section 7.

In order to define the fluid stress tensor Π_f we rewrite the Stokes system (1.3) in the reference configuration. As it is usual it is on this system we will work since the original Stokes equations are set in an unknown domain $\Phi(d_s(t))(\Omega_f)$.

To that aim, we introduce the mappings F, G and H defined (in Sobolev spaces that will be defined later on) by

$$\begin{array}{rcl} F(d_s(t)) & = & (\nabla(\Phi(d_s(t))))^{-1}, \\ G(d_s(t)) & = & \mathrm{cof}(\nabla(\Phi(d_s(t)))), \\ H(d_s(t)) & = & F(d_s(t))G(d_s(t)). \end{array} \tag{1.8}$$

The matrix $F(d_s(t))$ (and *a fortiori* the matrix $H(d_s(t))$) is well defined whenever $\Phi(d_s(t))$ is, for instance, a C^1-diffeomorphism, which will be the case for small enough displacements in well chosen spaces. Then, using the definitions of the mappings F, G and H, it follows from a change of variables that the Stokes equations written in the reference configuration of the fluid is

$$\begin{array}{rcll} -\mu_f \mathrm{div}\,\big((H(d_s(t))\nabla)w_f(t) + F(d_s(t))^T\nabla w_f(t)^T G(d_s(t))\big) & & & \\ +G(d_s(t))\nabla q_f(t) & = & 0 & \text{in } \Omega_f, \\ \mathrm{div}(G(d_s(t))^T w_f(t)) & = & 0 & \text{in } \Omega_f, \\ w_f(t) & = & 0 & \text{on } \Gamma_f, \end{array} \tag{1.9}$$

Moreover, the fluid stress tensor written in the reference configuration at time t is defined by

$$\begin{array}{rcl} \Pi_f(w_f(t), q_f(t)) & = & \mu_f\Big((H(d_s(t))\nabla)w_f(t) \\ & & +F(d_s(t))^T\nabla w_f(t)^T G(d_s(t))\Big) \\ & & -q_f(t)G(d_s(t)). \end{array} \tag{1.10}$$

Remark 1.1. The tensor Π_f is the Piola transform of the fluid stress tensor σ_f.

We now briefly outline the content of the present article. In section 2 we introduce some notations and prove some preliminary results. In particular, we show that, for sufficiently small displacements of the structure, the mapping F, defined by (1.8), is well-defined, such that the fluid problem in the reference configuration (1.9) is also well-defined. In section 3, we state our main result, namely the existence and the uniqueness (locally in time) of a regular strong solution to equations (1.1), (1.2), (1.9), (1.4), (1.5), (1.6) if the data are sufficiently small and if the structure is at rest initially. The proof is done thanks to Banach's fixed point Theorem, constructing a mapping that iterates between the resolution of the structure problem and the fluid problem. The particularity of the present quasi-static problem compared to steady problems (as in [19] for instance) relies on the kinematic condition (1.4) on the interface Γ, which is an unsteady condition. So that, in order to prove the convergence of the iterative process we have to split the fluid-structure problem by solving the structure problem with the Dirichlet boundary condition (1.7) and the fluid problem with the Neumann boundary condition (1.5). By doing so we ensure compactness in time, whereas if one solve the structure problem with the Neumann boundary condition (1.5) and the fluid problem with the Dirichlet boundary condition (1.4), then we loose time regularity in the iterative process. To that aim, we study in section 4 and in section 5, the structure and fluid problems independently. In section 6, the actual fixed point procedure is conducted, showing that the aforementioned mapping goes from a ball into itself and is a contraction. Finally, in section 7, we extend our result to the case where the internal forcing does not counterbalance the exterior load initially, i.e. $f_s(0) \neq \mathrm{div}\,(\Sigma^*(0))$.

2 Notations and preliminaries

2.1 Technical lemma

In this subsection, Ω denotes an open connected bounded subset of $\mathbb{R}^n$ of class $C^{k-1,1}$, for $k \geq 1$. For $r \geq 0$, the space $H^r(\Omega)$ denotes a standard Sobolev space associated to the L^2-norm. Moreover, the same notation is used whenever it is a space of real-valued functions or vector-valued functions.

If Γ is a part of the boundary of Ω, we say that Γ is a disjoint part of $\partial\Omega$ if Γ is non empty and $\overline{\Gamma} \cap (\overline{\partial\Omega \setminus \Gamma}) = \emptyset$. Then, if Γ is a disjoint part of $\partial\Omega$, we denote by γ_Γ the trace operator on Γ, which is continuous and surjective from $H^r(\Omega)$ onto $H^{r-1/2}(\Gamma)$ for all $\frac{1}{2} < r \leq k$ (see [9, chap. 3, sec. 2.5.3]). In particular, there exists a continuous lifting operator from $H^{r-1/2}(\Gamma)$ into $H^r(\Omega)$. Using this notations, we denote by $H^1_\Gamma(\Omega)$ the space defined by

$$H^1_\Gamma(\Omega) = \left\{ u \in H^1(\Omega); \gamma_\Gamma(u) = 0 \right\}.$$

For $s \geq 0$, the space $H^s(0,T;H^r(\Omega))$ denotes the space of Sobolev-valued functions in the time interval $(0,T)$, with $T > 0$. For $s = 0$ we denote this space by $L^2(0,T;H^r(\Omega))$.

The constant C that appears through the text always denotes a positive constant that can change from line to line. However, its dependencies on domains, variables or parameters will always be made clear.

We start by giving a technical lemma.

Lemma 2.1. *Let Ω be a Lipschitz open connected bounded subset of $\mathbb{R}^n$, with $n \geq 1$, and consider Γ, a part of its boundary.*

i) Let $r > \frac{n}{2}$. If u and v belong to $H^r(\Omega)$, then the product uv belongs to $H^r(\Omega)$ and there exists a constant $C(\Omega)$ which depends on the domain Ω such that,

$$\|uv\|_{H^r(\Omega)} \leq C(\Omega)\|u\|_{H^r(\Omega)}\|v\|_{H^r(\Omega)}.$$

ii) Let $r \geq 0$ and $s > \max(\frac{n}{2}, r)$. If u belongs to $H^s(\Omega)$ and v belongs to $H^r(\Omega)$, then the product uv belongs to $H^r(\Omega)$ and there exists a constant $C(\Omega)$ which depends on the domain Ω such that,

$$\|uv\|_{H^r(\Omega)} \leq C(\Omega)\|u\|_{H^s(\Omega)}\|v\|_{H^r(\Omega)}.$$

Moreover, we will call $H^s(\Omega)$ a multiplier space of $H^r(\Omega)$.

iii) Let $r > \frac{1}{2}$ and $T > 0$. Moreover, suppose that Ω is of class $C^{r-1,1}$. If w belongs to $L^2(0,T;H^r(\Omega))$, then the function δ defined by

$$\delta(t) = \int_0^t \gamma_\Gamma(w(s))ds, \quad \forall t \in [0,T],$$

belongs to $H^1(0,T;H^{r-1/2}(\Gamma))$, and there exists a constant $C(\Omega)$ which depends on the domain Ω such that,

$$\|\delta\|_{L^\infty(0,T;H^{r-1/2}(\Gamma))} \leq C(\Omega)T^{1/2}\|w\|_{L^2(0,T;H^r(\Omega))}.$$

Proof. The proof of point $i)$ relies on Sobolev injections and we refer to [1] for detailed information. The essential point here is that the space $H^r(\Omega)$ is a Banach algebra, because $2r$ is greater than the dimension n.

Point $ii)$ is a consequence of [3, Theorem 7.5] and also relies on embedding theorems for Sobolev spaces.

For the point $iii)$, a straightforward computation gives us that, for all t in $[0,T]$,

$$\|\delta(t)\|_{H^{r-1/2}(\Gamma)} = \| \int_0^t \gamma_\Gamma(\omega(s))ds\|_{H^{r-1/2}(\Gamma)} \leq C(\Omega)t^{1/2}\|\omega\|_{L^2(0,T;H^r(\Omega))},$$

where $C(\Omega)$ is a constant coming from the continuity of the trace operator from $H^r(\Omega)$ onto $H^{r-1/2}(\Gamma)$. It follows that

$$\|\delta\|_{L^\infty(0,T;H^{r-1/2}(\Gamma))} \leq C(\Omega)T^{1/2}\|\omega\|_{L^2(0,T;H^r(\Omega))}.$$

Moreover, because $\frac{\partial\delta}{\partial t} = \gamma_\Gamma(\omega)$ belongs to $L^2(0,T;H^{r-1/2}(\Gamma))$, we can conclude that δ belongs to $H^1(0,T;H^{r-1/2}(\Gamma))$. $\square$

2.2 Assumptions and preliminary results

In all that follows, we assume that the following assumptions hold true.

$(\mathbf{H_1})$	Domain Ω is an open connected bounded subset of $\mathbb{R}^n$ ($n \in \{2,3\}$) divided in two open connected bounded sets Ω_f and Ω_s by an interface Γ.
$(\mathbf{H_2})$	The interface Γ is of class $C^{3,1}$, is non empty and does not encounter the boundary of Ω, i.e. $\overline{\Gamma} \cap \overline{\partial\Omega} = \emptyset$.
$(\mathbf{H_3})$	The remaining boundaries are denoted $\Gamma_f = \partial\Omega_f \setminus \Gamma$ and $\Gamma_s = \partial\Omega_s \setminus \Gamma$ and are of class $C^{2,1}$.
$(\mathbf{H_4})$	The boundary Γ_f is such that $\|\Gamma_f\| \neq 0$, whereas the boundary Γ_s could be such that $\|\Gamma_s\| = 0$.

Remark 2.1. Note that at many steps one could only assume a $C^{2,1}$ regularity of the domains, yet the required $C^{3,1}$ regularity on Γ is used to prove the elliptic regularity of the solution of the fluid problem with mixed Dirichlet and Neumann boundary conditions, which requires more regularity than when considering only a Dirichlet boundary condition (see [9]).

Remark 2.2. Under assumptions $(\mathbf{H_1})$-$(\mathbf{H_4})$, we see that boundaries Γ, Γ_f and Γ_s are all disjoint parts of $\partial\Omega_f$ and $\partial\Omega_s$. An example of such a domain is given in Figure 1.

Let $T > 0$. The fluid problem written in the reference configuration, defined by (1.9), is well-defined if, for almost every t in $(0, T)$, the displacement of the structure at time t, $d_s(t)$, is sufficiently regular and if the deformation $\Phi(d_s(t))$ is a C^1-diffeomorphism that maps Ω_f into $\Phi(d_s(t))(\Omega_f)$. In the next lemma, we show that this is true if the displacement of the structure at time t belongs to the ball $\mathcal{B}^S_{\mathcal{M}_0}$ of $H^3(\Omega_s)$, defined by

$$\mathcal{B}^S_{\mathcal{M}_0} = \{b \in H^3(\Omega_s); \|b\|_{H^3(\Omega_s)} \leq \mathcal{M}_0\}, \tag{2.1}$$

if the constant $\mathcal{M}_0$ is sufficiently small.

Let $\mathcal{R}$ be a linear lifting operator from $H^{5/2}(\Gamma)$ to $H^3(\Omega_f) \cap H^1_{\Gamma_f}(\Omega_f)$ and we recall that γ_Γ is the trace operator on the interface Γ. Since the domains are of class $C^{2,1}$, they are both continuous operators. We have the following result

Lemma 2.2. *There exists a constant $\mathcal{M}_0 > 0$ such that for all b in $\mathcal{B}^S_{\mathcal{M}_0}$, we have*

i) $\nabla(\mathcal{I} + \mathcal{R}(\gamma_\Gamma(b))) = I + \nabla(\mathcal{R}(\gamma_\Gamma(b)))$ *is an invertible matrix in* $H^2(\Omega_f)$*,*

ii) $\Phi(b) = \mathcal{I} + \mathcal{R}(\gamma_\Gamma(b))$ *is one to one on* $\bar{\Omega}_f$*,*

iii) $\Phi(b)$ *is a* C^1*-diffeomorphism from* Ω_f *onto* $\Phi(b)(\Omega_f)$*,*

Proof. It is clear that $\Phi(b) = \mathcal{I} + \mathcal{R}(\gamma_\Gamma(b))$ belongs to $H^3(\Omega_f)$ for all b in $H^3(\Omega_s)$. From Lemma 2.1, point i), we know that $H^2(\Omega_f)$ is a Banach algebra. Thus, if $\mathcal{M}_0$ is chosen such that

$$\|b\|_{H^3(\Omega_s)} \leq \mathcal{M}_0 \implies \|\nabla(\mathcal{R}(\gamma_\Gamma(b)))\|_{H^2(\Omega_f)} < \frac{1}{C(\Omega_f)},$$

where $C(\Omega_f)$ is defined in Lemma 2.1, then $I + \nabla(\mathcal{R}(\gamma_\Gamma(b)))$ is an invertible matrix in $H^2(\Omega_f)$ and i) is proven.

