First and Second Order Moments of the Normalized Sample Covariance Matrix of Spherically Invariant Random Vectors

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Abstract

Under Gaussian assumptions, the Sample Covariance Matrix (SCM) is encountered in many covariance based processing algorithms. In case of impulsive noise, this estimate is no more appropriate. This is the reason why when the noise is modeled by Spherically Invariant Random Vectors (SIRV), a natural extension of the SCM is extensively used in the literature: the well-known Normalized Sample Covariance Matrix (NSCM) which estimates the covariance of SIRV. Indeed, this estimate gets rid of a fluctuating noise power and is widely used in radar applications. The aim of this paper is to derive closed-form expressions of the first and second order moments of the NSCM.

Index Terms

SIRV, NSCM, estimation, performance analysis.

I. Introduction

Given independent identically distributed observations of a zero-mean complex Gaussian random vector, the Sample Covariance Matrix (SCM) is the Maximum Likelihood estimate of the data covariance matrix.

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It is well-known that the SCM is complex Wishart distributed, unbiased, and its second order moments have simple expressions [1]. The full statistical characterization of the SCM allowed performance analysis of numerous algorithms relying on this estimate. However, this widespread estimate is no more appropriate when observations are not Gaussian. For instance, this is the case for radar clutter returns [2], radio fading analysis [3], sonar interferences [4]... In these contexts, Spherically Invariant Random Vectors (SIRV) have been appropriately used in modeling non-Gaussian problems. A SIRV is a complex compound Gaussian process with random power. More precisely, a SIRV $\mathbf{c}$ [5] is the product of the square root of a positive random variable τ, called the *texture*, and a m-dimensional independent zero mean complex Gaussian vector $\mathbf{x}$ with covariance matrix $\boldsymbol{\Sigma} = \mathbb{E}\left[\mathbf{x}\mathbf{x}^H\right]$ normalized according to $\mathrm{tr}\left(\boldsymbol{\Sigma}\right) = m$, where $\mathbb{E}[.]$ stands for the statistical mean, $\mathrm{tr}(.)$ is the trace of a matrix, and ^{H} denotes the transpose conjugate operator,

$$\mathbf{c} = \sqrt{\tau}\mathbf{x}.$$

The notation $\mathbf{x} \sim \mathbb{CN}\left(\mathbf{0}, \boldsymbol{\Sigma}\right)$ means that $\mathbf{x}$ is a zero mean complex Gaussian vector with covariance matrix $\boldsymbol{\Sigma}$. In this paper, we consider the estimation scheme of $\boldsymbol{\Sigma}$ from N independent SIRV observations, $\mathbf{c}_k = \sqrt{\tau_k}\mathbf{x}_k$, for $k = 1, \ldots, N$. In this context, we analyze the statistical properties of the well-known Normalized Sample Covariance Matrix (NSCM), introduced in [6], and defined by

$$\widehat{\boldsymbol{\Sigma}} = \frac{m}{N}\sum_{k=1}^{N}\frac{\mathbf{c}_k\mathbf{c}_k^H}{\mathbf{c}_k^H\mathbf{c}_k} = \frac{m}{N}\sum_{k=1}^{N}\frac{\mathbf{c}_k\mathbf{c}_k^H}{\|\mathbf{c}_k\|^2} = \frac{m}{N}\sum_{k=1}^{N}\frac{\mathbf{x}_k\mathbf{x}_k^H}{\|\mathbf{x}_k\|^2}. \tag{1}$$

Notice that, in Eq. (1), the NSCM does not depend on the texture. Moreover, the Central Limit Theorem (CLT) ensures that the NSCM is asymptotically Gaussian, but, first and second order moments of this estimate never appeared in the literature. Thus, the goal of this paper is to fill these gaps when the $\boldsymbol{\Sigma}$-eigenvalues are distinct, *i.e.*, the most common and realistic case.

II. First and second order moments of the NSCM

This section is devoted to the presentation of the main results while computational details will be provided in Appendix.

