Investigation of the sum of orbital angular momentum generated by conical diffraction

A. Brenier

Univ Lyon, Université Claude Bernard Lyon 1, CNRS, Institut Lumière Matière, F-69622, LYON, France

e-mail address : alain.brenier@univ-lyon1.fr

Abstract : The conical diffraction (CD) of a wave propagating along the optical axis of a biaxial crystal is an intriguing phenomenon with nowadays applications including imaging or optical communications. Moreover the emerging light can have a fractional or integer orbital angular momentum (OAM) along the propagation direction per unit thickness of a transverse section. The measurement of the OAM of a beam is not an easy task, especially for non-integer values. We operated the CD of two 1053 and 1047 nm beams with a KGd(WO₂)₄ (KGW) crystal, in various input and output polarization states. Then the sum-frequency has been obtained with a KTiOPO₄ (KTP) crystal located near the second Raman spike. The integer or fractional OAM of both the fundamental and sum-frequency waves have been visualized with the cylindrical lens method. All the experimental patterns: far field, near field and field in the focal plane of the cylindrical lens, are quite well described by a self-consistent model based on plane waves propagation and their interferences. Sum of OAM due to quadratic nonlinear conversion is evidenced from experimental and calculated patterns.

1. INTRODUCTION

The optical angular momentum (OAM) of an optical beam results of its linear momentum acting off-axis with respect to its centre [1]. For example Laguerre-Gaussian modes have $l \hbar/\text{photon}$ OAM related to their $\exp(il\varphi)$ transverse phase variation. This momentum is due to the helical phase-front structure while the spin angular momentum is due to the polarization. Several devices can be used to generate beams with OAM: insertion in the beam path of a phase-plate with a thickness increasing with the azimuthal angle [2], insertion of cylindrical lenses [3] or a spatial light modulator based on liquid crystals whose refractive index is modified by an electric field [4]... Moreover, two OAM of first harmonic (FH) beam modes can be added as a consequence of phase matching during a quadratic nonlinear interaction. This was proved for second harmonic (SH) generation of Laguerre-Gaussian beams [5]. More generally, matter waves can carry OAM, the most investigated ones being

electron beams: exact Bessel-beam solutions of the Dirac equation and having OAM have been exhibited [6], beams with a tight focusing carry fractional OAM [7], coupling of the OAM of a laser field with the total AM of relativistic electrons-vortex beams has been calculated [8]. Bessel matter waves with vortex and non-zero OAM constituted of light two-level atoms (hydrogen and alkali-metal atoms) have been constructed [9].

The measurement of the OAM of a beam is not an easy task, especially for non-integer values. Integer values can be studied by interference with a reference beam. 2π phase variation around the centre of the beam correspond to one surplus fringe. Going further, a quantitative OAM measurement has been demonstrated in [10] with the help of a single stationary cylindrical lens and a camera. It is based on the fact that the lens transforms the photon momentum to a position in the focal plane where the camera is located.

In the present work we obtain and study OAM thanks to the conical refraction (CR) of focused Gaussian beams. Let us briefly recall that the propagation along the optical axis of a biaxial crystal leads to CR of the beam (instead of the usual double-refraction) and so it emerges as a hollow cylinder [11, 12]. Because a more brilliant description is provided by wave interferences rather than ray propagation, we will speak hereby of conical diffraction (CD). Applications of the CD phenomenon are carefully listed and explained in [13], such as optical trapping, metrology or the super resolution microscopy... The OAM provided by the crystal from CD was calculated by Berry [14]. For an input field with a circular polarization, the output field is a superposition of a B_0 component with a null OAM (charge 0 component) and a B_1 component with $1 \hbar/\text{photon}$ OAM (charge 1 component). They can be easily separated with a linear polarizer and a quarter-wave plate. The theoretical predictions were nicely experimentally verified in the case of the centrosymmetric $\text{KGd}(\text{WO}_2)_4$ (KGW) crystal [15] and the non-centrosymmetric $\text{Bi}_2\text{ZnOB}_2\text{O}_6$ (BZBO) oxy-borate crystal [16] in which the CD is modified by the optical activity due to the crystal bi-anisotropy (Pasteur medium).