For point ii), we know from [13, Theorem 5.5-1] that there exists a constant $C > 0$ such that for all ϕ in $C^1(\bar{\Omega}_f)$,

$$\|\nabla\phi\|_{C^0(\bar{\Omega}_f)} \leq C \implies \begin{cases} \det(\nabla(\mathcal{I} + \phi))(x) > 0, \ \forall x \in \bar{\Omega}_f, \\ \mathcal{I} + \phi \text{ is injective on } \bar{\Omega}_f. \end{cases}$$

Then, because of the continuous embedding of $H^3(\Omega_f)$ into $C^1(\bar{\Omega}_f)$ (see [1, Theorem 6.3, part III]), and if $\mathcal{M}_0$ is chosen small enough, this result can be applied to $\mathcal{R}(\gamma_\Gamma(b))$ and we obtain point i). Finally, using the continuous embedding of $H^3(\Omega_f)$ into $C^1(\bar{\Omega}_f)$, the fact that $\det(\nabla\Phi(b))(x) > 0$, $\forall x \in \bar{\Omega}_f$ and point ii), we can apply the inverse function theorem and prove point iii). □

With the previous lemma we know that F and H, defined by (1.8), are well-defined in $H^2(\Omega_f)$ for all d_s in $\mathcal{B}^S_{\mathcal{M}_0}$. Now we state another lemma, dealing with the mappings F, G and H.

Lemma 2.3. *The mapping G defined from $H^3(\Omega_s)$ into $H^2(\Omega_f)$ is of class C^∞. The mappings F and H are defined from $\mathcal{B}^S_{\mathcal{M}_0}$ into $H^2(\Omega_f)$ and are infinitely differentiable everywhere in $\mathcal{B}^S_{\mathcal{M}_0}$.*

Proof. Let b be in $\mathcal{B}^S_{\mathcal{M}_0}$, then $G(b)$ belongs to $H^2(\Omega_f)$ because $H^2(\Omega_f)$ is a Banach algebra according to Lemma 2.1. The fact that $F(b)$ and $H(d)$ belong to $H^2(\Omega_f)$ is also due to the fact that $H^2(\Omega_f)$ is a Banach algebra, along with the invertible property of $\nabla\Phi(b)$ in $H^2(\Omega_f)$. The mapping G is of class C^∞ by composition of C^∞ mappings (such as γ, $\mathcal{R}$, ∇, det, cof). For the regularity of the mappings F and H, it is sufficient to use the fact that the mapping

$$\begin{array}{rcl} H^2(\Omega_f) & \to & H^2(\Omega_f) \\ M & \mapsto & M^{-1} \end{array}$$

is infinitely differentiable for all invertible matrix in $H^2(\Omega_f)$ (see [12, chap. I]). □

Corollary 2.1. *For all b_1 and b_2 in $\mathcal{B}^S_{\mathcal{M}_0}$, we have the following estimates,*

$$\begin{array}{rcl} \|F(b_1)-F(b_2)\|_{H^2(\Omega_f)} & \leq & C(\mathcal{M}_0)\|b_1-b_2\|_{H^3(\Omega_s)}, \\ \|G(b_1)-G(b_2)\|_{H^2(\Omega_f)} & \leq & C(\mathcal{M}_0)\|b_1-b_2\|_{H^3(\Omega_s)}, \\ \|H(b_1)-H(b_2)\|_{H^2(\Omega_f)} & \leq & C(\mathcal{M}_0)\|b_1-b_2\|_{H^3(\Omega_s)}. \end{array}$$

where $C(\mathcal{M}_0)$ are positive constants which depend on $\mathcal{M}_0$.

Proof. This result is a straightforward application of Lemma 2.3 and the mean value inequality (see [12, Thm. 3.3.2]). The constants appearing in these inequalities are in fact given by

$$\begin{array}{l} \sup\limits_{b\in\mathcal{B}^S_{\mathcal{M}_0}} \|DF(b)\|_{\mathcal{L}(H^3(\Omega_s),H^2(\Omega_f))}, \\ \sup\limits_{b\in\mathcal{B}^S_{\mathcal{M}_0}} \|DG(b)\|_{\mathcal{L}(H^3(\Omega_s),H^2(\Omega_f))}, \\ \sup\limits_{b\in\mathcal{B}^S_{\mathcal{M}_0}} \|DH(b)\|_{\mathcal{L}(H^3(\Omega_s),H^2(\Omega_f))}. \end{array}$$

For the sake of simplicity in the upcoming computations, we denote them all by $C(\mathcal{M}_0)$. □

3 Main result

In this section, we state the existence of a local (in time) solution for the fluid-structure interaction system with an active stress term, for small enough applied forces and a small enough internal activity of the structure.

Theorem 3.1. *Let the domains and frontiers Ω_f, Ω_s, Γ_f, Γ_s and Γ be defined by* $(\mathbf{H_1})$-$(\mathbf{H_4})$ *and let $T>0$. Consider the force f_s in $L^\infty(0,T;H^1(\Omega_s))$ and the internal activity Σ^* in $L^\infty(0,T;H^2(\Omega_s))$, the data of the problem, such that $f_s(0)=\mathrm{div}\,(\Sigma^*(0))$. Let us introduce the solution of a Stokes problem, denoted by (w^0_f,q^0_f), which satisfies the equations*

$$\begin{array}{rcll} -\mathrm{div}\left(\sigma_f(w^0_f,q^0_f)\right) & = & 0 & \text{in } \Omega_f, \\ \mathrm{div}(w^0_f) & = & 0 & \text{in } \Omega_f, \\ w^0_f & = & 0 & \text{on } \Gamma_f, \\ \sigma_f(w^0_f,q^0_f)\cdot n_f & = & -\Sigma^*(0)\cdot n_f & \text{on } \Gamma. \end{array} \tag{3.1}$$

Let $\mathcal{M}_1>0$ and consider the ball $\mathcal{B}^F_{\mathcal{M}_1}$ defined by

$$\begin{aligned} \mathcal{B}^F_{\mathcal{M}_1} = \Big\{ & (\omega,\pi)\in L^2(0,T;H^3(\Omega_f)\cap H^1_{\Gamma_f}(\Omega_f))\times L^2(0,T;H^2(\Omega_f)); \\ & \|\omega-w^0_f\|_{L^2(0,T,H^3(\Omega_f))}+\|\pi-q^0_f\|_{L^2(0,T,H^2(\Omega_f))}\leq\mathcal{M}_1\Big\}. \end{aligned} \tag{3.2}$$

If the data f_s and Σ^*, the time T and the constant $\mathcal{M}_1$ are sufficiently small, then there exists a unique solution (w_f, q_f, d_s) of (1.1), (1.2), (1.9), (1.4) and (1.5), with (w_f, q_f) in $\mathcal{B}^F_{\mathcal{M}_1}$ and d_s in $L^\infty(0,T;\mathcal{B}^S_{\mathcal{M}_0} \cap H^1_{\Gamma_s}(\Omega_s))$.

Remark 3.1. To be precise, the data f_s and Σ^*, the time T and the constant $\mathcal{M}_1$ are considered to be sufficiently small in order to apply the previous theorem, if they satisfy the following conditions, that will appear in the proof of Theorem 3.1, detailed in section 6:

$$\begin{aligned} &\| f_s - \mathrm{div}\,(\Sigma^*)\, \|_{L^\infty(0,T,H^1(\Omega_s))} \\ &\qquad + R_0 \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))} + C_1 \mathcal{M}_1 T^{1/2} \leq R_1, \end{aligned} \tag{3.3}$$

$$C^1_s \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))} < 1, \tag{3.4}$$

$$\begin{aligned} &\| f_s - \mathrm{div}\,(\Sigma^*)\, \|_{L^\infty(0,T,H^1(\Omega_s))} \\ &\qquad + (R_0 + C_1 C_f T) \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))} + C_1 \mathcal{M}_1 T^{1/2} \leq \frac{\mathcal{M}_0}{C^2_s}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &C_2 \Big((1+T) T^{1/2} \|\Sigma^*\|^2_{L^\infty(0,T;H^2(\Omega_s))} \\ &+ T^{1/2} \| f_s - \mathrm{div}\,(\Sigma^*)\, \|_{L^\infty(0,T;H^1(\Omega_s))} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\ &+ (\mathcal{M}_1 + T^{1/2}) \| f_s - \mathrm{div}\,(\Sigma^*)\, \|_{L^\infty(0,T;H^1(\Omega_s))} \\ &+ (T^{3/2} + \mathcal{M}_1 T + T^{1/2} + \mathcal{M}_1) \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\ &+ (T^{1/2} + \mathcal{M}_1) \mathcal{M}_1 T^{1/2} \Big) \leq \mathcal{M}_1, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &C_3 \Big(\| f_s - \mathrm{div}\,(\Sigma^*)\, \|_{L^\infty(0,T;H^1(\Omega_s))} + (1+T) \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\ &+ \mathcal{M}_1 T^{1/2} + \frac{T + \mathcal{M}_1 T^{1/2} + T \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}}{1 - C^1_s \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}} \Big) < 1. \end{aligned} \tag{3.7}$$

where R_0, R_1, $\mathcal{M}_0$, C^1_s, C^2_s, C_f, C_1, C_2 and C_3 are positive constants which only depend on the domains Ω_f and Ω_s, the viscosity of the fluid μ_f and the elasticity parameters μ_s and λ_s of the structure.

The proof of Theorem 3.1 is based on Banach's fixed point theorem (see [10, Theorem V.7]). In this scope, we construct a mapping $\mathcal{S}$ defined from $\mathcal{B}^F_{\mathcal{M}_1}$, the ball defined by equation (3.2), into the space for the velocity and pressure of the fluid $L^2(0,T;H^3(\Omega_f)) \times L^2(0,T;H^2(\Omega_f))$. This mapping is defined through a composition of mappings that takes a couple (ω, π) in $\mathcal{B}^F_{\mathcal{M}_1}$, constructs a boundary condition δ in $H^1(0,T;H^{5/2}(\Gamma))$, solves an elasticity problem with Dirichlet boundary conditions associated to the data (f_s, Σ^*, δ) and, finally, solves a fluid problem with mixed Dirichlet and Neumann boundary conditions. In fact, $\mathcal{S}$ can be represented as the following composition of mappings:

$$\mathcal{S} = \mathcal{O}_3 \circ \mathcal{O}_2 \circ \mathcal{O}_1, \tag{3.8}$$

where each mapping writes

$$\mathcal{O}_1 \quad : \quad \begin{array}{ccl} \mathcal{B}^F_{\mathcal{M}_1} & \to & H^1(0,T;H^{5/2}(\Gamma)) \\ (\omega,\pi) & \mapsto & \delta \end{array} ,$$

$$\mathcal{O}_2 \quad : \quad \begin{array}{rcl} H^1(0,T;H^{5/2}(\Gamma)) & \to & L^\infty(0,T;H^3(\Omega_s)) \\ \delta & \mapsto & d_s \end{array} ,$$

$$\mathcal{O}_3 \quad : \quad \begin{array}{rcl} L^\infty(0,T;H^3(\Omega_s)) & \to & L^2(0,T;H^3(\Omega_f)) \times L^2(0,T;H^2(\Omega_f)) \\ d_s & \mapsto & (w_f, q_f) \end{array} ,$$

and will now be defined.

Given a couple (ω, π) in $\mathcal{B}^F_{\mathcal{M}_1}$, the boundary condition δ is constructed, for all $t \in [0,T]$, by

$$\delta(t) = \int_0^t \gamma_\Gamma(\omega(s))ds,$$

which, according to Lemma 2.1, belongs to $H^1(0,T;H^{5/2}(\Gamma))$. This defines the mapping $\mathcal{O}_1$.

Then, with this boundary condition, we consider the following problem, to obtain an elastic displacement d_s such that, for almost every t in $(0,T)$,

$$\left\{ \begin{array}{rclcl} -\mathrm{div}\,(\Pi_s(d_s(t),t)) & = & f_s(t) & \text{in} & \Omega_s, \\ d_s(t) & = & \delta(t) & \text{on} & \Gamma, \\ d_s(t) & = & 0 & \text{on} & \Gamma_s. \end{array} \right. \tag{3.9}$$

This elasticity problem with internal activity is studied in section 4. As we will see, it admits a unique solution if the data (f_s, Σ^*, δ) are small enough, namely if conditions (3.3) and (3.4) are satisfied. This step defines the mapping $\mathcal{O}_2$.