First, let us introduce the eigenvalue decomposition of $\boldsymbol{\Sigma}$

$$\boldsymbol{\Sigma} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H = \sum_{k=1}^{m}\lambda_k\mathbf{u}_k\mathbf{u}_k^H, \tag{2}$$

where

- $\boldsymbol{\Lambda}$ is the diagonal matrix of the $\boldsymbol{\Sigma}$-eigenvalues, $\lambda_1 > \ldots > \lambda_m > 0\,$,

- $\mathbf{U}$ is the unitary matrix of the $\mathbf{\Sigma}$-eigenvectors.

Notice that we assume that all eigenvalues $\lambda_1, \ldots, \lambda_m$, are strictly positive and different, *i.e.*, their multiplicity order is 1.

Theorem 1 *The first order moment of the NSCM is given by*

$$\mathbb{E}\left[\widehat{\mathbf{\Sigma}}\right] = m\mathbf{U}\,\mathbf{\Delta}\,\mathbf{U}^H, \tag{3}$$

where

$$\delta_k = \sum_{\substack{n=1\\n\neq k}}^{m} d_n \lambda_k \left(\frac{\log\lambda_n - \log\lambda_k}{\lambda_n - \lambda_k} - \frac{1}{\lambda_n}\right), \tag{4}$$

$$d_n = \prod_{\substack{p=1\\p\neq n}}^{m} (1 - \lambda_p/\lambda_n)^{-1}, \tag{5}$$

and where $\mathbf{\Delta}$ *is the diagonal matrix of the* δ_k*'s, with* $\delta_1 > \cdots > \delta_m > 0$.

Proof: See Appendix I. ■

Remark 1 *This theorem provides as a by-product the eigen-decomposition of* $\mathbb{E}\left[\widehat{\mathbf{\Sigma}}\right]$. *It shows also that* $\mathbb{E}\left[\widehat{\mathbf{\Sigma}}\right]$ *and* $\mathbf{\Sigma}$ *share the same eigenvectors but have different eigenvalues. Consequently, the NSCM is a biased estimate of* $\mathbf{\Sigma}$.

Remark 2 *The NSCM preserves the ordering of the eigenvectors.*

Let us denote $\mathrm{vec}(.)$ the operator which reshapes a $m \times n$ matrix elements into a mn column vector and let us introduce the two matrices

$$\mathbf{V}_1 = \mathbb{E}\left[\mathrm{vec}\left(\widehat{\mathbf{\Sigma}}\right)\,\mathrm{vec}\left(\widehat{\mathbf{\Sigma}}\right)^H\right] \quad \text{and} \quad \mathbf{V}_2 = \mathbb{E}\left[\mathrm{vec}\left(\widehat{\mathbf{\Sigma}}\right)\,\mathrm{vec}\left(\widehat{\mathbf{\Sigma}}\right)^T\right], \tag{6}$$

from which the covariances of the real and imaginary parts of the NSCM are straightforwardly derived.

Theorem 2 *The NSCM is asymptotically Gaussian and*

$$\begin{aligned}\mathbf{V}_1 = {} & \frac{m^2}{N}\sum_{p=1}^{m}\sum_{k=1}^{m}\left(w_{pk} + (N-1)\,\delta_p\,\delta_k\right)\,\mathrm{vec}\left(\mathbf{u}_p\,\mathbf{u}_p^H\right)\,\mathrm{vec}\left(\mathbf{u}_k\,\mathbf{u}_k^H\right)^H \\ & + \frac{m^2}{N}\sum_{p=1}^{m}\sum_{\substack{k=1\\k\neq p}}^{m} w_{pk}\,\mathrm{vec}\left(\mathbf{u}_p\,\mathbf{u}_k^H\right)\,\mathrm{vec}\left(\mathbf{u}_p\,\mathbf{u}_k^H\right)^H,\end{aligned} \tag{7}$$

$$\begin{aligned}\mathbf{V}_2 &= \frac{m^2}{N}\sum_{p=1}^{m}\sum_{k=1}^{m}\left(w_{pk}+(N-1)\,\delta_p\,\delta_k\right)\,\mathrm{vec}\left(\mathbf{u}_p\,\mathbf{u}_p^H\right)\,\mathrm{vec}\left(\mathbf{u}_k\,\mathbf{u}_k^H\right)^T\\ &+\frac{m^2}{N}\sum_{p=1}^{m}\sum_{\substack{k=1\\k\neq p}}^{m} w_{pk}\,\mathrm{vec}\left(\mathbf{u}_p\,\mathbf{u}_k^H\right)\,\mathrm{vec}\left(\mathbf{u}_k\,\mathbf{u}_p^H\right)^T, \qquad (8)\end{aligned}$$