Of course SH generation of CD beams from quadratic nonlinearity is drastically more complex than for Gaussian beams and require specific investigations. Crescent-like patterns were exhibited by frequency doubling a 1064 nm linearly polarized beam from a YAG:Nd laser [17], with a single KTP crystal being used for both frequency doubling and CD. Refined results were obtained in [18] with two separate crystals: KGW for CD and a lithium triborate (LBO) for SH generation. The LBO crystal was displaced inside the infrared beam in order to explore its structure after CD: mainly the Poggendorff rings (two brilliant rings separated by a dark one) located in the focal plane between the two axial spikes (Raman spots) which are places where the light collapses. Selecting infrared beam with different charges and moving

the camera, different annular SH patterns are recorded, which are explained by modelling the fundamental by its more intense ring component. The complicated weak substructure is left out and an additional background is necessary.

In this work we investigate the OAM of both the FH of a CD beam in various polarization states and its SH and sum-frequency (SF) of two FH beams. The integer or fractional OAM of both the fundamental and sum-frequency waves have been visualized with the cylindrical lens method. All the experimental patterns are quite well described by a self-consistent model including any substructure without the need of an additional background. Sum of OAM is evidenced from experimental and calculated patterns.

2. EXPERIMENTAL METHODS

FIG. 1 Experimental set-up.

The main source was a compact home-made diode-pumped $\text{LiYF}_4:\text{Nd}$ laser emitting two orthogonally polarized output beams at $\lambda_1=1053$ and $\lambda_2=1047$ nm with tuneable relative intensity and directly collinear [19]. Their polarization was rotated with a half-wave plate in order that equal 1053 and 1047 nm intensities go through the vertical P1 polarizer. If only the 1053 nm is wanted, the 1047 nm polarization was rotated horizontally and so it was suppressed by the vertical P1 polarizer. Q-switching the laser cavity with an acousto-optics modulator leads to simultaneous pulses with a duration less than 10 ns adjusted here at 6.67 kHz repetition rate.

The beam, focused into a $10\text{ }\mu\text{m}$ waist spot behind an optical axis oriented KGW sample (0.3 cm thickness with its xz-principal plane horizontal) with a 3 cm focal length doublet L1, is schematically represented in Fig. 1. The near field (red circle) resulting of CD behind the

crystal is deported with no magnification through the L2 and L3 doublets (10 cm focal length each and arranged as an afocal telescope) in order to reproduce exactly in amplitude, phase and structure the CD phenomenon (Raman spikes and Poggendorff ring) at 10 cm distance of L3. A quarter-wave plate QW1 is introduced in the path of the incident beam, allowing this latter to be in various polarization states: right (RC) or left (LC) circular or elliptical, and linear polarized (LP). In addition, another set QW2 and Pol2 are placed after the KGW crystal to select the wanted output polarization, this latter set being used as an input for second harmonic generation.

SH of the 1053 nm emission or SF of 1053 and 1047 nm emissions were obtained thanks to a commercially available KTP crystal from Eksma Optics. This later one was type II oriented for phase-matching the SH of the fundamental at 1064 nm. This means that the fundamental must propagate in the horizontal pm plane while the g-axis is vertical (noting down p, m, g the KTP principal axes). Type II demands that the fundamental has both a g-axis component of its polarization (this component is then an ordinary (“o”) wave) and a component polarization in the pm-plane (this is then an extraordinary (“e”) wave). The resulting SH is an extraordinary “e” wave polarized in the pm plane. This is summarized by labelling “oe-e” the type II interaction. The FH polarizations are obtained thanks to the Pol2 polarizer oriented at 45°. A rotation around the g-axis (calculated to be 17.1°) is necessary to phase-match the 1053 nm fundamental (a slight different angle is necessary for phase matching the SF). The KTP was inserted about 10 cm after the L3 doublet and be displaced around the L3 focal plane. More the fast axis of the QW2 wave-plate can be rotated to vary the OAM of the beam incident inside the KTP (see sub-section 4.1).