Next, to define the mapping $\mathcal{O}_3$, we consider a fluid problem, obtained from the fluid equations written in the reference configuration (1.9) through a perturbation argument, which writes: find (w_f, q_f) such that, for almost every t in $(0,T)$,

$$\left\{ \begin{array}{cl} -\mathrm{div}\,(\sigma_f(w_f(t),q_f(t))) = -\mu_f \mathrm{div}\,(((I-H(d_s(t))\nabla)\omega(t)) & \\ -\mu_f \mathrm{div}\left(\nabla\omega(t)^T - F(d_s(t))^T\nabla\omega(t)^T G(d_s(t))\right) & \\ +(I-G(d_s(t)))\nabla\pi(t) & \text{in } \Omega_f, \\ \mathrm{div}\,(w_f(t)) = -\mathrm{div}\left((I-G(d_s(t))^T)\omega(t)\right) & \text{in } \Omega_f, \\ w_f(t) = 0 & \text{on } \Gamma_f, \\ \sigma_f(w_f(t),q_f(t))n_f = \Pi_s(d_s(t),t)n_f & \\ +\mu_f\,((I-H(d_s(t))\nabla)\omega(t))\,n_f & \\ +\mu_f\left(\nabla\omega(t)^T - F(d_s(t))^T\nabla\omega(t)^T G(d_s(t))\right)n_f & \\ -\left(\pi(t)(I-G(d_s(t)))\right)n_f & \text{on } \Gamma. \end{array} \right. \tag{3.10}$$

This is a Stokes problem with mixed Dirichlet and Neumann boundary conditions, which is studied in section 5. Moreover, condition (3.5) ensures that, for almost all t in $(0,T)$, the displacement $d_s(t)$ belongs to the ball $\mathcal{B}^S_{\mathcal{M}_0}$ (defined by (2.1)), such that problem (3.10) is well-defined.

Finally, under condition (3.6) the image by $\mathcal{S}$ of the ball $\mathcal{B}^F_{\mathcal{M}_1}$ is included in $\mathcal{B}^F_{\mathcal{M}_1}$ (i.e. $\mathcal{S}(\mathcal{B}^F_{\mathcal{M}_1}) \subset \mathcal{B}^F_{\mathcal{M}_1}$) and under condition (3.7), $\mathcal{S}$ is a contraction mapping such that we can apply Banach's fixed point theorem.

Therefore, this is a solution of the fluid-structure interaction problem since each fixed point of $\mathcal{S}$ in $\mathcal{B}^F_{\mathcal{M}_1}$ is a solution of (1.1), (1.2), (1.9), (1.4) and (1.5).

Remark 3.2. In the stationary case studied in [19], the fixed point procedure is done by solving the fluid problem in a given geometry with Dirichlet boundary conditions and the solid problem with Neumann boundary conditions and performing the fixed point on the geometry. However, in the present study, where we consider a quasi-static time dependent model, the fixed point is conducted on the velocity and the pressure of the fluid by solving the structure with Dirichlet boundary conditions and the fluid problem with mixed Dirichlet and Neumann boundary conditions. Our choice on fluid-structure splitting is due to the fact that we need time regularity on the solution. Indeed, because of the kinematic coupling condition (1.4), we instantly lose time regularity in the decoupling process if the Dirichlet boundary conditions are applied to the fluid.

Remark 3.3. The condition $f_s(0) = \mathrm{div}\,(\Sigma^*(0))$ appearing in Theorem 3.1 along with the hypothesis that the displacement of the structure is null on the interface Γ at $t=0$, ensures that the structure is at rest initially. Indeed this implies that $d_s(0) = 0$. Moreover if $\gamma(d_s(0)) \neq 0$, the linearization of the fluid problem in the reference configuration (1.9), that we considered in problem (3.10), has to be done around the initial geometrical configuration given by $d_s(0)$, instead of the reference one.

The remaining of this article is the following. In section 4, we consider system (3.9) and prove its well-posedness and the regularity of its solution. In section 5, we study equations (3.10) and prove existence, uniqueness and regularity results. In section 6, the proof of Theorem 3.1 is done using a fixed point procedure on the mapping $\mathcal{S}$. Finally, in section 7, an extension of Theorem 3.1 with more general data is given.

4 Structure equations

In this section we study the two or three-dimensional elasticity equations with non-homogeneous Dirichlet boundary conditions, where the solid is described by the nonlinear Saint Venant-Kirchhoff law, and with an additional active stress. The domain Ω_s satisfies $(\mathbf{H_1})$-$(\mathbf{H_4})$. For a given body force f_s, a given Dirichlet boundary condition δ on Γ and a given active stress tensor Σ^*, the considered structure problem writes:

$$\left\{\begin{array}{rcll} -\mathrm{div}((I+\nabla d)(\Sigma_s(d)-\Sigma^*)) & = & f_s & \text{in} \quad \Omega_s, \\ d & = & \delta & \text{on} \quad \Gamma, \\ d & = & 0 & \text{on} \quad \Gamma_s, \end{array}\right. \tag{4.1}$$

with the passive stress tensor Σ_s defined by (1.2). Then, the following lemma states that problem (4.1) admits a unique solution d in $H^3(\Omega_s)$, for small enough data.

Lemma 4.1. *Let Ω_s, Γ_s and Γ be defined by $(\mathbf{H_1})$-$(\mathbf{H_4})$ and suppose that f_s belongs to $H^1(\Omega_s)$, δ belongs to $H^{5/2}(\Gamma)$ and Σ^* belongs to $H^2(\Omega_s)$. There exist three real positive constants R_0, R_1 and C_s^1, that only depend on the domain Ω_s and the elasticity parameters μ_s and λ_s such that, if the data satisfy the conditions*

$$\|f_s - \mathrm{div}\,(\Sigma^*)\,\|_{H^1(\Omega_s)} + R_0\|\Sigma^*\|_{H^2(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)} \leq R_1, \tag{4.2}$$

$$C_s^1\|\Sigma^*\|_{H^2(\Omega_s)} < 1, \tag{4.3}$$

then there exists a unique solution d of (4.1) in a neighborhood of 0 in the space $H^3(\Omega_s)\cap H^1_{\Gamma_s}(\Omega_s)$. Moreover, there exists a positive constant C_s^2 that only depends on Ω_s, μ_s and λ_s such that the solution can be estimated with respect to the data:

$$\|d\|_{H^3(\Omega_s)} \leq C_s^2(\|f_s - \mathrm{div}(\Sigma^*)\|_{H^1(\Omega_s)} + R_0\|\Sigma^*\|_{H^2(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)}). \tag{4.4}$$

Proof. The proof is based on Banach's fixed point Theorem. We construct a mapping $\mathcal{T}$ defined from $H^3(\Omega_s)\cap H^1_{\Gamma_s}(\Omega_s)$ into itself which, to all u in $H^3(\Omega_s)\cap H^1_{\Gamma_s}(\Omega_s)$, associates the solution of the following passive elasticity problem: find a structure displacement d such that

$$\left\{\begin{array}{rl} -\mathrm{div}((I+\nabla d)\Sigma_s(d)) = f_s - \mathrm{div}\,(\Sigma^*) - \mathrm{div}\,(\nabla u\Sigma^*) & \text{in } \Omega_s, \\ d=\delta & \text{on } \Gamma, \\ d=0 & \text{on } \Gamma_s. \end{array}\right. \tag{4.5}$$

More precisely, $\mathcal{T}$ is the following mapping:

$$\begin{array}{rcccl} \mathcal{T} & : & H^3(\Omega_s)\cap H^1_{\Gamma_s}(\Omega_s) & \to & H^3(\Omega_s)\cap H^1_{\Gamma_s}(\Omega_s) \\ & & u & \mapsto & d. \end{array}$$

The objective is to show that the mapping $\mathcal{T}$ has a fixed point d in the space $H^3(\Omega_s)\cap H^1_{\Gamma_s}(\Omega_s)$, which in turns solves problem (4.1).

The rest of the proof is divided in three parts. Firstly, we prove that, under condition (4.2) on the data, the mapping $\mathcal{T}$ is well-defined, i.e. that problem (4.5) admits a unique solution in the space $H^3(\Omega_s)\cap H^1_{\Gamma_s}(\Omega_s)$. Secondly, we show that, under condition (4.3) on the data, $\mathcal{T}$ is a contraction from a ball in $H^3(\Omega_s)\cap H^1_{\Gamma_s}(\Omega_s)$ into itself. Finally, we apply Banach's fixed point Theorem and obtain the desired estimate.

Let us show that (4.5) admits a unique solution in a neighborhood of 0 in $H^3(\Omega_s) \cap H^1_{\Gamma_s}(\Omega_s)$ if the data are sufficiently small. A study of elasticity problems similar to problem (4.5) has been conducted in [13] in Sobolev spaces $W^{2,p}$ with $p > n$ and considering homogeneous Dirichlet boundary conditions; our proof essentially uses the same arguments. We introduce the following nonlinear operator of passive elasticity:

$$\begin{array}{rcccc} A & : & H^3(\Omega_s) \cap H^1_{\Gamma_s}(\Omega_s) & \rightarrow & H^1(\Omega_s) \times H^{5/2}(\Gamma), \\ & & d & \mapsto & (-\mathrm{div}((I + \nabla d)\Sigma_s(d)), \gamma_\Gamma(d)), \end{array}$$

where we recall that γ_Γ is the trace operator on Γ. The mapping A is defined since $H^2(\Omega_s)$ is an algebra in both two or three space dimensions (see Lemma 2.1) and is infinitely differentiable since it is a sum of continuous multilinear mappings. As a consequence, problem (4.5) can be written in term of operator: find d such that

$$A(d) = (\tilde{f}_s, \delta),$$

where

$$\tilde{f}_s := f_s - \mathrm{div}(\Sigma^*) - \mathrm{div}\left(\nabla u \Sigma^*\right).$$

We can observe that $d = 0$ is a particular solution corresponding to the data $(\tilde{f}_s, \delta) = (0, 0)$. Thus, a natural idea consists in showing that the mapping A is locally invertible in a neighborhood of this particular solution. In order to prove it, we need to check that the differential of A at 0 is an isomorphism between $H^3(\Omega_s) \cap H^1_{\Gamma_s}(\Omega_s)$ and $H^1(\Omega_s) \times H^{5/2}(\Gamma)$, to be able to use the implicit function Theorem. This differential at point 0 is given by

$$\mathcal{D}A(0) \cdot d = \left(-\mathrm{div}\left(2\mu_s D(d) + \lambda_s \mathrm{div}\,(d)\, I\right), \gamma_\Gamma(d)\right),$$

where $D(d) = \frac{1}{2}(\nabla d + \nabla d^T)$ is the symmetric gradient of d. The operator $\mathcal{D}A(0)$ is the linearized elasticity operator and is an isomorphism if for all $\tilde{f}_s$ in $H^1(\Omega_s)$ and for all δ in $H^{5/2}(\Gamma)$, there exists a unique solution d in $H^3(\Omega_s) \cap H^1_{\Gamma_s}(\Omega_s)$ to the problem: find d such that

$$\left\{ \begin{array}{rcccc} -\mathrm{div}\left(2\mu_s D(d) + \lambda_s \mathrm{div}(d) I\right) & = & \tilde{f}_s & \text{in} & \Omega_s, \\ d & = & \delta & \text{on} & \Gamma, \\ d & = & 0 & \text{on} & \Gamma_s. \end{array} \right. \tag{4.6}$$

Because δ belongs to $H^{5/2}(\Gamma)$ and Ω_s is of class $C^{2,1}$, there exists a lifting of δ in $H^3(\Omega_s) \cap H^1_{\Gamma_s}(\Omega_s)$, denoted by $\tilde{\delta}$, such that $\gamma_\Gamma(\tilde{\delta}) = \delta$. Then, the function $\tilde{d}$ defined by

$$\tilde{d} = d - \tilde{\delta}$$

is solution of the linearized elasticity problem with homogeneous Dirichlet boundary conditions: find $\tilde{d}$ such that,

$$\left\{ \begin{array}{rl} -\mathrm{div}\left(2\mu_s D(\tilde{d}) + \lambda_s \mathrm{div}(\tilde{d}) I\right) = & \\ \tilde{f}_s + \mathrm{div}\left(2\mu_s D(\tilde{\delta}) + \lambda_s \mathrm{div}(\tilde{\delta}) I\right) & \text{in } \Omega_s, \\ \tilde{d} = 0 & \text{on } \partial\Omega_s. \end{array} \right. \tag{4.7}$$

Problem (4.7) is known as the *linearized pure displacement problem* and has been studied for instance in [13, Theorem 6.3-6]. Because the boundary of Ω_s is of class $C^{2,1}$ and the right-hand side in the first equation of (4.7) belongs to $H^1(\Omega_s)$, it follows that problem (4.7) admits a unique solution in the space $H^3(\Omega_s) \cap H^1_0(\Omega_s)$ (see [20, Theorem 2.5.1.1]), with the estimate

$$\|\tilde{d}\|_{H^3(\Omega_s)} \leq C(\Omega_s, \mu_s, \lambda_s)(\|\tilde{f}_s\|_{H^1(\Omega_s)} + \|\tilde{\delta}\|_{H^3(\Omega_s)}).$$

Furthermore, problem (4.6) admits $\tilde{d} + \tilde{\delta}$ as unique solution.