where

$$w_{kk} = \sum_{\substack{n=1\\n\neq k}}^{m} d_n\lambda_k\left(\frac{2\,\lambda_k\,\log(\lambda_k/\lambda_n)}{(\lambda_k-\lambda_n)^2}-\frac{1}{\lambda_n}\,\frac{\lambda_k+\lambda_n}{\lambda_k-\lambda_n}\right), \qquad (9)$$

$$w_{pk} = \sum_{\substack{n=1\\n\neq p\\n\neq k}}^{m} d_n\,\tilde{w}_{pkn},\ \textit{for}\ p\neq k, \qquad (10)$$

with

$$\begin{aligned}\tilde{w}_{pkn} &= \left[\frac{\lambda_n\,(\lambda_p+\lambda_k)-2\,\lambda_p\,\lambda_k}{\lambda_n^2\,(\lambda_p-\lambda_k)^2}\right]\lambda_p\,\lambda_k-\left[\frac{\lambda_p\,\lambda_k}{(\lambda_n-\lambda_p)\,(\lambda_n-\lambda_k)}\right]\log\frac{\lambda_n}{\lambda_p}\\ &+\left[\frac{\lambda_p\,\lambda_k^2\,(\lambda_n-\lambda_p)}{\lambda_n^2\,(\lambda_n-\lambda_k)(\lambda_p-\lambda_k)^3}\right](2\,\lambda_p\,\lambda_n-\lambda_p\,\lambda_k-\lambda_k^2)\log\frac{\lambda_k}{\lambda_p}, \qquad (11)\end{aligned}$$

where δ_n and d_n are respectively defined in equations (4) and (5).

Proof: See Appendix II, III, IV. ■

III. CONCLUSION

The closed-form expressions of the first and second order moments of the NSCM for SIRV modeling have been provided in this paper with full detailed proofs.

These analytical equations are essential for analyzing performance of signal processing methods based on NSCM: detection schemes in radar applications, direction of arrivals (DOA), estimation in array processing, ...

APPENDIX I
PROOF OF THEOREM 1

Using the eigen-decomposition of Eq. (2), let us whiten $\mathbf{x}$ according to $\mathbf{y}=\mathbf{\Lambda}^{-1/2}\,\mathbf{U}^H\,\mathbf{x}$. Hence, $\mathbf{y}\sim\mathbb{CN}\,(\mathbf{0},\mathbf{I})$ and consequently

$$\frac{\mathbf{x}\,\mathbf{x}^H}{\|\mathbf{x}\|^2}=\mathbf{U}\,\mathbf{\Lambda}^{-1/2}\,\frac{\mathbf{y}\,\mathbf{y}^H}{\mathbf{y}^H\,\mathbf{\Lambda}^{-1}\,\mathbf{y}}\mathbf{\Lambda}^{-1/2}\,\mathbf{U}^H.$$

The NSCM statistical mean can be rewritten as

$$\mathbb{E}\left[\widehat{\mathbf{\Sigma}}\right] = \mathbb{E}\left[\frac{m}{N}\sum_{k=1}^{N}\frac{\mathbf{x}_k\,\mathbf{x}_k^H}{\|\mathbf{x}_k\|^2}\right] = m\,\mathbb{E}\left[\frac{\mathbf{x}\,\mathbf{x}^H}{\|\mathbf{x}\|^2}\right] = m\,\mathbf{U}\,\mathbf{\Lambda}^{-1/2}\,\mathbb{E}\left[\frac{\mathbf{y}\,\mathbf{y}^H}{\mathbf{y}^H\,\mathbf{\Lambda}^{-1}\,\mathbf{y}}\right]\,\mathbf{\Lambda}^{-1/2}\,\mathbf{U}^H. \tag{I.12}$$