An Beamage Gentec CCD camera was placed at different distances to visualize the output near field for both FH, SH, or sum-frequency, selected with adequate filters. The near field OAM were visualized inserting a 7.5 cm focal length cylindrical lens, vertically or horizontally oriented, at 7.5 cm of the camera. The far fields were recorded by locating the camera in the focal plane of a fourth L4 doublet (15 cm focal length, not shown in Fig. 1).

3. THEORETICAL BACKGROUND

3.1 CD of the First Harmonic beam

Because in this sub-section we deal with the linearity of the propagation of an electromagnetic wave through space and matter, we can claim that the emerging field $[\hat{\mathbf{E}}(L, k_{ix''}, k_{iy''})]_{x''y''}$ of a plane wave from a crystal (with a thickness L) is obtained as a linear operator $\hat{\mathbf{U}}$ acting on the $[\hat{\mathbf{E}}(0, k_{ix''}, k_{iy''})]_{x''y''}$ input field. Here $k_{ix''}, k_{iy''}$ are the two transverse components of the wave-vector of one plane wave. The motivation of this section is first the calculation of this operator $\hat{\mathbf{U}}$. Then the output field in a given plane is obtained by interferences of all the plane waves.

The electric field $\mathbf{E}$ (in the real space) with a given polarization of the beam at the entrance face of the crystal is known in the (x'', y'', z'') frame, where x'' and y'' are the axes on the entrance KGW face and z'' is perpendicular to this face. So z'' , which is also the beam axis, is parallel to the KGW optical axis (see Fig. 1). Its decomposition in plane waves is obtained from its 2D-Fourier transform $[\hat{\mathbf{E}}(0, k_{ix''}, k_{iy''})]_{x''y''}$. Because we use typically Gaussian waves, $\hat{\mathbf{E}}(0)$ is proportional to $\exp(-w_0^2(k_{x''}^2 + k_{y''}^2)/4)$ (but not limited to), where w_0 is the beam waist. Each plane wave propagates inside the crystal after refraction according to a calculation detailed in Ref. [20] including the following steps:

-First: with adequate frame rotations we obtain the components of $\hat{\mathbf{E}}(0)$ in the $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ transverse frame of the refracted plane wave, which is the frame such that the $\mathbf{x}_3$ axis is parallel to the refracted wave-vector.

-Second: The two eigen-modes (labelled $\pm$) in the propagation direction of each plane wave (after refraction) are obtained thanks to the KGW linear permittivity tensor. Their propagation constant is calculated by solving the Fresnel equation, leading to an additional phase factor $e^{ik_{\pm}L}$. With the inverse rotation, the $\hat{\mathbf{E}}(L)$ transverse components for each plane wave at the output crystal face after a path L (sample thickness) are calculated.

-Third: we go back to the (x'', y'', z'') frame which leads to the $[\hat{\mathbf{E}}(L, k_{ix''}, k_{iy''})]_{x''y''}$ electric field of the plane wave at the exit of the KGW. The pattern obtained with the full range of the $k_{ix''}, k_{iy''}$ transverse components is the far field.

After propagation in air over a distance d behind the crystal, all the plane waves constituting the far field interfere, in other words the electric near field at the distance d is obtained as an inverse 2D-Fourier transform:

$$[\mathbf{E}(x'', y'', d)]_{x''y''} = iFT_{2D} \left\{ \exp(ik_{iz''}d) [\hat{\mathbf{E}}(L, k_{ix''}, k_{iy''})]_{x''y''} \right\} \quad (1)$$

For that latter operation we have used the inverse Fast Fourier Transform iFFT2 algorithm of the Matlab package. Formula (1) describes quite well the full CD beam structure in the z'' -planes of interest, in particular the Raman spikes and the Poggendorff rings.

A projection on various polarization states with different optical elements can be performed at this step.

3.2 Modified CD First Harmonic beam through KTP

In this sub-section the above $[\hat{\mathbf{E}}(L, k_{ix''}, k_{iy''})]_{x''y''}$ CD far field is used as the input field for the KTP crystal. The procedure to calculate the propagation inside KTP is the same than in 3.1 sub-section, except that the crystal is rotated for phase matching the SH or the SF, so an additional frame (x', y', z') linked to the crystal is necessary. We operate first the $(x'', y'', z'') \rightarrow (x', y', z')$ rotation and follow the same three steps than 3.1 sub-section. More, the KTP linear permittivity tensor expressed in (x', y', z') includes the $(\mathbf{p}, \mathbf{z}') = \theta = 23.5^\circ$ rotation around z_{KTP} from the KTP (p, m, g) principal axes for phase matching the 1064 nm SH.