Hence, the linear continuous operator

$$\mathcal{D}A(0) : H^3(\Omega_s) \cap H^1_{\Gamma_s}(\Omega_s) \rightarrow H^1(\Omega_s) \times H^{5/2}(\Gamma))$$

is bijective and its inverse is also continuous, by the closed graph Theorem. Thus, we can apply the implicit function Theorem. Consequently, there exists $\mathcal{V}_0$ a neighborhood of 0 in $H^3(\Omega_s) \cap H^1_{\Gamma_s}(\Omega_s)$

and $\mathcal{V}_1$ a neighborhood of $(0,0)$ in $H^1(\Omega_s)\times H^{5/2}(\Gamma)$ such that the mapping A is a C^1-diffeomorphism from $\mathcal{V}_0$ to $\mathcal{V}_1$. In particular, there exist two positive constants R_0 and R_1 such that the ball

$$\mathcal{B}^S_{R_0} \quad = \quad \left\{u \in H^3(\Omega_s)\cap H^1_{\Gamma_s}(\Omega_s); \|u\|_{H^3(\Omega_s)} \le R_0\right\},$$

satisfies

$$A^{-1}\left(\left\{(f,\delta)\in H^1(\Omega_s)\times H^{3/2}(\Gamma); \|f\|_{H^1(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)} \le R_1\right\}\right) \subset \mathcal{B}^S_{R_0}.$$

Going back to the nonlinear problem (4.5), for all u in $\mathcal{B}^S_{R_0}$, if the couple $(\tilde{f}_s,\delta)$ belongs to the space

$$\left\{(f,\delta)\in H^1(\Omega_s)\times H^{3/2}(\Gamma); \|f\|_{H^1(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)} \le R_1\right\},$$

i.e. if the inequality (4.2) is satisfied, there exists a unique solution d in $\mathcal{B}^S_{R_0}$ of problem (4.5). Moreover, this solution can be estimated with respect to the data:

$$\|d\|_{H^3(\Omega_s)} \quad \le \quad C^2_s\left(\|f_s - \operatorname{div}\left((I+\nabla u)\Sigma^*\right)\|_{H^1(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)}\right), \tag{4.8}$$

where the constant C^2_s is defined by

$$C^2_s = \sup_{\|(f,\delta)\|\le R_1} \|\mathcal{D}A^{-1}(f,\delta)\|_{\mathcal{L}(H^1(\Omega_s)\times H^{5/2}(\Gamma),H^3(\Omega_s))}.$$

Thus, we just proved that the mapping $\mathcal{T}$ is well-defined from $\mathcal{B}^S_{R_0}$ into itself, if the data satisfy condition (4.2).

Now, let us show that, under condition (4.3), $\mathcal{T}$ is a contraction from $\mathcal{B}^S_{R_0}$ into itself. Under condition (4.2) on the data, we have that, for all u_1 and u_2 in $\mathcal{B}^S_{R_0}$,

$$\begin{aligned}\mathcal{T}(u_1) &= A^{-1}(f_s - \operatorname{div}\left((I+\nabla u_1)\Sigma^*\right),\delta),\\ \mathcal{T}(u_2) &= A^{-1}(f_s - \operatorname{div}\left((I+\nabla u_2)\Sigma^*\right),\delta).\end{aligned}$$

Since the mapping A^{-1} is a C^1-diffeomorphism from $\mathcal{V}_1$ to $\mathcal{V}_0$, the mean value inequality can be applied. From the mean value inequality and Lemma 2.1, we obtain the inequality

$$\begin{aligned}\|\mathcal{T}(u_1)-\mathcal{T}(u_2)\|_{H^3(\Omega_s)} \le\ & C^2_s\|(\operatorname{div}\left((\nabla u_1 - \nabla u_2)\Sigma^*\right),0)\|_{H^1(\Omega_s)\times H^{5/2}(\Gamma)},\\ \le\ & C^1_s\|\Sigma^*\|_{H^2(\Omega_s)}\|u_1-u_2\|_{H^3(\Omega_s)},\end{aligned}$$

where $C^1_s = C^2_s C(\Omega_s)$. It follows that $\mathcal{T}$ is a contraction from $\mathcal{B}^S_{R_0}$ into itself if the active stress Σ^* satisfies inequality (4.3).

So, under conditions (4.2) and (4.3) on the data, the mapping $\mathcal{T}$ is a contraction from $\mathcal{B}^S_{R_0}$ into itself. Thus, Banach's fixed point theorem implies that $\mathcal{T}$ has a unique fixed point d in $\mathcal{B}^S_{R_0}$, which proves that there exists a unique solution $d\in H^3(\Omega_s)\cap H^1_{\Gamma_s}(\Omega_s)$ of (4.1), under smallness assumptions on the data. Moreover, replacing the fixed point d in (4.8) and using the fact that d belongs to $\mathcal{B}^S_{R_0}$ we obtain the estimate

$$\|d\|_{H^3(\Omega_s)} \quad \le \quad C^2_s(\|f_s - \operatorname{div}(\Sigma^*)\|_{H^1(\Omega_s)} + R_0\|\Sigma^*\|_{H^2(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)}).$$

□

Remark 4.1. In the particular case where $f_s = \operatorname{div}(\Sigma^*)$ and $\delta = 0$, if Σ^* satisfies conditions (4.2) and (4.3), then Lemma 4.1 implies that $d_s = 0$ is the unique solution of problem (4.1).

Using the notation introduced in the proof of Lemma 4.1, we also state the following corollary, which gives a continuity estimate of the solutions of (4.1) with respect to the boundary data.

Corollary 4.1. *Let f_s be in $H^1(\Omega_s)$, Σ^* be in $H^2(\Omega_s)$, and δ_1 and δ_2 be in $H^{5/2}(\Gamma)$. Moreover, suppose that these data satisfy the following conditions:*

$$\begin{array}{rcl} \|f_s - \mathrm{div}\,(\Sigma^*)\,\|_{H^1(\Omega_s)} + R_0\|\Sigma^*\|_{H^2(\Omega_s)} + \|\delta_1\|_{H^{5/2}(\Gamma)} & \leq & R_1, \\ \|f_s - \mathrm{div}\,(\Sigma^*)\,\|_{H^1(\Omega_s)} + R_0\|\Sigma^*\|_{H^2(\Omega_s)} + \|\delta_2\|_{H^{5/2}(\Gamma)} & \leq & R_1, \\ C_s^1\|\Sigma^*\|_{H^2(\Omega_s)} & < & 1, \end{array}$$

where R_0, R_1 and C_s^1 have been defined in Lemma 4.1. Then there exists a unique solution d_1 of (4.1) in a neighborhood of 0 in $H^3(\Omega_s) \cap H^1_{\Gamma_s}(\Omega_s)$, associated to the data $(f_s, \Sigma^*, \delta_1)$ and there exists a unique solution d_2 to the same problem associated to the data $(f_s, \Sigma^*, \delta_2)$. Furthermore, we have the following estimate:

$$(1 - C_s^1\|\Sigma^*\|_{H^2(\Omega_s)})\|d_1 - d_2\|_{H^3(\Omega_s)} \quad \leq \quad C_s^2\|\delta_1 - \delta_2\|_{H^{5/2}(\Gamma)},$$

where C_s^1 and C_s^2 have been defined in Lemma 4.1.

Proof. The existence and uniqueness of d_1 and d_2 is a direct consequence of Lemma 4.1. Moreover, using the fact that the mapping A^{-1} is everywhere differentiable in $\mathcal{V}_1$ and according to Lemma 2.1, we have that

$$\begin{aligned} & \|d_1 - d_2\|_{H^3(\Omega_s)} \\ & = \Big\| A^{-1}(f - \mathrm{div}\,(\Sigma^*) - \mathrm{div}\,(\nabla d_1 \Sigma^*)\,, \delta_1) \\ & \qquad - A^{-1}(f - \mathrm{div}\,(\Sigma^*) - \mathrm{div}\,(\nabla d_2 \Sigma^*)\,, \delta_2) \Big\|_{H^3(\Omega_s)}, \\ & \leq C_s^2 \|(\mathrm{div}\,((\nabla d_1 - \nabla d_2)\Sigma^*)\,, \delta_1 - \delta_2)\|_{H^1(\Omega_s)\times H^{5/2}(\Gamma)}, \\ & \leq C_s^1\|\Sigma^*\|_{H^2(\Omega_s)}\|d_1 - d_2\|_{H^3(\Omega_s)} + C_s^2\|\delta_1 - \delta_2\|_{H^{5/2}(\Gamma)}. \end{aligned}$$

Then, condition (4.3) enables us to obtain the inequality

$$0 \leq (1 - C_s^1\|\Sigma^*\|_{H^2(\Omega_s)})\|d_1 - d_2\|_{H^3(\Omega_s)} \leq C_s^2\|\delta_1 - \delta_2\|_{H^{5/2}(\Gamma)}.$$

□

5 Fluid equations

In this section, we study the two or three-dimensional Stokes equations with mixed non-homogeneous Dirichlet and Neumann boundary conditions, where the domain Ω_f is defined by $(\mathbf{H_1})$-$(\mathbf{H_4})$. The cases of pure Dirichlet or Neumann boundary conditions have been, for example, treated in [9], where existence, uniqueness and regularity have been obtained for the solution of the Stokes problem, depending on the regularity of the domain and the data.

Let f be a given body force, g be a given divergence constraint and h be a given Neumann boundary condition. We consider the Stokes problem: find (u, p) such that

$$\left\{ \begin{array}{rcll} -\mathrm{div}\,(\sigma_f(u,p)) & = & f & \text{in} \quad \Omega_f, \\ \mathrm{div}(u) & = & g & \text{in} \quad \Omega_f, \\ u & = & 0 & \text{on} \quad \Gamma_f, \\ \sigma_f(u,p)n_f & = & h & \text{on} \quad \Gamma, \end{array} \right. \tag{5.1}$$

where $\sigma_f(u, p)$ is the fluid stress tensor defined by

$$\begin{array}{rcl} \sigma_f(u,p) & = & 2\mu_f D(u) - pI, \\ D(u) & = & \dfrac{1}{2}(\nabla u + \nabla u^T). \end{array}$$

In the following lemma, we state an existence, uniqueness and regularity result for the solution of problem (5.1). The key argument here is the fact that the boundary Γ_f, where the homogeneous Dirichlet boundary condition is applied, and the boundary Γ, where the Neumann boundary condition is applied, are disjoint parts of $\partial\Omega_f$, i.e. $\overline{\Gamma_f} \cap \overline{\Gamma} = \emptyset$. Actually, this assumption on the domain Ω_f enables us to easily obtain the regularity of the solution, using existing regularity results on the solution of the pure Dirichlet boundary problem and the pure Neumann boundary problem, that can be found in [9].

Lemma 5.1. *Let Ω_f, Γ_f and Γ be defined by* $(\mathbf{H_1})$-$(\mathbf{H_4})$ *and suppose that f belongs to $H^1(\Omega_f)$, g to $H^2(\Omega_f)$ and h to $H^{3/2}(\Gamma)$. Then, the Stokes problem* (5.1) *admits a unique solution in $(H^3(\Omega_f) \cap H^1_{\Gamma_f}(\Omega_f)) \times H^2(\Omega_f)$. Moreover, there exists a positive constant C_f depending only on Ω_f and μ_f such that*

$$\|u\|_{H^3(\Omega_f)} + \|p\|_{H^2(\Omega_f)} \leq C_f(\|f\|_{H^1(\Omega_f)} + \|g\|_{H^2(\Omega_f)} + \|h\|_{H^{3/2}(\Gamma)}).$$

To prove this lemma, we start by giving a preliminary result concerning the divergence operator, which is based on Bogovskii's result in [4].

Lemma 5.2. *Let Ω_f and Γ_f be defined by* $(\mathbf{H_1})$-$(\mathbf{H_4})$ *and suppose that g is a function in $L^2(\Omega_f)$. Then, there exists a function u in $H^1_{\Gamma_f}(\Omega_f)$ such that* $\mathrm{div}(u) = g$. *Moreover, there exists a constant $C(\Omega_f)$ which depends on Ω_f such that,*

$$\|u\|_{H^1(\Omega_f)} \leq C(\Omega_f)\|g\|_{L^2(\Omega_f)}.$$

Proof. We consider an extension of Ω_f, denoted by Ω^*, which strictly contains Ω_f and whose part of its boundary coincides with Γ_f. We define the function g^* by

$$g^*(x) = \begin{cases} g(x) & \text{if} \quad x \in \Omega_f, \\ \dfrac{|\Omega_f|}{|\Omega^* \setminus \Omega_f|} \displaystyle\int_{\Omega_f} g & \text{if} \quad x \in \Omega^* \setminus \Omega_f, \end{cases}$$

where $|\Omega_f|$ (resp. $|\Omega^* \setminus \Omega_f|$) is the volume of the domain Ω_f (resp. $\Omega^* \setminus \Omega_f$). Then, g^* belongs to $L^2(\Omega^*)$ and has zero mean value over Ω^*, thus we can apply Bogovskii's result [4], which states that there exists a function u^* in $H^1_0(\Omega^*)$ such that $\mathrm{div}(u^*) = g^*$. Moreover, there exists a constant that only depends on the domain Ω^* (then depends on Ω_f) such that

$$\|u^*\|_{H^1(\Omega^*)} \leq C(\Omega_f)\|g^*\|_{L^2(\Omega^*)}.$$

Now, defining u as the restriction of u^* over Ω_f, we obtain that $\mathrm{div}\,(u) = g$ in Ω_f. Moreover, we have the following estimate:

$$\|u\|_{H^1(\Omega_f)} \leq C(\Omega_f)\|g\|_{L^2(\Omega_f)}.$$

□

We are now able to prove Lemma 5.1.