Each component y_k of the vector $\mathbf{y}$ is a zero-mean unit variance circular complex Gaussian variable and can be expressed as:

$$y_k = \sqrt{\frac{1}{2}\chi_k^2\,\exp(i\theta_k)}\,,$$

where χ_k^2 is Chi-squared-distributed with 2 degrees of freedom, θ_k is uniformly distributed on $[0, 2\pi]$. All the χ_k^2's and θ_k's are two-by-two independent. It follows that Eq. (I.12) yields

$$\mathbb{E}\left[\widehat{\mathbf{\Sigma}}\right] = m\sum_{k=1}^{m}\lambda_k\,\mathbb{E}\left[\chi_k^2\Big/\sum_{n=1}^{m}\lambda_n\,\chi_n^2\right]\,\mathbf{u}_k\,\mathbf{u}_k^H.$$

Let us set

$$\delta_k = \mathbb{E}\left[\lambda_k\,\chi_k^2\Big/\sum_{n=1}^{m}\lambda_n\,\chi_n^2\right] = \mathbb{E}\left[\frac{1}{1+X_2/X_1}\right], \tag{I.13}$$

where $X_1 = \lambda_k\,\chi_k^2$ and $X_2 = \sum_{\substack{n=1\\n\neq k}}^{m}\lambda_n\,\chi_n^2$.

To complete the proof, the PDF of X_2 has to be derived. Since all χ_k^2's are independent, the characteristic function of X_2 is

$$\phi_{X_2}(u) = \prod_{\substack{n=1\\n\neq k}}^{m}(1-2\,i\,\lambda_n u)^{-1} = \sum_{\substack{n=1\\n\neq k}}^{m}\frac{c_n}{1-2\,i\,\lambda_n u},$$

where $c_n = \prod_{\substack{p=1\\p\neq n\\p\neq k}}^{m}\left(1-\frac{\lambda_p}{\lambda_n}\right)^{-1}$. Thus, the PDF of X_2 follows

$$p_{X_2}(x) = \frac{1}{2}\sum_{\substack{n=1\\n\neq k}}^{m}\frac{c_n}{\lambda_n}\exp\left(-\frac{x}{2\lambda_n}\right),\quad x \geq 0. \tag{I.14}$$

So, the density of X_2 is obtained by the weighted sum of the densities of $\lambda_n\chi_n^2$ by the coefficient c_n. Now, the PDF of the ratio X_2/X_1 is a weighted sum of F laws (Fisher-Snedecor)

$$p_{X_2/X_1}(x) = \sum_{\substack{n=1\\n\neq k}}^{m} c_n\frac{\lambda_k}{\lambda_n}\left(1+\frac{\lambda_k}{\lambda_n}x\right)^{-2},\quad x \geq 0, \tag{I.15}$$

and after some manipulations, Eq. (I.13) yields

$$\delta_k = \sum_{\substack{n=1\\n\neq k}}^{m} c_n \left(\frac{\lambda_n/\lambda_k}{(1-\lambda_n/\lambda_k)^2} \log(\lambda_n/\lambda_k) + \frac{1}{1-\lambda_n/\lambda_k} \right).$$

It remains to show that $\delta_1 > \cdots > \delta_m > 0$. First, the δ_k's, defined in Eq. (I.13), are strictly positive. Now, let us consider the function

$$f_w(x,y) = \frac{1}{4}\int_{\mathbb{R}_+^2} \frac{x\,u}{x\,u + y\,v + w} \exp\left(-\frac{u+v}{2}\right) du\,dv,$$

where it follows from (I.13) that we have $\delta_k = \mathbb{E}_w\left[f_w(\lambda_k,\lambda_p)\right]$ and $\delta_p = \mathbb{E}_w\left[f_w(\lambda_p,\lambda_k)\right]$, for $w = \sum_{\substack{n=1\\n\neq k\\n\neq p}}^{m} \lambda_n \chi_n^2$, and where $\mathbb{E}_w[.]$ stands for the statistical mean related to w.

To show that $\delta_k < \delta_p$, we prove that $f_w(\lambda_k,\lambda_p) < f_w(\lambda_p,\lambda_k)$ for all w, assuming $\lambda_k < \lambda_p$.