The calculation provides two plane waves far fields: $[\hat{\mathbf{E}}_{KTP}(l, k_{ix'}, k_{iy'})]_{x'y'z'}$ which is the FH inside the KTP after a path l propagation, and $[\hat{\mathbf{E}}_{OUT}(L, k_{ix''}, k_{iy''})]_{x''y''}$ which is the FH outside the KTP after a path L propagation. The first field is used in 3.3 subsection for SH calculation, the second one is used for far field FH imaging outside the KTP or near field imaging after inverse Fourier Transform.

3.3 Sum or Second Harmonic Generation

The CD FH electric field $[\mathbf{E}_{KTP}]_{x'y'z'}$ inside the KTP is the inverse Fourier Transform of the above $[\hat{\mathbf{E}}_{KTP}(l, k_{ix'}, k_{iy'})]_{x'y'z'}$ plane wave field and provides the type II quadratic nonlinear polarization of the oe-e interaction through the d_{15} and d_{24} nonlinear coefficients. For this purpose we can calculate the polarization going first in the (p, m, g) KTP dielectric frame, then we return back to the (x', y', z') frame. The result is (we illustrate here the SH case):

$$[\mathbf{P}_{NL}]_{x'y'z'} = 2 \begin{bmatrix} (d_{15}\sin^2(\theta) + d_{24}\cos^2(\theta))E_{x'}E_{y'} + (d_{15} - d_{24})\cos(\theta)\sin(\theta)E_{z'}E_{y'} \\ 0 \\ (d_{15}\sin^2(\theta) + d_{24}\cos^2(\theta))E_{z'}E_{y'} + (d_{15} - d_{24})\cos(\theta)\sin(\theta)E_{x'}E_{y'} \end{bmatrix}_{x'y'z'} \quad (2)$$

where $E_{x'y'z'}$ is a short notation for the CD 1F $[\mathbf{E}_{KTP}]_{x'y'z'}$.

The SH electric field inside KTP (extraordinary polarization projected on the x' axis) propagates according to the inhomogeneous Helmholtz equation [21]:

$$\nabla^2 E_{x'}^{2\omega} - 2ik^{2\omega} \operatorname{tg}(\beta) \frac{\partial E_{x'}^{2\omega}}{\partial x'} + (k^{2\omega})^2 E_{x'}^{2\omega} = -\mu_0(2\omega)^2 [\mathbf{P}_{NL}]_{x'} \quad (3)$$

where the transverse part of the ∇^2 operator describes the SH e-wave diffraction, the second terms describes the β -angle walk-off of the e-wave and the right side is the SH wave source. Making a 2D-Fourier Transform we find:

$$\frac{\partial^2 \hat{E}_{x'}^{2\omega}}{\partial z'^2} + ((k_{z'}^{2\omega})^2 + 2k^{2\omega} \operatorname{tg}(\beta) k_{x'}^{2\omega}) \hat{E}_{x'}^{2\omega} = -\mu_0(2\omega)^2 [\hat{\mathbf{P}}_{NL}]_{x'} \quad (4)$$

Using the property that the Fourier transform of a product of two functions is the convolution product $*$ of the two Fourier transforms of each function, the right hand side of Eq. (4) is:

$$[\hat{\mathbf{P}}_{NL}]_{x'} = 2(d_{15}\sin^2(\theta) + d_{24}\cos^2(\theta))\hat{E}_{x'} * \hat{E}_{y'} + (d_{15} - d_{24})\cos(\theta)\sin(\theta)\hat{E}_{z'} * \hat{E}_{y'} \quad (5)$$

where the convolution product $*$ has been calculated with the conv2 Matlab algorithm.