Proof of Lemma 5.1. We define the saddle-point formulation of problem (5.1) to be the following problem:

$$\left\{ \begin{array}{lcll} \multicolumn{4}{l}{\text{find } (u,p) \text{ in } H^1_{\Gamma_f}(\Omega_f) \times L^2(\Omega_f) \text{ such that,}} \\ a(u,v) + (p, Bv)_{L^2(\Omega_f)} & = & l(v) & \forall v \in H^1_{\Gamma_f}(\Omega_f), \\ (q, Bu)_{L^2(\Omega_f)} & = & (q,g)_{L^2(\Omega_f)} & \forall q \in L^2(\Omega_f), \end{array} \right. \tag{5.2}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega_f)$. The bilinear and linear forms a and l are defined by

$$\begin{aligned} a(u,v) &= \mu_f \int_{\Omega_f} D(u) : D(v), \\ l(v) &= \int_{\Omega_f} f \cdot v + \int_\Gamma h \cdot v. \end{aligned}$$

Moreover, B is the divergence operator defined by

$$\begin{array}{rccl} B & : & H^1_{\Gamma_f}(\Omega_f) & \to & L^2(\Omega_f) \\ & & u & \mapsto & \mathrm{div}\,(u)\,. \end{array}$$

It is easy to check that a is a symmetric coercive continuous bilinear form on $H^1_{\Gamma_f}(\Omega_f) \times H^1_{\Gamma_f}(\Omega_f)$ and that l is a continuous linear form on $H^1_{\Gamma_f}(\Omega_f)$. Moreover, the operator B is a linear continuous operator from $H^1_{\Gamma_f}(\Omega_f)$ into $L^2(\Omega)$ and Lemma 5.2 ensures that B is surjective. Thus, according to Brezzi's result on saddle-point problems (see [11]), there exists a unique solution (u,p) of problem (5.2). Moreover, there exists a constant which depends on Ω_f such that

$$\|u\|_{H^1(\Omega_f)} + \|p\|_{L^2(\Omega_f)} \leq C(\Omega_f)(\|f\|_{H^1(\Omega_f)} + \|g\|_{H^2(\Omega_f)} + \|h\|_{H^{3/2}(\Gamma)}).$$

Hence, since the boundaries Γ_f and Γ are disjoint and because Γ_f is of class $C^{2,1}$ and Γ is of class $C^{3,1}$, the rest of the proof, i.e. the regularity of the solution, follows from the method of translations. For local (interior) regularity we refer to Theorem IV.6.1 in [9]. For tangential regularity on the boundary Γ_f, where Dirichlet boundary conditions are applied, we refer to the proof of Theorem IV.5.8 in [9]. For the tangential regularity on the boundary Γ, where Neumann boundary conditions are applied, we refer to the proof of Theorem IV.7.4 in [9], which is presented in three space dimension but is still true in two space dimension. Then, we use the tangential and normal coordinates in a tubular neighborhood of $\partial\Omega_f$ in Ω_f, to deduce the regularity up to the boundary of the solution from the tangential regularity. The desired estimate follows from these results. □

Remark 5.1. On one hand, in [9], no assumption on the regularity of the domain is necessary to apply the result on the interior regularity of the solution, stated in Theorem IV.6.1, which is also independent of the chosen boundary conditions. On the other hand, Theorem IV.5.8, dealing with pure Dirichlet boundary conditions, and Theorem IV.7.4, dealing with pure Neumann boundary conditions, require hypotheses on the regularity of the domain and compatibility conditions on the data due to the particular choice of boundary conditions. Yet, these compatibility conditions only appear in the case of pure Dirichlet or pure Neumann boundary conditions, which is not the case here. Nevertheless, the proofs for the regularity of the solution apply in the same way, since the boundaries where Dirichlet and Neumann boundary conditions are applied are disjoint.

6 Fixed point procedure. Proof of Theorem 3.1

In this section, we go back to the coupled fluid-structure problem and apply the results of sections 4 and 5 to prove Theorem 3.1.

We recall that we consider $T > 0$, a positive constant $\mathcal{M}_1$, a force f_s in the space $L^\infty(0,T;H^1(\Omega_s))$ and an internal activity of the structure Σ^* in the space $L^\infty(0,T;H^2(\Omega_s))$, that satisfy conditions (3.3), (3.4), (3.5), (3.6) and (3.7). Moreover, the structure is supposed to be at equilibrium initially, i.e. we suppose that $f_s(0) = \mathrm{div}\,(\Sigma^*(0))$ and that $d_0 = 0$, which implies that $d_s(0) = 0$ (see Remark 4.1). However, because the active stress tensor Σ^* is not necessarily zero at $t = 0$, neither are the velocity and pressure of the fluid. That is why we introduce the initial solution of the fluid problem (3.10), denoted by (w_f^0, q_f^0), which satisfies equations (3.1), that we recall:

$$\left\{\begin{array}{rcll} -\mathrm{div}\left(\sigma_f(w_f^0,q_f^0)\right) & = & 0 & \text{in} \quad \Omega_f, \\ \mathrm{div}\left(w_f^0\right) & = & 0 & \text{in} \quad \Omega_f, \\ w_f^0 & = & 0 & \text{on} \quad \Gamma_f, \\ \sigma_f(w_f^0,q_f^0)n_f & = & -\Sigma^*(0)n_f & \text{on} \quad \Gamma. \end{array}\right.$$

This Stokes problem is similar to the one studied in Lemma 5.1. Then, because the domain Ω_f is of class $C^{3,1}$ and $\Sigma^*(0)\cdot n_f$ is in $H^{3/2}(\Gamma)$, it follows that equations (3.1) admit a unique solution denoted (w_f^0, q_f^0) in the space $(H^3(\Omega_f)\cap H^1_{\Gamma_f}(\Omega_f)) \times H^2(\Omega_f)$, which satisfies the following inequality:

$$\|w_f^0\|_{H^3(\Omega_f)} + \|q_f^0\|_{H^2(\Omega_f)} \leq C_f\|\Sigma^*(0)\|_{H^2(\Omega_s)}.$$

Furthermore, by taking the L^2-norm in time between 0 and T we obtain:

$$\|w_f^0\|_{L^2(0,T;H^3(\Omega_f))} + \|q_f^0\|_{L^2(0,T;H^2(\Omega_f))} \leq C_f T^{1/2}\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}. \tag{6.1}$$

As explained in section 3, the fixed-point procedure will be done on the fluid velocity and pressure, in a neighborhood of the initial fluid state (w_f^0, q_f^0). That is why we introduced the ball $\mathcal{B}^F_{\mathcal{M}_1}$, defined in (3.2), for which we recall the definition:

$$\mathcal{B}^F_{\mathcal{M}_1} = \Big\{(\omega, \pi) \in L^2(0,T; H^3(\Omega_f) \cap H^1_{\Gamma_f}(\Omega_f)) \times L^2(0,T; H^2(\Omega_f)); \\ \|\omega - w_f^0\|_{L^2(0,T,H^3(\Omega_f))} + \|\pi - q_f^0\|_{L^2(0,T,H^2(\Omega_f))} \leq \mathcal{M}_1\Big\}.$$

Remark 6.1. From (6.1), we deduce that a given couple (ω, π) in $\mathcal{B}^F_{\mathcal{M}_1}$ can be estimated with respect to the constant $\mathcal{M}_1$, the time T and the norm of Σ^*:

$$\|\omega\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi\|_{L^2(0,T;H^2(\Omega_f))} \\ \leq \mathcal{M}_1 + C_f T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}. \tag{6.2}$$

The remaining of the proof is divided in three steps. First, we show that the mapping $\mathcal{S}$, defined in (3.8), is well-defined from $\mathcal{B}^F_{\mathcal{M}_1}$ into itself under condition (3.6). Then, we prove that $\mathcal{S}$ is a contraction mapping if the data satisfy condition (3.7). Finally, we conclude using Banach's fixed point theorem.

Step 1. Let us show that the mapping $\mathcal{S}$ is well-defined from $\mathcal{B}^F_{\mathcal{M}_1}$ into itself under conditions (3.6). We consider a couple (ω, π) in $\mathcal{B}^F_{\mathcal{M}_1}$ and we construct, for all $t \in [0,T]$,

$$\delta(t) = \int_0^t \gamma_\Gamma(\omega(s))ds.$$

According to Lemma 2.1, δ belongs to $H^1(0,T; H^{5/2}(\Gamma))$, with the following estimate,

$$\|\delta\|_{L^\infty(0,T;H^{5/2}(\Gamma))} \leq C_1 T^{1/2} \|\omega\|_{L^2(0,T;H^3(\Omega_f))}, \tag{6.3}$$

where $C_1 = C(\Omega_f)$.

Now, we consider the elasticity problem (3.9) associated to the data δ, that writes: find d_s such that, for almost every $t \in (0,T)$,

$$\left\{\begin{array}{rcll} -\mathrm{div}\,(\Pi_s(d_s(t), t)) &=& f_s(t) & \text{in} \quad \Omega_s, \\ d_s(t) &=& \delta(t) & \text{on} \quad \Gamma, \\ d_s(t) &=& 0 & \text{on} \quad \Gamma_s. \end{array}\right.$$

Conditions (3.3) and (3.4) guarantee that f_s, Σ^* and δ satisfy conditions (4.2) and (4.3) for almost every t in $(0,T)$:

$$\begin{array}{rcl} \|f_s(t) - \mathrm{div}\,(\Sigma^*(t))\,\|_{H^1(\Omega_s)} + R_0\|\Sigma^*(t)\|_{H^2(\Omega_s)} + \|\delta(t)\|_{H^{5/2}(\Gamma)} &\leq& R_1, \\ C_s^1 \|\Sigma^*(t)\|_{H^2(\Omega_s)} &<& 1, \end{array}$$

where R_0, R_1 and C_s^1 have been introduced in Lemma 4.1. Then, for almost every t in $(0,T)$, Lemma 4.1 ensures the existence of a unique solution d_s in $\mathcal{B}^S_{R_0}$ of problem (3.9). Moreover, it can be estimated with respect to the data:

$$\begin{array}{rcl} \|d_s(t)\|_{H^3(\Omega_s)} &\leq& C_s^2 \Big(\|f_s(t) - \mathrm{div}\,(\Sigma^*(t))\,\|_{H^1(\Omega_s)} \\ && +R_0\|\Sigma^*(t)\|_{H^2(\Omega_s)} + \|\delta(t)\|_{H^{5/2}(\Gamma)}\Big), \end{array}$$

where C_s^2 has been introduced in Lemma 4.1. It follows, using (6.3) and (6.2), that

$$\begin{aligned} \|d_s\|_{L^\infty(0,T;H^3(\Omega_s))} \leq & C_s^2 \big(\|f_s - \mathrm{div}\,(\Sigma^*)\,\|_{L^\infty(0,T;H^1(\Omega_s))} \\ & + R_0 \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\ & + C_1 T^{1/2} \|\omega\|_{L^2(0,T;H^3(\Omega_f))}\big), \\ \leq & C_s^2 \big(\|f_s - \mathrm{div}\,(\Sigma^*)\,\|_{L^\infty(0,T;H^1(\Omega_s))} \\ & + (R_0 + C_1 C_f T) \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\ & + C_1 T^{1/2} \mathcal{M}_1\big). \end{aligned} \tag{6.4}$$

The next step is the study of the fluid problem (3.10), that we recall:

$$\left\{\begin{array}{ll} -\mathrm{div}\,(\sigma_f(w_f(t),q_f(t))) = -\mu_f \mathrm{div}\,(((I-H(d_s(t)))\nabla)\omega(t)) & \\ \quad -\mu_f \mathrm{div}\,\big(\nabla\omega(t)^T - F(d_s(t))^T\nabla\omega(t)^T G(d_s(t))\big) & \\ \quad +(I-G(d_s(t)))\nabla\pi(t) & \text{in } \Omega_f, \\ \mathrm{div}\,(w_f(t)) = -\mathrm{div}\,\big((I-G(d_s(t))^t)\omega(t)\big) & \text{in } \Omega_f, \\ w_f(t) = 0 & \text{on } \Gamma_f, \\ \sigma_f(w_f(t),q_f(t))n_f = \Pi_s(d_s(t),t)n_f & \\ \quad +\mu_f\,((I-H(d_s(t)))\nabla)\omega(t))\,n_f & \\ \quad +\mu_f\,\big(\nabla\omega(t)^T - F(d_s(t))^T\nabla\omega(t)^T G(d_s(t))\big)\,n_f & \\ \quad -\,(\pi(t)(I-G(d_s(t))))\,n_f & \text{on } \Gamma. \end{array}\right.$$