Let us define the functions

$$\begin{aligned} f_1(t) &= f_w((1-t)\lambda_p + t\lambda_k, \lambda_p), \\ f_2(t) &= f_w(\lambda_p, (1-t)\lambda_p + t\lambda_k), \end{aligned}$$

which verify $f_1(0) = f_2(0)$, $f_1(1) = \delta_k$ and $f_2(1) = \delta_p$. To demonstrate that $\delta_k < \delta_p$, we show hereafter that f_1 and f_2 are respectively strictly decreasing and strictly increasing functions of t on the interval $[0,1]$. To proceed, we compute and study the signs of the partial derivatives of f_w,

$$\begin{aligned} \frac{\partial f_w}{\partial x}(x,y) &= \frac{1}{4}\int_{\mathbb{R}_+^2} \frac{u\,(y\,v + w)}{(x\,u + y\,v + w)^2} \exp\left(-\frac{u+v}{2}\right) du\,dv > 0, \\ \frac{\partial f_w}{\partial y}(x,y) &= -\frac{1}{4}\int_{\mathbb{R}_+^2} \frac{v\,x\,u}{(x\,u + y\,v + w)^2} \exp\left(-\frac{u+v}{2}\right) du\,dv < 0, \end{aligned}$$

which allow to find the signs of the derivatives of f_1 and f_2,

$$\begin{aligned} \frac{df_1}{dt} &= \left(\frac{\partial f_w}{\partial x}\right)_{((1-t)\lambda_p + t\lambda_k, \lambda_p)} (\lambda_k - \lambda_p) < 0, \\ \frac{df_2}{dt} &= \left(\frac{\partial f_w}{\partial y}\right)_{(\lambda_p, (1-t)\lambda_p + t\lambda_k)} (\lambda_k - \lambda_p) > 0. \end{aligned}$$

In summary, $\delta_k < \delta_p$ for any k, p such that $\lambda_k < \lambda_p$. This completes the proof of theorem 1.

APPENDIX II

PROOF OF THE EQS. (7), (8) AND (9) OF THEOREM 2

By expressing the variance of the NSCM as a linear combination of functions of the $\mathbf{\Sigma}$-eigenvectors, we compute the statistical means of the coefficients. Eqs. (1), (3), (6) and (I.12) leads to

$$\begin{aligned}\mathbf{V}_1 &= \frac{m^2}{N^2}\sum_{k=1}^{m}\sum_{p=1}^{m}\mathbb{E}\left[\mathrm{vec}\left(\frac{\mathbf{x}_k\,\mathbf{x}_k^H}{\|\mathbf{x}_k\|^2}\right)\,\mathrm{vec}\left(\frac{\mathbf{x}_p\,\mathbf{x}_p^H}{\|\mathbf{x}_p\|^2}\right)^H\right],\\
&= \frac{m^2}{N}\mathbb{E}\left[\mathrm{vec}\left(\frac{\mathbf{x}\,\mathbf{x}^H}{\|\mathbf{x}\|^2}\right)\,\mathrm{vec}\left(\frac{\mathbf{x}\,\mathbf{x}^H}{\|\mathbf{x}\|^2}\right)^H\right]+m^2\,\frac{N-1}{N}\mathrm{vec}\left(\mathbb{E}\left[\frac{\mathbf{x}\,\mathbf{x}^H}{\|\mathbf{x}\|^2}\right]\right)\,\mathrm{vec}\left(\mathbb{E}\left[\frac{\mathbf{x}\,\mathbf{x}^H}{\|\mathbf{x}\|^2}\right]\right)^H,\\
&= \frac{m^2}{N}\sum_{p=1}^{m}\sum_{j=1}^{m}\sum_{n=1}^{m}\sum_{k=1}^{m}\left(\omega_{pjnk}+(N-1)\,\delta_p\,\delta_n\,\delta(p-j)\,\delta(n-k)\right)\mathrm{vec}\left(\mathbf{u}_p\,\mathbf{u}_j^H\right)\,\mathrm{vec}\left(\mathbf{u}_n\,\mathbf{u}_k^H\right)^H,\end{aligned}$$

