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# Harmonious sets

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## Abstract

We correct a flaw found in *Algebraic numbers and harmonic analysis*, Elsevier (1972).

## 1 Erratum

Lemma 5 in Chapter 2 of *Algebraic numbers and harmonic analysis* [1] is nonsense. The correct statement is the one given below (Lemma 1.1). Here are the notations used in Lemma 1.1.  $\Gamma$  is a locally compact abelian group. The Bohr compactification of  $\Gamma$  is denoted by  $\tilde{\Gamma}$ . We have  $\Gamma \subset \tilde{\Gamma}$  and the topology on  $\Gamma$  which is induced by the topology of  $\tilde{\Gamma}$  is denoted by  $\mathcal{T}$ . If  $E \subset \Gamma, F \subset \Gamma$ , then  $E - F$  denotes the set of all  $x - y, x \in E, y \in F$ . Similarly for  $E + F$ .

A set  $E \subset \Gamma$  is relatively dense in  $\Gamma$  if there exists a compact set  $K$  such that  $K + E = \Gamma$ . A set  $\Lambda \subset \mathbb{R}^n$  is harmonious if for any positive  $\epsilon$  the set

$$\Lambda_\epsilon = \{x; |\exp(2\pi i x \cdot y) - 1| \leq \epsilon, \forall y \in \Lambda\} \quad (1)$$

is relatively dense. Let  $H$  the additive subgroup of  $\mathbb{R}^n$  generated by  $\Lambda$ . Then  $\Lambda$  is harmonious if for any homomorphism  $\chi : H \mapsto \mathbb{T}$  and any positive  $\epsilon$  there exists a  $y \in \mathbb{R}^n$  such that

$$\sup_{x \in \Lambda} |\chi(x) - \exp(2\pi i x \cdot y)| \leq \epsilon. \quad (2)$$

Here  $\mathbb{T}$  is the multiplicative group  $\{z; |z| = 1\}$ . If  $\Lambda$  is harmonious, then its closure  $M$  in  $\mathbb{R}^n$  equipped with the topology  $\mathcal{T}$  is still harmonious. Moreover  $M - M$  is harmonious. This provides an example where the hypotheses of Lemma 1.1 are satisfied. The following lemma shall replace Lemma 5 in [1] Chapter 2.

**Lemma 1.1** *Let  $M$  be a relatively dense subset of  $\Gamma$  and let  $M_1$  be the closure of  $M - M$  in  $\Gamma$  for the topology  $\mathcal{T}$ . Let  $\Omega \subset \Gamma$  be an open set for the topology  $\mathcal{T}$ . Let us assume  $0 \in \Omega$ . Then  $\Lambda = M_1 \cap \Omega$  is relatively dense in  $\Gamma$ .*

**Corollary 1.1** *Let us assume  $\Gamma = \mathbb{R}^n$  and let  $\Lambda \subset \Gamma$  be a discrete set of points. Let us assume that for any positive  $\epsilon$  there exists a finite subset  $F_\epsilon$  of  $\Lambda$  such that the set  $M_\epsilon = \{x; |\exp(2\pi i x \cdot y) - 1| \leq \epsilon, \forall y \in \Lambda \setminus F_\epsilon\}$  is relatively dense. Then  $\Lambda$  is harmonious.*

We first prove the corollary. It suffices to show that for any positive  $\epsilon$  the set  $\Lambda_\epsilon$  defined by

$$\Lambda_\epsilon = \{x; |\exp(2\pi i x \cdot y) - 1| \leq \epsilon, \forall y \in \Lambda\} \quad (3)$$

is relatively dense. We set

$$\Omega_\epsilon = \{x; |\exp(2\pi i x \cdot y) - 1| < \epsilon, \forall y \in F_\epsilon\}. \quad (4)$$

Then  $\Omega_\epsilon$  is open for the topology  $\mathcal{T}$ . On one hand  $M_{\epsilon/2}$  is closed the topology  $\mathcal{T}$ . On the other hand  $M_{\epsilon/2} - M_{\epsilon/2} \subset M_\epsilon$ . It suffices to conclude to observe that  $M_\epsilon \cap \Omega_\epsilon \subset \Lambda_\epsilon$  and to apply Lemma 1.1 to  $M = M_{\epsilon/2}$  and  $\Omega = \Omega_\epsilon$ .

The proof of Lemma 1.1 follows the argument given in [1] for proving Lemma 5. Everything takes place on  $\tilde{\Gamma}$ . Let us repeat that  $\Gamma \subset \tilde{\Gamma}$ . For any  $E \subset \Gamma$  we denote by  $\overline{E}$  the closure of  $E$  in  $\tilde{\Gamma}$ . Since  $\tilde{\Gamma}$  is compact we have  $\overline{M - M} = \overline{M} - \overline{M}$ . For any subset  $E \subset M$  we have

$$(E + \Omega) \cap \overline{M} \subset E + \Omega \cap (\overline{M} - \overline{M}). \quad (5)$$

Indeed if  $x + r = y$  with  $x \in E \subset M$ ,  $r \in \Omega$ , and  $y \in \overline{M}$  it implies

$$r = y - x \in \overline{M} - \overline{M} \quad (6)$$

and  $y = x + r \in E + \Omega \cap (\overline{M} - \overline{M})$ . Since  $\Omega$  is a neighborhood of 0 we have

$$\overline{M} \subset M + \Omega. \quad (7)$$

Since  $\overline{M}$  is compact and since  $\Omega$  is open there exists a finite subset  $A$  of  $M$  such that

$$\overline{M} \subset A + \Omega. \quad (8)$$

Therefore  $\overline{M} = (A + \Omega) \cap \overline{M}$  and (5) yields

$$\overline{M} \subset A + \Omega \cap (\overline{M} - \overline{M}). \quad (9)$$

Since  $M$  is relatively dense there exists a compact  $K \subset \Gamma$  such that  $M + K = \Gamma$ . This together with (9) implies

$$\tilde{\Gamma} = K + A + \Omega \cap (\overline{M} - \overline{M}) \quad (10)$$

and

$$\Gamma = K + A + \Gamma \cap \Omega \cap (\overline{M} - \overline{M}). \quad (11)$$

Therefore  $\Lambda = \Gamma \cap \Omega \cap (\overline{M} - \overline{M})$  is relatively dense and the proof of Lemma 1.1 is completed.

## References

- [1] Y. Meyer. Algebraic numbers and harmonic analysis. Elsevier (1972)