Problem (3.10) is well-defined if $d_s(t)$ belongs to the ball $\mathcal{B}^S_{\mathcal{M}_0}$ (defined by (2.1)) for almost every t. Indeed, the matrix $F(d_s(t))$ is well-defined under this condition (see Lemma 2.3). From estimate (6.4), we see that condition (3.5) ensures that $d_s(t)$ belongs to $\mathcal{B}^S_{\mathcal{M}_0}$, hence that problem (3.10) is well-defined. Moreover, in order to apply Lemma 5.1, we must show that every term in the right-hand side of problem (3.10) is regular enough. The term

$$\begin{aligned} f(t) = &-\mu_f \mathrm{div}\,(((I-H(d_s(t)))\nabla)\omega(t)) \\ &-\mu_f \mathrm{div}\,\big(\nabla w_f(t)^T - F(d_s(t))^T\nabla w_f(t)^T G(d_s(t))\big) \\ &+(I-G(d_s(t)))\nabla\pi(t), \end{aligned}$$

belongs to $H^1(\Omega_f)$ because $F(d_s(t))$, $G(d_s(t))$ and $H(d_s(t))$ are in $H^2(\Omega_f)$ and because $H^2(\Omega_f)$ is a Banach algebra and a multiplier space of $H^1(\Omega_f)$ (see Lemma 2.1). Moreover $f \in L^2(0,T;H^1(\Omega_f))$. For the same reasons and because of the $H^2(\Omega_f)$ regularity of $\Sigma^*(t)$ a. e. in t, the term

$$\begin{aligned} h(t) = &\Pi_s(d_s(t),t)n_f + \mu_f((I-H(d_s(t)))\nabla)\omega(t)n_f \\ &+\mu_f\,\big(\nabla\omega(t)^T - F(d_s(t))^T\nabla\omega(t)^T G(d_s(t))\big)\,n_f \\ &-(\pi(t)(I-G(d_s(t))))n_f \end{aligned}$$

belongs to $H^{3/2}(\Gamma)$, a. e in t and $h \in L^2(0,T;H^{3/2}(\Gamma))$. In addition, because $G(d_s(t)) = \mathrm{cof}(\nabla\Phi(d_s(t)))$, the Piola identity (see [13, Chapter I, p 39]) implies that the term

$$g(t) = -\mathrm{div}\,\big((I-G(d_s(t))^t)\omega(t)\big) = (I-G(d_s(t))^t) : \nabla\omega(t)$$

belongs to $H^2(\Omega_f)$ a.e. in t and $g \in L^2(0,T;H^2(\Omega_f))$. Moreover, the domain Ω_f satisfies assumptions $(\mathbf{H_1})$-$(\mathbf{H_4})$. As a consequence, using Lemma 5.1, problem (3.10) admits a unique solution $(w_f(t),q_f(t))$ in $(H^3(\Omega_f)\cap H^1_{\Gamma_f}(\Omega_f))\times H^2(\Omega_f)$ for almost every t in $(0,T)$. Therefore, due to the linearity of the Stokes equations (3.10), the couple $(w_f(t)-w_f^0, q_f(t)-q_f^0)$, is also solution of a Stokes problem. Using once more time Lemma 5.1, we have the following estimate for almost every t in $(0,T)$:

$$\begin{aligned} &\|w_f(t)-w_f^0\|_{H^3(\Omega_f)} + \|q_f(t)-q_f^0\|_{H^2(\Omega_f)} \\ &\quad \le C_f\Big(\|\mu_f \mathrm{div}\,(((I-H(d_s(t)))\nabla)\omega(t))\,\|_{H^1(\Omega_f)} \\ &\qquad +\|\mu_f \mathrm{div}\,\big(\nabla\omega(t)^T - F(d_s(t))^T\nabla\omega(t)^T G(d_s(t))\big)\,\|_{H^1(\Omega_f)} \\ &\qquad +\|(I-G(d_s(t)))\nabla\pi(t)\|_{H^1(\Omega_f)} \\ &\qquad +\|(I-G(d_s(t))^t) : \nabla\omega(t)\|_{H^2(\Omega_f)} \\ &\qquad +\|(\Pi_s(d_s(t),t)+\Sigma^*(0))n_f\|_{H^{3/2}(\Gamma)} \\ &\qquad +\|\mu_f(((I-H(d_s(t)))\nabla)\omega(t))n_f\|_{H^{3/2}(\Gamma)} \\ &\qquad +\|\mu_f\,\big(\nabla\omega(t)^T - F(d_s(t))^T\nabla\omega(t)^T G(d_s(t))\big)\,n_f\|_{H^{3/2}(\Omega_f)} \\ &\qquad +\|(\pi(t)(I-G(d_s(t))))n_f\|_{H^{3/2}(\Gamma)}\Big). \end{aligned} \tag{6.5}$$

Now, we estimate each term appearing in the right-hand side of the previous inequality. For the first term of the right-hand side of estimate (6.5), using Lemma 2.1 and Corollary 2.1, since $d_s(t)$ belongs to

$\mathcal{B}^S_{\mathcal{M}_0}$, we have

$$\begin{aligned}&\|\mu_f \mathrm{div}\left(((I-H(d_s(t)))\nabla)\omega(t)\right)\|_{H^1(\Omega_f)}\\ &\quad\le C(\Omega_f,\mu_f)\|\nabla\omega(t)\|_{H^2(\Omega_f)}\|I-H(d_s(t))\|_{H^2(\Omega_f)},\\ &\quad\le C(\Omega_f,\mathcal{M}_0,\mu_f)\|d_s(t)\|_{H^3(\Omega_s)}\|\omega(t)\|_{H^3(\Omega_f)}.\end{aligned}$$

For the second term of the right-hand side of estimate (6.5), we remark that

$$\begin{aligned}&\nabla\omega(t)^T-F(d_s(t))^T\nabla\omega(t)^TG(d_s(t))\\ &=(I-F(d_s(t))^T)\nabla\omega(t)^TG(d_s(t))-\nabla\omega(t)^T(G(d_s(t))-I).\end{aligned}\tag{6.6}$$

Then, using (6.6), Lemma 2.1, Corollary 2.1 and the fact that $G(d_s(t))$ is bounded because $d_s(t)$ belongs to $\mathcal{B}^S_{\mathcal{M}_0}$, it follows that

$$\begin{aligned}&\|\mu_f \mathrm{div}\left(\nabla\omega(t)^T-F(d_s(t))^T\nabla\omega(t)^TG(d_s(t))\right)\|_{H^1(\Omega_f)}\\ &\quad\le C(\Omega_f,\mathcal{M}_0,\mu_f)\Big(\|(I-F(d_s(t))^T)\nabla\omega(t)^T\|_{H^2(\Omega_f)}\\ &\qquad+\|\nabla\omega(t)^T(I-G(d_s(t)))\|_{H^2(\Omega_f)}\Big)\\ &\quad\le C(\Omega_f,\mathcal{M}_0,\mu_f)\|d_s(t)\|_{H^3(\Omega_s)}\|\omega(t)\|_{H^3(\Omega_f)}.\end{aligned}$$

Similarly, for the third, fourth, sixth, seventh and eighth terms of the right-hand side of estimate (6.5), using Lemma 2.1, Corollary 2.1 and equation (6.6) (for the seventh term) leads to

$$\|(I-G(d_s(t)))\nabla\pi(t)\|_{H^1(\Omega_f)}\le C(\Omega_f,\mathcal{M}_0)\|d_s(t)\|_{H^3(\Omega_s)}\|\pi(t)\|_{H^2(\Omega_f)},$$

$$\|(I-G(d_s(t)^t)):\nabla\omega(t)\|_{H^2(\Omega_f)}\le C(\Omega_f,\mathcal{M}_0)\|d_s(t)\|_{H^3(\Omega_s)}\|\omega(t)\|_{H^3(\Omega_f)}.$$

$$\|\mu_f\nabla\omega(t)(I-F(d_s(t)))n_f\|_{H^{3/2}(\Gamma)}\le C(\Omega_f,\mathcal{M}_0,\mu_f)\|d_s(t)\|_{H^3(\Omega_s)}\|\omega(t)\|_{H^3(\Omega_f)},$$

$$\begin{aligned}&\|\mu_f\left(\nabla\omega(t)^T-F(d_s(t))^T\nabla\omega(t)^TG(d_s(t))\right)n_f\|_{H^{3/2}(\Omega_f)}\\ &\quad\le C(\Omega_f,\mathcal{M}_0,\mu_f)\|d_s(t)\|_{H^3(\Omega_s)}\|\omega(t)\|_{H^3(\Omega_f)},\end{aligned}$$

$$\|(\pi(t)(I-G(d_s(t))))n_f\|_{H^{3/2}(\Gamma)}\le C(\Omega_f,\mathcal{M}_0)\|d_s(t)\|_{H^3(\Omega_s)}\|\pi(t)\|_{H^2(\Omega_f)}.$$

Finally, for the fifth term of the right-hand side of estimate (6.5), we use the continuity of the trace operator from $H^2(\Omega_s)$ to $H^{3/2}(\Gamma)$, the multilinearity property of the operator A, defined in the proof of Lemma 4.1, and Lemma 2.1. We obtain

$$\begin{aligned}&\|(\Pi_s(d_s(t),t)+\Sigma^*(0))n_f\|_{H^{3/2}(\Gamma)}\\ &\le C(\Omega_s)\|\Pi_s(d_s(t),t)+\Sigma^*(0)\|_{H^2(\Omega_s)},\\ &\le C(\Omega_s)\Big(\|(I+\nabla d_s(t))\Sigma_s(d_s(t))\|_{H^2(\Omega_s)}+\|\nabla d_s(t)\Sigma^*(t)\|_{H^2(\Omega_s)}\\ &\qquad+\|\Sigma^*(t)-\Sigma^*(0)\|_{H^2(\Omega_s)}\Big),\\ &\le C(\Omega_s,R_0,\mu_s,\lambda_s)\|d_s(t)\|_{H^3(\Omega_s)}+C(\Omega_s)\|\nabla d_s(t)\|_{H^2(\Omega_s)}\|\Sigma^*(t)\|_{H^2(\Omega_s)}\\ &\qquad+C(\Omega_s)\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))},\\ &\le C(\Omega_s,R_0,\mu_s,\lambda_s)\|d_s(t)\|_{H^3(\Omega_s)}+C(\Omega_s)\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\|d_s(t)\|_{H^3(\Omega_s)}\\ &\qquad+C(\Omega_s)\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}.\end{aligned}$$

Then, replacing each term in (6.5), yields

$$\begin{aligned}&\|w_f(t)-w_f^0\|_{H^3(\Omega_f)}+\|q_f(t)-q_f^0\|_{H^2(\Omega_f)}\\ &\le C\Big(\big(\|\omega(t)\|_{H^3(\Omega_f)}+\|\pi(t)\|_{H^2(\Omega_f)}\big)\|d_s(t)\|_{H^3(\Omega_s)}\\ &\qquad+\big(1+\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\big)\|d_s(t)\|_{H^3(\Omega_s)}\\ &\qquad+\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\Big),\end{aligned}$$

where $C=C(\Omega_f,\Omega_s,R_0,\mathcal{M}_0,\mu_f,\mu_s,\lambda_s)$. Then, taking the L^2-norm in time leads to

$$\begin{aligned}&\|w_f-w_f^0\|_{L^2(0,T;H^3(\Omega_f))}+\|q_f-q_f^0\|_{L^2(0,T;H^2(\Omega_f))}\\ &\le C\Big(\big(\|\omega\|_{L^2(0,T;H^3(\Omega_f))}+\|\pi\|_{L^2(0,T;H^2(\Omega_f))}\big)\|d_s\|_{L^\infty(0,T;H^3(\Omega_s))}\\ &\qquad+T^{1/2}\big(1+\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\big)\|d_s\|_{L^\infty(0,T;H^3(\Omega_s))}\\ &\qquad+T^{1/2}\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\Big).\end{aligned}$$

Moreover, using estimate (6.2), we obtain

$$\begin{aligned}
&\|w_f - w_f^0\|_{L^2(0,T;H^3(\Omega_f))} + \|q_f - q_f^0\|_{L^2(0,T;H^2(\Omega_f))} \\
&\leq C\Big(\big(T^{1/2} + \mathcal{M}_1 + (1+C_f)T^{1/2}\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\big)\|d_s\|_{L^\infty(0,T;H^3(\Omega_s))} \\
&\quad + T^{1/2}\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\Big).
\end{aligned} \tag{6.7}$$