where $\omega_{pjnk}=(\lambda_p\,\lambda_j\,\lambda_n\,\lambda_k)^{1/2}\,\mathbb{E}\left[\frac{(\chi_p^2\,\chi_j^2\,\chi_n^2\,\chi_k^2)^{1/2}}{\left(\sum_{t=1}^{m}\lambda_t\,\chi_t^2\right)^2}\right]\mathbb{E}\left[\exp\left(i(\theta_p-\theta_j+\theta_k-\theta_n)\right)\right]$ and $\delta(.)$ is the Kronecker delta. The θ's being independent uniform variables, the last term of previous equations is zero if $p\neq j$ and $n\neq k$, or if $p\neq n$ and $k\neq j$, which leads to

$$\begin{aligned}\mathbf{V}_1 &= \frac{m^2}{N}\sum_{p=1}^{m}\sum_{k=1}^{m}(w_{pk}+(N-1)\,\delta_p\,\delta_k)\,\mathrm{vec}\left(\mathbf{u}_p\,\mathbf{u}_p^H\right)\,\mathrm{vec}\left(\mathbf{u}_k\,\mathbf{u}_k^H\right)^H\\
&+\frac{m^2}{N}\sum_{p=1}^{m}\sum_{\substack{k=1\\k\neq p}}^{m}w_{pk}\,\mathrm{vec}\left(\mathbf{u}_p\,\mathbf{u}_k^H\right)\,\mathrm{vec}\left(\mathbf{u}_p\,\mathbf{u}_k^H\right)^H\end{aligned}$$

where

$$w_{pk}=\lambda_p\,\lambda_k\,\mathbb{E}\left[\chi_p^2\,\chi_k^2\Big/\left(\sum_{n=1}^{m}\lambda_n\,\chi_n^2\right)^2\right], \tag{II.16}$$

This is Eq. (7) of theorem 2 and Eq. (8) is derived from the same reasoning. Concerning Eq. (9), one has, from Eq. (II.16), for $p=k$, $w_{kk}=\mathrm{E}\left[(1+X_2/X_1)^{-2}\right]$, where X_1 and X_2 are defined in (I.13). Thus, $w_{kk}=\int_0^{+\infty}(1+x)^{-2}\,p_{X_2/X_1}(x)\,dx$. Eq. (10) will be derived in Appendix IV. Its proof needs some results on exponential integrals, recaped in Appendix III.

APPENDIX III

EXPONENTIAL INTEGRALS AND RELATED FUNCTIONS

This section contains some mathematical tools used in Appendix IV. From pp. 228 of [7], let us recall the definition of the exponential integral

$$\begin{aligned} E_{\mathrm{n}}(z) &= \int_1^{+\infty} \frac{e^{-z\,t}}{t^n} dt\,,\; n \in \mathbb{N}\,,\; \Re(z) > 0, \\ E_1(z) &= \int_1^{+\infty} \frac{e^{-z\,t}}{t} dt = \int_z^{+\infty} \frac{e^{-t}}{t} dt = -\gamma - \ln z - \sum_{n=1}^{+\infty} \frac{(-1)^n\, z^n}{n\, n!}, \end{aligned} \tag{III.17}$$

where $\Re(z)$ denotes the real part of z and $\gamma \simeq 0.57721$ is Euler's constant. Let us introduce the function

$$F_n(a,z) = \int_z^{+\infty} t^n\, e^{-a\,t}\, E_1(t)\, dt\,,\; n \in \mathbb{N}\,,\; \Re(z) > 0\,,\; a > -1\,, \tag{III.18}$$

which gives for $n = 0$ and $n = 1$

$$\begin{aligned} F_0(a,z) &= \frac{1}{a} e^{-a\,z}\, E_1(z) - \frac{1}{a} E_1\left((1+a)\, z\right), \\ F_1(a,z) &= \frac{1+a\,z}{a^2} e^{-a\,z}\, E_1(z) - \frac{1}{a^2} E_1((1+a)\,z) - \frac{1}{a\,(1+a)} e^{-(1+a)\,z}. \end{aligned}$$

In particular, for $a > -1$, we will use the values

$$\begin{aligned} F_1(a,0) &= \frac{1}{a}\left[\frac{1}{a}\ln(1+a) - \frac{1}{1+a}\right], \\ F_2(a,0) &= \frac{2}{a^3}\ln(1+a) - \frac{3\,a+2}{a^2\,(1+a)^2}. \end{aligned} \tag{III.19}$$