3.4 OAM from the cylindrical lens method

This method is based on the fact that a cylindrical lens transforms the linear momentum of one photon to a position in its focal plane. If the transverse axis of the cylindrical lens is vertical, the image is the distribution of the horizontal linear component of the momentum. Following [10] with the notations of the present work, the z''

component of the OAM of the full beam recorded in the $x''y''$ -plane situated at the focal length l of the lens is the average:

$$L_{z''} = \frac{2\pi\hbar}{f\lambda} (\langle x''y'' \rangle_H - \langle x''y'' \rangle_V) \quad (7)$$

where $\langle x''y'' \rangle_{V,H}$ are the averages of the photon distribution ($x'' = fk_{x''}/k$, $y'' = fk_{y''}/k$).

The $\langle x''y'' \rangle_V$ average can of course be expressed from the CCD recorded electric field intensity:

$$\langle x''y'' \rangle_V = \frac{\iint_{-\infty}^{\infty} |\mathbf{E}_V(x'', y'')|^2 x'' y'' dx'' dy''}{\iint_{-\infty}^{\infty} |\mathbf{E}_V(x'', y'')|^2 dx'' dy''} \quad (8)$$

In this expression the electric field $\mathbf{E}_V$ at the point $(x'' = fk_{x''}/k, y'')$ results of the contribution of the plane waves with all the $k_{iy''}$ values, i. e. it is the 1D-inverse Fourier Transform ($D=y''$):

$$[\mathbf{E}_V(x'', y'', d)]_{x'', y''} = iFT_{1D=y''} \left\{ [\hat{\mathbf{E}}(k_{ix''}, k_{iy''})]_{x'', y''} \right\} \quad (9)$$

where $\hat{\mathbf{E}}$ in the right hand side is calculated in sub-section 3.1 for the FH and in sub-section 3.3 for the SH (k has to be replaced by $k^2\omega$).

4. RESULTS

4.1 OAM of the first harmonic beam

In this sub-section we detail the results with the 1053 nm FH (identical results are obtained with both the two 1053 and 1047 nm beams), without the KTP crystal, the camera was located at 12.5 cm after the L3 lens in order to record the FH near field resulting of the CD. The fast axis of the QW1 was oriented at 45° in such a way that the input field had a RC polarization. The output Pol2 was fixed at 45° and the fast axis of the QW2 was varied in order to project the CD mode on a polarization state chosen from right to left circular, in passing by elliptical and linear. Typical angles of rotation were 90° (RC), 62.5° (elliptical), 45° (LP), 22.5° (elliptical) and 0° (LC).

Fig. 2 Columns 1 (experimental) and 2 (theoretical): FH (1053 nm) near field intensity; columns 3 (experimental) and 4 (theoretical): FH (1053 nm) horizontal component of the linear momentum; columns 5 (experimental) and 6 (theoretical): FH (1053 nm) vertical component of the linear momentum; the output polarization is: line 1: RC, line 2: 45° LP, line 3: LC.

In Fig. 2 we present results only for 90° (line 1), 45° (line 2) and 0° (line 3). Column 1 is the experimental near field intensity and column 2 the theoretical one calculated from Eq. 1. Along propagation beyond the Raman spike these fields transform towards the far fields with only a slight change except that they expand.

Column 3 is the experimental visualization of the horizontal component of the linear momentum obtained by introducing on the output beam the cylindrical lens with its axis vertical; column 4 is the calculated image from Eq. 9. Column 5 and 6 are corresponding images obtained with the cylindrical lens having its axis horizontal. We have emphasized with a few black arrows the linear momentum in the regions of peak intensity but of course the full distribution contributes to the OAM.

We notice that all the theoretical images in Fig. 2 reproduce quite well the experimental ones. The 2-D numerical calculation with Eq. (7) leads to the calculated FH OAM gathered in Table 1 for the different fast axis rotation of the QW2. All these OAM, including the fractional ones, have been calculated by respect to the vortex at the centre of the left circular projection (line 3 first and fourth columns in Fig. 2).