Now, we estimate the first term in the right-hand side of (6.7), using (6.4):

$$\begin{aligned}
&\big(T^{1/2} + \mathcal{M}_1 + (1+C_f)T^{1/2}\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\big)\|d_s\|_{L^\infty(0,T;H^3(\Omega_s))} \\
&\leq C_s^2\Big((1+C_f)(R_0 + C_1C_fT)T^{1/2}\|\Sigma^*\|^2_{L^\infty(0,T;H^2(\Omega_s))} \\
&\quad + (1+C_f)T^{1/2}\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))}\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\
&\quad + (\mathcal{M}_1 + T^{1/2})\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} \\
&\quad + \big(C_1C_fT^{3/2} + C_1(1+2C_f)\mathcal{M}_1T + R_0T^{1/2} + R_0\mathcal{M}_1\big)\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\
&\quad + C_1(T^{1/2} + \mathcal{M}_1)\mathcal{M}_1T^{1/2}\Big),
\end{aligned}$$

which rewrites

$$\begin{aligned}
&\big(T^{1/2} + \mathcal{M}_1 + (1+C_f)T^{1/2}\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\big)\|d_s\|_{L^\infty(0,T;H^3(\Omega_s))} \\
&\leq C'\Big((1+T)T^{1/2}\|\Sigma^*\|^2_{L^\infty(0,T;H^2(\Omega_s))} \\
&\quad + T^{1/2}\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))}\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\
&\quad + (\mathcal{M}_1 + T^{1/2})\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} \\
&\quad + (T^{3/2} + \mathcal{M}_1T + T^{1/2} + \mathcal{M}_1)\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\
&\quad + (T^{1/2} + \mathcal{M}_1)\mathcal{M}_1T^{1/2}\Big),
\end{aligned} \tag{6.8}$$

where $C' = C(R_0, C_1, C_f, C_s^2)$. Replacing (6.8) in (6.7) we find that

$$\begin{aligned}
&\|w_f - w_f^0\|_{L^2(0,T;H^3(\Omega_f))} + \|q_f - q_f^0\|_{L^2(0,T;H^2(\Omega_f))} \\
&\leq C_2\Big((1+T)T^{1/2}\|\Sigma^*\|^2_{L^\infty(0,T;H^2(\Omega_s))} \\
&\quad + T^{1/2}\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))}\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\
&\quad + (\mathcal{M}_1 + T^{1/2})\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} \\
&\quad + (T^{3/2} + \mathcal{M}_1T + T^{1/2} + \mathcal{M}_1)\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\
&\quad + (T^{1/2} + \mathcal{M}_1)\mathcal{M}_1T^{1/2}\Big),
\end{aligned}$$

where $C_2 = C(\Omega_f, \Omega_s, R_0, \mathcal{M}_0, \mu_f, \mu_s, \lambda_s)$. Therefore, condition (3.6) guarantees that the solution (w_f, q_f) of the fluid problem belongs to $\mathcal{B}^F_{\mathcal{M}_1}$. Hence, the mapping $\mathcal{S}$ is well-defined from $\mathcal{B}^F_{\mathcal{M}_1}$ into $\mathcal{B}^F_{\mathcal{M}_1}$ if the data f_s and Σ^*, the time T and the constant $\mathcal{M}_1$ satisfy condition (3.6).

Step 2. Now, let us show that $\mathcal{S}$ is a contraction mapping. Let (ω_1, π_1) and (ω_2, π_2) be given in $\mathcal{B}^F_{\mathcal{M}_1}$. We built δ_1 and δ_2 such that, for all $t \in [0, T]$,

$$\begin{aligned}
\delta_1(t) &= \int_0^t \gamma_\Gamma(\omega_1(s))ds, \\
\delta_2(t) &= \int_0^t \gamma_\Gamma(\omega_2(s))ds.
\end{aligned}$$

Applying Lemma 2.1 to the difference $\delta_1 - \delta_2$, it comes

$$\|\delta_1 - \delta_2\|_{L^\infty(0,T;H^{5/2}(\Gamma))} \leq C_1T^{1/2}\|\omega_1 - \omega_2\|_{L^2(0,T;H^3(\Omega_f))}, \tag{6.9}$$

where C_1 has been introduced in (6.3) and depends on the domain Ω_f.

As before, conditions (3.3) and (3.4) ensure that the data $(f_s, \Sigma^*, \delta_1)$ and $(f_s, \Sigma^*, \delta_2)$ are sufficiently small to apply Lemma 4.1. Thus, there exists a unique solution $d_{s,1}(t)$ to problem (3.9) associated to the data $(f_s, \Sigma^*, \delta_1)$ and a unique solution $d_{s,2}(t)$ associated to the data $(f_s, \Sigma^*, \delta_2)$. Moreover, according to Corollary 4.1 and using (6.9), we also have the estimate:

$$
\begin{aligned}
&\|d_{s,1} - d_{s,2}\|_{L^\infty([0,T],H^3(\Omega_s))} \\
&\qquad \leq \frac{C_s^2 C_1 T^{1/2}}{1 - C_s^1 \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}} \|\omega_1 - \omega_2\|_{L^2(0,T;H^3(\Omega_f))}.
\end{aligned} \tag{6.10}
$$

Furthermore, condition (3.5) ensures that $d_{s,1}(t)$ and $d_{s,2}(t)$ belong to $\mathcal{B}^S_{\mathcal{M}_0}$ for almost every t in $(0,T)$ and, as before, the two fluid problems of type (3.10), associated with the data $(d_{s,1}, \omega_1, \pi_1)$ and $(d_{s,2}, \omega_2, \pi_2)$ are well-defined for almost every t in $(0,T)$. According to Lemma 5.1, it follows that they both admit a unique solution in $(H^3(\Omega_f) \cap H^1_{\Gamma_f}(\Omega_f)) \times H^2(\Omega_f)$, respectively denoted by $(w_{f,1}(t), q_{f,1}(t))$ and $(w_{f,2}(t), q_{f,2}(t))$. By linearity of the Stokes problem (3.10), the couple $(w_{f,1}(t) - w_{f,2}(t), q_{f,1}(t) - q_{f,2}(t))$ is also solution of a Stokes problem, which writes

$$
\left\{
\begin{array}{rcll}
-\mu_f \Delta(w_{f,1}(t) - w_{f,2}(t)) + \nabla(q_{f,1}(t) - q_{f,2}(t)) &=& \bar{f}(t) & \text{in } \Omega_f, \\
\operatorname{div}(w_{f,1}(t) - w_{f,2}(t)) &=& \bar{g}(t) & \text{in } \Omega_f, \\
w_{f,1}(t) - w_{f,2}(t)) &=& 0 & \text{on } \Gamma_f, \\
\sigma_f(w_{f,1}(t) - w_{f,2}(t), q_{f,1}(t) - q_{f,2}(t)) n_f &=& \bar{h}(t) & \text{on } \Gamma,
\end{array}
\right. \tag{6.11}
$$

where, $\bar{f}$, $\bar{g}$ and $\bar{h}$ are defined, for almost all t in $(0,T)$ by

$$
\begin{array}{rcl}
\bar{f}(t) &=& -\mu_f \operatorname{div}(((I - H(d_{s,1}(t))\nabla)(\omega_1(t) - \omega_2(t))) \\
&& +\mu_f \operatorname{div}(((H(d_{s,1}(t)) - H(d_{s,2}(t))\nabla)\omega_2(t)) \\
&& -\mu_f \operatorname{div}\left(\nabla\omega_1(t)^T - F(d_{s,1}(t))^T \nabla\omega_1(t)^T G(d_{s,1}(t))\right) \\
&& +\mu_f \operatorname{div}\left(\nabla\omega_2(t)^T - F(d_{s,2}(t))^T \nabla\omega_2(t)^T G(d_{s,2}(t))\right) \\
&& +(I - G(d_{s,1}(t))\nabla(\pi_1(t) - \pi_2(t)) \\
&& -(G(d_{s,1}(t)) - G(d_{s,2}(t))\nabla\pi_2(t),
\end{array} \tag{6.12}
$$

$$
\begin{array}{rcl}
\bar{g}(t) &=& -\operatorname{div}\left((I - G(d_{s,1}(t))^T)(\omega_1(t) - \omega_2(t))\right) \\
&& +\operatorname{div}\left((G(d_{s,1}(t))^T - G(d_{s,2}(t))^T)\omega_2(t)\right),
\end{array}
$$

$$
\begin{array}{rcl}
\bar{h}(t) &=& (\Pi_s(d_{s,1}(t)) - \Pi_s(d_{s,2}(t)))n_f \\
&& +\mu_f((I - H(d_{s,1}(t))\nabla)(\omega_1(t) - \omega_2(t)))n_f \\
&& -\mu_f((H(d_{s,1}(t)) - H(d_{s,2}(t)))\nabla)\omega_2(t)n_f \\
&& +\mu_f\left(\nabla\omega_1(t)^T - F(d_{s,1}(t))^T \nabla\omega_1(t)^T G(d_{s,1}(t))\right) n_f \\
&& -\mu_f\left(\nabla\omega_2(t)^T - F(d_{s,2}(t))^T \nabla\omega_2(t)^T G(d_{s,2}(t))\right) n_f \\
&& -(\pi_1(t) - \pi_2(t))(I - G(d_{s,1}(t)))n_f \\
&& +\pi_2(t)(G(d_{s,1}(t)) - G(d_{s,2}(t)))n_f.
\end{array}
$$

We recall that matrices $F(d_{s,1}(t))$, $F(d_{s,2}(t))$, $G(d_{s,1}(t))$, $G(d_{s,2}(t))$, $H(d_{s,1}(t))$ and $H(d_{s,2}(t))$ are in $H^2(\Omega_f)$ for a. e. $t \in (0,T)$. Moreover the space $H^2(\Omega_f)$ is a Banach algebra and a multiplier space of $H^1(\Omega_f)$. Therefore we can show that $\bar{f}(t)$ belongs to $H^1(\Omega_f)$, $\bar{g}(t)$ belongs to $H^2(\Omega_f)$ and the function $\bar{h}(t)$ belongs to $H^{5/2}(\Gamma)$ for a. e. $t \in (0,T)$. Thus, applying Lemma 5.1, the solution of problem (6.11) is unique and satisfies the estimate

$$
\begin{array}{l}
\|w_{f,1}(t) - w_{f,2}(t)\|_{H^3(\Omega_f)} + \|q_{f,1}(t) - q_{f,2}(t)\|_{H^2(\Omega_f)} \\
\leq C_f(\|\bar{f}(t)\|_{H^1(\Omega_1)} + \|\bar{g}(t)\|_{H^2(\Omega_f)} + \|\bar{h}(t)\|_{H^{3/2}(\Gamma)}).
\end{array} \tag{6.13}
$$

Let us now estimate each term in the right-hand side of (6.13). First, we remark that

$$
\begin{array}{c}
\nabla\omega_1(t)^T - F(d_{s,1}(t))^T \nabla\omega_1(t)^T G(d_{s,1}(t)) \\
-\nabla\omega_2(t)^T + F(d_{s,2}(t))^T \nabla\omega_2(t)^T G(d_{s,2}(t)) \\
= (I - F(d_{s,1}(t))^T)(\nabla\omega_1(t)^T - \nabla\omega_2(t)^T)G(d_{s,1}(t)) \\
+(F(d_{s,2}(t))^T - F(d_{s,1}(t))^T)\nabla\omega_2(t)^T G(d_{s,1}(t)) \\
-F(d_{s,2}(t))^T \nabla\omega_2(t)^T (G(d_{s,1}(t)) - G(d_{s,2}(t))) \\
+(\nabla\omega_1(t)^T - \nabla\omega_2(t)^T)(I - G(d_{s,1}(t))).
\end{array} \tag{6.14}
$$

Then, injecting (6.14) in (6.12) and using Lemma 2.1, Corollary 2.1 and the fact that $F(d_{s,2}(t))$ and $G(d_{s,1}(t))$ are bounded since $d_{s,1}(t)$ and $d_{s,2}(t)$ belong to $\mathcal{B}^S_{\mathcal{M}_0}$, $\bar{f}$ satisfies

$$\begin{gathered}\|\bar{f}(t)\|_{H^1(\Omega_f)} \leq \\ C(\Omega_f, \mathcal{M}_0, \mu_f)\Big(\|d_{s,1}(t)\|_{H^3(\Omega_s)}\left(\|\omega_1(t)-\omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_1(t)-\pi_2(t)\|_{H^2(\Omega_f)}\right) \\ +\|d_{s,1}(t)-d_{s,2}(t)\|_{H^3(\Omega_s)}\left(\|\omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_2(t)\|_{H^2(\Omega_f)}\right)\Big).\end{gathered}$$

Using Lemma 2.1 and Corollary 2.1, $\bar{g}$ can be estimated as

$$\begin{aligned}\|\bar{g}(t)\|_{H^2(\Omega_f)} \ &\leq \ C(\Omega_f, \mathcal{M}_0)\Big(\|d_{s,1}(t)\|_{H^3(\Omega_s)}\|\omega_1(t)-\omega_2(t)\|_{H^3(\Omega_f)} \\ &\qquad +\|d_{s,1}(t)-d_{s,2}(t)\|_{H^3(\Omega_s)}\|\omega_2(t)\|_{H^3(\Omega_f)}\Big).\end{aligned}$$

Then, using (6.14), the continuity property of the trace operator from $H^2(\Omega_s)$ into $H^{3/2}(\Gamma)$ and from $H^2(\Omega_f)$ into $H^{3/2}(\Gamma)$, Lemma 2.1 and Corollary 2.1, $\bar{h}$ can be bounded

$$\begin{gathered}\|\bar{h}(t)\|_{H^{3/2}(\Gamma)} \leq \\ C(\Omega_s, R_0, \mu_s, \lambda_s)\left(1+\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\right)\|d_{s,1}(t)-d_{s,2}(t)\|_{H^3(\Omega_s)} \\ +C(\Omega_f, \mathcal{M}_0, \mu_f)\Big(\|d_{s,1}(t)\|_{H^3(\Omega_s)}\left(\|\omega_1(t)-\omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_1(t)-\pi_2(t)\|_{H^2(\Omega_f)}\right) \\ +\|d_{s,1}(t)-d_{s,2}(t)\|_{H^3(\Omega_s)}\left(\|\omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_2(t)\|_{H^2(\Omega_f)}\right)\Big).\end{gathered}$$