APPENDIX IV

END OF PROOF OF THEOREM 2 (w_{pk} FOR $p \neq k$, SEE EQ. (10))

At the end of Appendix II, it remained to compute Eq. (II.16) to complete the proof of theorem 2. A PDF decomposition similar to Eq. (I.14) provides

$$w_{pk} = \sum_{\substack{n=1\\ n\neq p\\ n\neq k}}^{m} \delta_{pkn} \prod_{\substack{j=1\\ j\neq n\\ j\neq p\\ j\neq k}}^{m} \left(1 - \frac{\lambda_j}{\lambda_n}\right)^{-1}, \tag{IV.20}$$

where $\delta_{pkn} = \lambda_p\, \lambda_k\, \mathbb{E}\left[\frac{\chi_p^2\, \chi_k^2}{\left(\lambda_p\, \chi_p^2 + \lambda_k\, \chi_k^2 + \lambda_n\, \chi_n^2\right)^2}\right]$ can be rewritten as

$$\delta_{pkn} = \frac{\lambda_p\, \lambda_k}{8} \int_{\mathbb{R}_+^3} \frac{x_p\, x_k}{\left(\lambda_p\, x_p + \lambda_k\, x_k + \lambda_n\, x_n\right)^2} \exp\left(-\frac{1}{2}(x_p + x_k + x_n)\right) dx_p\, dx_k\, dx_n. \tag{IV.21}$$

An analytic expression of δ_{pkn} is obtained by computing the above integral. Eq. (IV.21) is rewritten as

$$\delta_{pkn} = \frac{\lambda_p\,\lambda_k}{8} \int_0^{+\infty}\int_0^{+\infty} x_p\,x_k\,\exp\left(-\frac{1}{2}(x_p+x_k)\right) t_1\,dx_p\,dx_k, \tag{IV.22}$$

where $t_1 = \int_0^{+\infty} \frac{\exp(-x_n/2)}{(\lambda_p\,x_p+\lambda_k\,x_k+\lambda_n\,x_n)^2}\,dx_n$. Then, by setting $C = \lambda_p\,x_p + \lambda_k\,x_k$, t_1 is rewritten as

$$t_1 = \frac{1}{\lambda_n\,C}\exp\left(\frac{C}{2\,\lambda_n}\right) E_2\left(\frac{C}{2\,\lambda_n}\right) = \frac{1}{\lambda_n\,C}\left[1 - \frac{C}{2\,\lambda_n}\exp\left(\frac{C}{2\,\lambda_n}\right) E_1\left(\frac{C}{2\,\lambda_n}\right)\right].$$

where E_1 and E_2 are defined in (III.17). Now, by replacing t_1 in Eq. (IV.22), we obtain

$$\delta_{pkn} = \frac{\lambda_p\,\lambda_k}{8\,\lambda_n}\left(t_2 - \frac{1}{2\,\lambda_n}t_3\right), \tag{IV.23}$$

where

$$t_2 = \int_0^{+\infty}\int_0^{+\infty} \frac{x_p\,x_k}{(\lambda_p\,x_p+\lambda_k\,x_k)}\exp\left((x_p+x_k)/2\right) dx_p\,dx_k\,,$$

$$t_3 = \int_0^{+\infty}\int_0^{+\infty} x_p\,x_k\exp\left(-\frac{1}{2}\left(x_p+x_k-\frac{\lambda_p\,x_p+\lambda_k\,x_k}{\lambda_n}\right)\right) E_1\left(\frac{\lambda_p\,x_p+\lambda_k\,x_k}{2\,\lambda_n}\right) dx_p\,dx_k\,.$$