Increasing from 0 $\hbar$ /photon the absolute value of the OAM (by decreasing from 90° the fast axis angle with the horizontal of the QW2 plate) we observe a continuous deformation of the

field: crescent-like dark areas appear instead of dark circles. Inside these areas, phase vortices of the fields (points where the phase is not defined) are located in eccentric positions. Some of them have been marked with white arrows in the second column of Fig. 2. For fractional OAM $-0.5 \hbar/\text{photon}$, the phase of the field is represented in Fig. 3 (1). The white arrows show that the vortices are linked by pairs with opposite ± 1 charge (following lines with identical phase). They are distributed on either side of the beam centre. These results remind but with some differences the prediction of Berry [22] for the propagation of a wave with an initial fractional phase step. In this latter case when the phase step is half-integer, the alternating vortices are along a radial line of low intensity. Berry's predictions were nicely found again experimentally with a spatial phase modulator [23]. They also can be described from a decomposition of a fractional OAM state on the basis of integer OAM states [24]. This is clearly the same thing in our case: the beam with a fractional OAM $-0.5 \hbar/\text{photon}$ results of the superposition after the KGW of RC and LC polarizations, which is nothing else than the coherent superposition of 0 and -1 integer charge states. Finally, let us add that in our case we observe some evolution of the vortices location along the propagation direction mainly near the focal plane (plane of the Poggendorff rings). Going further towards the far field the beam trends to keep its shape (except its expansion). In the past light carrying fractional OAM by a synthesis of Laguerre-Gaussian modes with a limited number of different Gouy phases in the superposition was produced. These beams can maintain their structural stability completely [25].

Fig. 3 Phase of the near field for fractional OAM: (1) $l = -0.5 \hbar/\text{photon}$; (2) $l = -1.5 \hbar/\text{photon}$.

4.2 The double role of KTP crystal

The first role of KTP in the present work is of course to provide SH or SF by inserting it in the output FH path in the vicinity of the focal plane of the L3 lens. This role is partly described in section 2 and will be completed in sub-section 4.3. Hereby in sub-section we exhibit inescapable and important modifications of the FH beam by KTP as the second role of this crystal. The first role is a nonlinear optical process, the second role is a linear process.

As it is well-known, in its focal plane, the FH CD beam exhibits Poggendorff rings and slightly before and after two Raman spikes. These structures in our set-up are reproduced faithfully near the focal plane of the L3 lens. In particular the second Raman spike (RS2) is of interest in the present work because of its high beam density useful for frequency conversion. A magnified image (X 13.3) of RS2 has been obtained with two additional lenses L4 and L5 (located farther than L3 and not shown in Fig. 1). In the case of the output LC polarization, an expected intense ring (44 μm diameter) is obtained, surrounded by less intense ones (Fig. 4 column 1; line 1: experimental, line 2: calculation from sub-section 3.1).

FIG. 4 FH (1053 nm); line 1: experimental, line 2: theoretical;

Column 1: Image of the RS2 Raman spike from RC input and LC output without the KTP crystal
column 2: same than column 1 but through the KTP crystal; column 3: same than column 2 but from
LC input and RC output.

When the KTP crystal is introduced in the beam path, we observe the expected double refraction of the image. We can see the two images having 72 μm separation, corresponding to two different linear polarizations, in Fig. 4 second column (line 1: experimental, line 2: calculation from sub-section 3.2). “e-pol” is horizontal polarization and “o-pol” is vertical due to crystal orientation. The key-point is that in each doubly refracted image the RS2 ring has been split mainly in two brilliant spots separated by a dark point. More, if the input beam is switched to left circular polarization and the output analyser to right circular, the two brilliant spots are 90° rotated (Fig. 4 third column).

The modification of the FH RS2 beam described experimentally as well as theoretically in this sub-section is very important in the present work because we will mainly locate the KTP crystal in this region.

4.3 OAM of the second harmonic and sum-frequency beams

In a preliminary step without the KGW crystal nor the QW1, P2 and QW2 optical elements, a 45° polarized FH Gaussian at 1053 nm beam was launched in the KTP crystal located at the focal point of the L3 lens. Type II oe-e phase matching was found with a rotation the crystal very close to 17.1° (which is the calculated angle based on KTP Sellmeier formulas). We obtained 3.5 mW SH power at 526.5 nm with 135 mW FH input.