Replacing them in inequality (6.13), yields

$$\begin{gathered}\|w_{f,1}(t)-w_{f,2}(t)\|_{H^3(\Omega_f)} + \|q_{f,1}(t)-q_{f,2}(t)\|_{H^2(\Omega_f)} \\ \leq C_f C(\Omega_s, R_0, \mu_s, \lambda_s)\left(1+\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\right)\|d_{s,1}(t)-d_{s,2}(t)\|_{H^3(\Omega_s)} \\ +C_f C(\Omega_f, \mathcal{M}_0, \mu_f)\Big(\|d_{s,1}(t)\|_{H^3(\Omega_s)}\left(\|\omega_1(t)-\omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_1(t)-\pi_2(t)\|_{H^2(\Omega_f)}\right) \\ +\|d_{s,1}(t)-d_{s,2}(t)\|_{H^3(\Omega_s)}\left(\|\omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_2(t)\|_{H^2(\Omega_f)}\right)\Big).\end{gathered}$$

Then, taking the L^2-norm in time we obtain that

$$\begin{gathered}\|w_{f,1}-w_{f,2}\|_{L^2(0,T;H^3(\Omega_f))} + \|q_{f,1}-q_{f,2}\|_{L^2(0,T;H^2(\Omega_f))} \\ \leq C\Big(T^{1/2}\left(1+\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\right)\|d_{s,1}-d_{s,2}\|_{L^\infty(0,T;H^3(\Omega_s))} \\ +\|d_{s,1}\|_{L^\infty(0,T;H^3(\Omega_s))}\left(\|\omega_1-\omega_2\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi_1-\pi_2\|_{L^2(0,T;H^2(\Omega_f))}\right) \\ +\|d_{s,1}-d_{s,2}\|_{L^\infty(0,T;H^3(\Omega_s))}\left(\|\omega_2\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi_2\|_{L^2(0,T;H^2(\Omega_f))}\right)\Big),\end{gathered}$$

where $C = C(\Omega_f, \Omega_s, \mathcal{M}_0, R_0, \mu_f, \mu_s, \lambda_s)$. Then, using estimate (6.2) on the couple (ω_2, π_2) which belongs to $\mathcal{B}^F_{\mathcal{M}_1}$,

$$\begin{gathered}\|w_{f,1}-w_{f,2}\|_{L^2(0,T;H^3(\Omega_f))} + \|q_{f,1}-q_{f,2}\|_{L^2(0,T;H^2(\Omega_f))} \\ \leq C\Big(T^{1/2}\left(1+\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}\right)\|d_{s,1}-d_{s,2}\|_{L^\infty(0,T;H^3(\Omega_s))} \\ +\|d_{s,1}\|_{L^\infty(0,T;H^3(\Omega_s))}\left(\|\omega_1-\omega_2\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi_1-\pi_2\|_{L^2(0,T;H^2(\Omega_f))}\right) \\ +\|d_{s,1}-d_{s,2}\|_{L^\infty(0,T;H^3(\Omega_s))}(\mathcal{M}_1 + C_f\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}T^{1/2})\Big).\end{gathered}$$

Finally, using estimate (6.10) applied to the difference $d_{s,1}-d_{s,2}$ and estimate (6.4) to the displacement $d_{s,1}$, we obtain

$$\begin{gathered}\|w_{f,1}-w_{f,2}\|_{L^2(0,T;H^3(\Omega_f))} + \|q_{f,1}-q_{f,2}\|_{L^2(0,T;H^2(\Omega_f))} \\ \leq C\Big(\left(\|\omega_1-\omega_2\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi_1-\pi_2\|_{L^2(0,T;H^2(\Omega_f))}\right) \\ \times C_s^2\left(\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} + (R_0 + C_1C_fT)\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} + C_1\mathcal{M}_1T^{1/2}\right) \\ +C_s^2C_1\frac{T+\mathcal{M}_1T^{1/2}+(1+C_f)T\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}}{1-C_s^1\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}}\|\omega_1-\omega_2\|_{L^2(0,T;H^3(\Omega_f))}\Big),\end{gathered}$$

which rewrites

$$\begin{aligned}
&\|w_{f,1} - w_{f,2}\|_{L^2(0,T;H^3(\Omega_f))} + \|q_{f,1} - q_{f,2}\|_{L^2(0,T;H^2(\Omega_f))} \\
&\leq C_3 \left(\|\omega_1 - \omega_2\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi_1 - \pi_2\|_{L^2(0,T;H^2(\Omega_f))} \right) \\
&\times \Big(\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} + (1+T)\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} + \mathcal{M}_1 T^{1/2} \\
&\quad + \frac{T + \mathcal{M}_1 T^{1/2} + T\|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}}{1 - C_s^1 \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}} \Big),
\end{aligned}$$

where $C_3 = C(\Omega_f, \Omega_s, R_0, \mathcal{M}_0, \mu_f, \mu_s, \lambda_s)$. Therefore, we see that condition (3.7) guarantees that the mapping $\mathcal{S}$ is a contraction.

Step 3. To conclude, we have proved that, *i)* under conditions (3.3), (3.4) and (3.5), the mapping $\mathcal{S}$ is well-defined from $\mathcal{B}^F_{\mathcal{M}_1}$ into $L^2(0,T;H^3(\Omega_f)) \times L^2(0,T;H^2(\Omega_f))$, *ii)* under condition (3.6), the image of $\mathcal{B}^F_{\mathcal{M}_1}$ by $\mathcal{S}$ is included in $\mathcal{B}^F_{\mathcal{M}_1}$ and *iii)* under condition (3.7), $\mathcal{S}$ is a contraction mapping. Moreover, $\mathcal{B}^F_{\mathcal{M}_1}$ is a bounded closed subset of a Banach space. Consequently, we apply Banach's fixed point theorem and conclude that the mapping $\mathcal{S}$ has a unique fixed point in $\mathcal{B}^F_{\mathcal{M}_1}$. Furthermore, this also proves the existence and the uniqueness, for small enough forces and a small enough time, of a regular solution to the fluid-structure interaction system (1.1), (1.2), (1.9), (1.4) and (1.5). □

Remark 6.2. Note that here we have both smallness conditions on the time and on the amplitude of the applied forces. It is due to the quasi-static nature of the problem we consider with elliptic problems coupled through a kinematic condition at the interface.

7 More general conditions on the data

In this section, let us relax the assumption on the initial configuration of the system, $f_s(0) = \operatorname{div}(\Sigma^*(0))$. This means that the structure is not at rest initially, even if it starts from its reference configuration. Then, we show that the result stated in Theorem 3.1 is still true, i.e. that, under some smallness conditions on the data, the time T and the constant $\mathcal{M}_1$, there exists a unique solution (w_f, q_f, d_s) to the coupled fluid-structure system (1.1), (1.2), (1.9), (1.4) and (1.5).

Theorem 7.1. *Let the domains Ω_f and Ω_s and the frontiers Γ_f, Γ_s and Γ be defined by assumptions* $(\mathbf{H_1})$*-*$(\mathbf{H_4})$ *and let $T > 0$ and $0 < \varepsilon < T$. Consider f_s in $L^\infty(0,T;H^1(\Omega_s))$ and Σ^* in the space $L^\infty(0,T;H^2(\Omega_s))$, the data of the problem. Let $\mathcal{M}_1 > 0$ and consider the ball $\mathcal{B}^\varepsilon_{\mathcal{M}_1}$ defined by*

$$\begin{aligned}
\mathcal{B}^{F,\varepsilon}_{\mathcal{M}_1} = \{&(\omega,\pi) \in L^2(0,T-\varepsilon;H^3(\Omega_f) \cap H^1_{\Gamma_f}(\Omega_f)) \times L^2(0,T-\varepsilon;H^2(\Omega_f)); \\
&\|\omega - w_f^0\|_{L^2(0,T-\varepsilon,H^3(\Omega_f))} + \|\pi - q_f^0\|_{L^2(0,T-\varepsilon,H^2(\Omega_f))} \leq \mathcal{M}_1\},
\end{aligned}$$

where (w_f^0, q_f^0) is the initial state of the fluid, solution of (3.1).

There exists positive constants R_0, R_1, $\mathcal{M}_0$, C_s^1, C_s^2, C_f, C_1, C_2 and C_3, which only depend on the domains Ω_f and Ω_s, the viscosity of the fluid μ_f and the elasticity parameters μ_s and λ_s of the structure such that, if the data f_s and Σ^*, the time T and the constant $\mathcal{M}_1$ satisfy conditions (3.3), (3.4), (3.5), (3.6) *and* (3.7)*, then, there exists a unique solution (w_f, q_f, d_s) of* (1.1), (1.2), (1.9), (1.4) *and* (1.5)*, with (w_f, q_f) in $\mathcal{B}^{F,\varepsilon}_{\mathcal{M}_1}$ and d_s in the space $L^\infty(0,T-\varepsilon;\mathcal{B}_{\mathcal{M}_0} \cap H^1_{\Gamma_s}(\Omega_s))$.*

The idea of the proof of Theorem 7.1 is inspired by the *incremental method*, usually used for the numerical resolution of elasticity problems involving large deformations. In [13, sec. 6.10], Ciarlet describes it as a method which consists in "letting the forces vary by small increments from zero to the given ones and to compute corresponding approximate solutions by successive linearization". Actually, in the context of numerical simulation, it enables to compute the displacement of a structure whose equilibrium position is "far" from its reference position, and which could not be obtained directly. Here, this trick is used to apply Theorem 3.1 on a slightly different problem whose data, f_s^ε and Σ^*_ε, satisfy the condition $f_s^\varepsilon(0) = \operatorname{div}(\Sigma^*_\varepsilon(0))$. Then, we recover the solution associated with the true data, f_s and Σ^*.

Proof. Let us introduce a body force f_s^ε and an internal activity Σ_ε^*, defined by

$$\begin{aligned} f_s^\varepsilon(t) &= \begin{cases} \frac{1}{\varepsilon} f_s(0)t & \text{if } t \leq \varepsilon, \\ f_s(t-\varepsilon) & \text{for almost every } t \text{ in } (\varepsilon, T), \end{cases} \\ \Sigma_\varepsilon^*(t) &= \begin{cases} \frac{1}{\varepsilon} \Sigma^*(0)t & \text{if } t \leq \varepsilon, \\ \Sigma^*(t-\varepsilon) & \text{for almost every } t \text{ in } (\varepsilon, T). \end{cases} \end{aligned}$$

With these definitions, f_s^ε belongs to the space $L^\infty(0,T;H^1(\Omega_s))$, Σ_ε^* belongs to the space $L^\infty(0,T;H^2(\Omega_s))$ and the condition $f_s^\varepsilon(0) = \text{div}\,(\Sigma_\varepsilon^*(0)) = 0$ is satisfied. Moreover,

$$\begin{aligned} \|f_s^\varepsilon - \text{div}\,(\Sigma_\varepsilon^*)\,\|_{L^\infty(0,T;H^1(\Omega_s))} &\leq \|f_s - \text{div}\,(\Sigma^*)\,\|_{L^\infty(0,T;H^1(\Omega_s))}, \\ \|\Sigma_\varepsilon^*\|_{L^\infty(0,T:H^2(\Omega_s))} &\leq \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}, \end{aligned}$$

then the data f_s^ε and Σ_ε^*, the time T and the constant $\mathcal{M}_1$ also satisfy the conditions (3.3), (3.4), (3.6) and (3.7) of Theorem 3.1. Thus, we can apply Theorem 3.1 to the fluid-structure system (1.1), (1.2), (1.9), (1.4) and (1.5) associated to the data f_s^ε and Σ_ε^*. Consequently, there exists a unique solution $(w_f^\varepsilon, q_f^\varepsilon, d_s^\varepsilon)$, with $(w_f^\varepsilon, q_f^\varepsilon)$ in $\mathcal{B}_{\mathcal{M}_1}$ and d_s^ε in $L^\infty(0,T;\mathcal{B}_{\mathcal{M}_0} \cap H^1_{\Gamma_s}(\Omega_s))$. Furthermore, if we choose $\varepsilon < T$, the triplet (w_f, q_f, d_s) defined almost everywhere in $(0, T-\varepsilon)$ by,

$$\begin{aligned} w_f(t) &= w_f^\varepsilon(t+\varepsilon), \\ q_f(t) &= q_f^\varepsilon(t+\varepsilon), \\ d_s(t) &= d_s^\varepsilon(t+\varepsilon), \end{aligned}$$

is solution to the fluid-structure system (1.1), (1.2), (1.9), (1.4) and (1.5), associated to the data f_s and Σ^*, but only almost everywhere in $(0, T-\varepsilon)$:

$$(w_f, q_f) \in \mathcal{B}^\varepsilon_{\mathcal{M}_1} \quad \text{and} \quad d_s \in L^\infty(0, T-\varepsilon; \mathcal{B}_{\mathcal{M}_0} \cap H^1_{\Gamma_s}(\Omega_s)).$$

□

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