Integrating firstly along x_k in t_2 allow to rewrite t_2 as

$$t_2 = \frac{8}{\lambda_k} - \frac{8\,\lambda_p}{\lambda_k^2}F_2\left(\frac{\lambda_k-\lambda_p}{\lambda_p}, 0\right), \tag{IV.24}$$

where the function $F_2(.)$ is defined in Appendix III. Now, let us compute t_3 as

$$t_3 = \int_0^{+\infty} x_p\,\exp\left(-\frac{x_p}{2}\left(1-\frac{\lambda_p}{\lambda_n}\right)\right) t_4\,dx_p\,,$$

with

$$t_4 = \int_0^{+\infty} x_k\exp\left(-\frac{x_k}{2}\left(1-\frac{\lambda_k}{\lambda_n}\right)\right) E_1\left(\frac{\lambda_p\,x_p+\lambda_k\,x_k}{2\lambda_n}\right) dx_k\,.$$

By a change of variable, t_4 is rewritten as

$$\begin{aligned} t_4 &= \frac{2\,\lambda_n}{\lambda_k^2}\int_{\frac{\lambda_p\,x_p}{2\,\lambda_n}}^{+\infty} (2\,\lambda_n\,t - \lambda_p\,x_p)\,\exp\left(-\frac{1}{2\,\lambda_k}(2\,\lambda_n\,t-\lambda_p\,x_p)\left(1-\frac{\lambda_k}{\lambda_n}\right)\right) E_1(t)\,dt\,,\\ &= \frac{2\,\lambda_n}{\lambda_k^2}\exp\left(\frac{\lambda_p}{2\,\lambda_k}\left(1-\frac{\lambda_k}{\lambda_n}\right)x_p\right)\left[2\,\lambda_n\,F_1\left(\frac{\lambda_n}{\lambda_k}-1, \frac{\lambda_p\,x_p}{2\,\lambda_n}\right) - \lambda_p\,x_p F_0\left(\frac{\lambda_n}{\lambda_k}-1, \frac{\lambda_p\,x_p}{2\,\lambda_n}\right)\right]. \end{aligned}$$

and can be simplified with the following relation

$$F_1(a,z) - z\,F_0(a,z) = \frac{1}{a^2}\,e^{-a\,z}\,E_1(z) + \left(\frac{z}{a} - \frac{1}{a^2}\right)\,E_1((1+a)\,z) - \frac{1}{a\,(1+a)}\,e^{-(1+a)\,z}\,,$$

where $a > -1\,,\ \Re(z) > 0\,.$ The simplified expression of t_4 allow to rewrite t_3 as

$$t_3 = \frac{16\,\lambda_n^2}{\lambda_n - \lambda_k}\left(\frac{\lambda_k}{\lambda_p^2\,(\lambda_n - \lambda_k)}\left[\lambda_n^2\,F_1\,(b_n, 0) - \lambda_k^2\,F_1\,(b_k, 0)\right] - \frac{1}{\lambda_n} + \frac{\lambda_k^2}{\lambda_n\,\lambda_p^2}\,F_2\,(b_k, 0)\right).$$

where $b_j = \dfrac{\lambda_j - \lambda_p}{\lambda_p}$, for $j = k, n\,.$

Finally, combining the previous result with Eqs. (III.19), (IV.23) and (IV.24), one has

$$\begin{aligned}\delta_{pkn} &= \lambda_p\,\lambda_k\left[\frac{\lambda_n\,(\lambda_p + \lambda_k) - 2\,\lambda_p\,\lambda_k}{(\lambda_n - \lambda_p)\,(\lambda_n - \lambda_k)\,(\lambda_p - \lambda_k)^2} - \frac{\lambda_n^2\,\log\lambda_n}{(\lambda_n - \lambda_p)^2(\lambda_n - \lambda_k)^2}\right] \\ &\quad + \frac{\lambda_p^2\,\lambda_k^2}{(\lambda_p - \lambda_k)^3}\left[\frac{2\,\lambda_p\,\lambda_n - \lambda_p\,\lambda_k - \lambda_k^2}{\lambda_p\,(\lambda_n - \lambda_k)^2}\,\log\lambda_k - \frac{2\,\lambda_k\,\lambda_n - \lambda_p\,\lambda_k - \lambda_p^2}{\lambda_k\,(\lambda_n - \lambda_p)^2}\,\log\lambda_p\right].\end{aligned}$$

Thanks to Eq. (IV.20), the previous equation provides Eqs. (10) and (11). This concludes the proof of theorem 2.

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