The main step was launching in the KGW crystal, now introduced in the beam path with the QW1, P2 and QW2 optical elements as shown in Fig. 1, the FH beam in a RC polarization state. Then the CD output beam is projected on various elliptical states by rotating the fast axis of the QW2 as it is explained in sub-section 4.1. This polarization-modified CD beam is now launched inside the KTP. Our goal in this sub-section is to study the SH, emitted by the KTP, of the RS2 Raman spike FH. Moreover, we take into account the linear optical process described in sub-section 4.2. So the KTP was initially located after the RS2 spike, producing a weak SH power. Then the KTP was moved towards the L3 lens until the RS2 was inside the KTP at about 0.2 cm of its entrance face. The SH power reached up to 0.6 mW. The SH far field was recorded through a L4 lens (15 cm focal length, not shown in Fig. 1) with the camera located at 15 cm of L4. The images obtained with the fast axis of the QW2 chosen at 0° (LC), 45° (LP) and 90° (RC) are presented in Fig. 5 (1), (2) and (3) respectively. We notice the two brilliant spots separated by two dark ones in Fig. 5 (1) while a brilliant spot is at the

centre of Fig. 5 (3). The image obtained with an input FH in LC polarization and the QW2 at 90° (RC output) is Fig. 5 (4) which is similar to 4 (1) but with 90° rotation.

FIG. 5 Line 1: experimental SH (526.5 nm) far field obtained for: FH (1053 nm) RC input and LC output (1), FH (1053 nm) RC input and LP output (2), FH (1053 nm) RC input and RC output (3), FH (1053 nm) LC input and RC output (4);

line 2: Fourier transform of the quadratic nonlinear polarization generated by the same FH (1053 nm) polarizations than line 1;

(9): SH (526.5 nm) $\rightarrow$ FH (1053 nm) theoretical conversion efficiency versus the x' and y' wave-vector components; (10): experimental SH (526.5 nm) near field for FH (1053 nm) RC input and LC output; (11) and (12): respectively horizontal and vertical component of the linear momentum of the SH (526.5 nm); (13): theoretical SH (526.5 nm) phase for FH (1053 nm) RC input and LC output; (14)

and (15): same than (11) and (12) but theoretical; (16): experimental SF (525 nm) of the two 1053 and 1047 nm fundamentals with the same polarizations than (10).

In favourable cases or after simplification, the FH as well as the SH can be a priori modelled with functions known in advanced, such as Bessel beams [18, 26] and the SH propagation results of the overlap between the beams. Fig. 5 (1), (2), (3) and (4) show that this procedure should be hard in the present work: no straightforward functions are evidenced. So the theoretical description of our experimental results start with the calculation of the Fourier transform $[\hat{\mathbf{P}}_{NL}]_{x'}$ nonlinear polarization from Eq. (5). For short z' -propagation values, let us say about $z'=0.2$ cm, $|\hat{\mathbf{P}}_{NL}|^2$ is represented in Fig. 5 (5-8) for the same polarization states than in the above paragraph. We can see that there is a clear similitude between the experimental SH far field in Fig. 5 (1), (2), (3) and (4) and the theoretical Fourier transform nonlinear polarization in Fig. 5 (5), (6), (7) and (8) respectively. This can be justified first because the short z' -propagation values constitute a restricted region inside which the FH beam is focused and is the main SH source. Elsewhere, the FH is expanded, its two orthogonal o and e-components separate due to birefringence, and this is also the case with the SH e-wave. Thanks to this occurrence and because Eq. (4) is rather complicated to solve (the full solution for any z' -propagation value is outside the present work), we tentatively use the approximation that the SH e-wave is boosted after a short propagation up to a value proportional to:

$$\hat{E}_{x'}^{2\omega} \approx [\hat{\mathbf{P}}_{NL}]_{x'}(z' \cong 0.2) \quad (6)$$

A second justification of Eq. (6) is provided by type II phase matching occurring in Eq. (4). For each plane wave direction of propagation the phase mismatch Δk which involves the two oe FH modes and the e SH mode can be calculated, likewise the corresponding coherence length L_c . Then the efficiency ε of the conversion is given by:

$$\varepsilon = \left(\frac{\sin(\pi z'/(2L_c))}{\pi z'/(2L_c)} \right)^2 \quad (7)$$

The efficiency is represented in Fig. 5 (9) versus the propagation direction and we can see that for the directions of interest it is $>95\%$. So only a small distortion of the SH wave is expected applying Eq. (6).

After that initialisation step each plane wave component of the SH propagates freely with the same kind of calculation for the FH in sub-section 3.2, and is refracted outside the KTP. The last step is the inversion of the Fourier Transform, leading to the $E_{x''}^{2\omega}$ electric near field.

The intensity of the latter was measured with the camera removing the L4 lens. It is represented in Fig. 5 (10). It is very similar to the calculated $E_{x''}^{2\omega}$ intensity which is not shown in Fig. 5 because it is also very similar to Fig. 5 (5) (except that the axes are graduated in μm and not in radians). However, it is instructive to show the calculated phase of the $E_{x''}^{2\omega}$ electric near field (Fig. 5 (13)). It reveals that the two dark spots near the centre of Fig. 5 (10) exhibit each a 2π rad phase variation in the same sense of rotation around. Let us add that in our model the two charge-1 dark spots are a direct and natural consequence of the two FH and SH wave interaction. We do not use in the calculation any additional small coherent background as it was mandatory for beams with vortices elsewhere [27, 28]. In another study devoted to self-frequency doubling [29], complex SH optical vortices have been exhibited. According to the authors, the multimode nature of the fundamental beam was responsible for this behaviour. The amplitude of the SH being the square of the fundamental one, the products of fields as in Eq. (2) provides the basis of the explanation without the help of an additional background.

For fractional $-1.5 \hbar/\text{photon}$ OAM, the phase of the field is represented in Fig. 3 (2). The white arrows show that the vortices are linked by pairs with opposite ± 1 charge (following lines with identical phase). These results confirm the discussion in sub-section 3.1 concerning the $-0.5 \hbar/\text{photon}$ OAM case and based on Ref. [22-25].

The experimental visualization of the horizontal component of the SH linear momentum is obtained by introducing on the output beam the cylindrical lens with its axis vertical: Fig. 5 (11) (experimental) and 5 (14) (theoretical) for FH right circular input and left circular output. Fig. 5 (12) and (15) are the same but the axis cylindrical lens is horizontal. We notice that these theoretical images with the cylindrical lenses reproduce quite well the experimental ones. Let us notice that the contribution to the OAM is anti-clockwise in Fig. 5 (11) and (14), while it is clockwise in Fig. 5 (12) and (15). From Eq. (7) the calculated SH OAM are gathered in Table 1 for different fast axis rotation of the QW2. All these OAM, including the fractional ones, have been calculated by respect to the barycentre of the pattern for 0° . They are twice the FH OAM found in subsection 4.1 within the uncertainty of our 2D-numerical

calculation, as a consequence of phase matching, making evidence of the sum of the OAM of the CD waves by the KTP quadratic nonlinearity.

Finally, we have launched simultaneously the two 1053 and 1047 nm beams (with the same polarizations) through the KGW and the KTP crystals. The SF, after slight KTP rotation for phase matching, was identified by its wavelength (525 nm) with a spectrometer. Very similar patterns than SH at 526.5 nm were obtained. For example, we show in Fig. 5 (16) the SF in KTP resulting of RC input and LC output in KGW: it is very close to Fig. 5 (10).

	Fast axis rotation of QW2 (°)				
	90	62.5	45	22.5	0
FH OAM ($\hbar$ /photon)	0	-0.214	-0.5	-0.85	-0.99
SH OAM ($\hbar$ /photon)	0	-0.52	-1.02	-1.55	-1.95

Table 1. Values of the OAM versus the rotation of the QW2 fast axis.

5. CONCLUSION

We operated the CD of two FH at 1053 and 1047 nm beams with a KGW optical axis oriented crystal, in various input and output polarization states. Then the SH/SF has been obtained with a KTP phase matched crystal located near the RS2 Raman spike. The integer or fractional OAM of both the FH and its SH/SF have been visualized with the cylindrical lens method. All the experimental patterns: far field, near field and field in the focal plane of the cylindrical lens, are quite well described by a self-consistent model based on plane waves propagation and their interferences, including the two KGW and KTP crystal optical properties. Sum of OAM due to FH—>SH/SF conversion is evidenced from experimental and calculated patterns.

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