Harmonious sets

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Abstract

We correct a flaw found in *Algebraic numbers and harmonic analysis*, Elsevier (1972).

1 Erratum

Lemma 5 in Chapter 2 of *Algebraic numbers and harmonic analysis* [1] is nonsense. The correct statement is the one given below (Lemma 1.1). Here are the notations used in Lemma 1.1. Γ is a locally compact abelian group. The Bohr compactification of Γ is denoted by $\widetilde{\Gamma}$. We have $\Gamma \subset \widetilde{\Gamma}$ and the topology on Γ which is induced by the topology of $\widetilde{\Gamma}$ is denoted by $\mathcal{T}$. If $E \subset \Gamma, F \subset \Gamma$, then $E - F$ denotes the set of all $x - y$, $x \in E, y \in F$. Similarly for $E + F$.

A set $E \subset \Gamma$ is relatively dense in Γ if there exists a compact set K such that $K + E = \Gamma$. A set $\Lambda \subset \mathbb{R}^n$ is harmonious if for any positive ϵ the set

$$\Lambda_\epsilon = \{x; |\exp(2\pi i x \cdot y) - 1| \leq \epsilon, \ \forall y \in \Lambda\} \tag{1}$$

is relatively dense. Let H the additive subgroup of $\mathbb{R}^n$ generated by Λ. Then Λ is harmonious if for any homomorphism $\chi : H \mapsto \mathbb{T}$ and any positive ϵ there exists a $y \in \mathbb{R}^n$ such that

$$\sup_{x \in \Lambda} |\chi(x) - \exp(2\pi i x \cdot y)| \leq \epsilon. \tag{2}$$

Here $\mathbb{T}$ is the multiplicative group $\{z; |z| = 1\}$. If Λ is harmonious, then its closure M in $\mathbb{R}^n$ equipped with the topology $\mathcal{T}$ is still harmonious. Moreover $M - M$ is harmonious. This provides an example where the hypotheses of Lemma 1.1 are satisfied. The following lemma shall replace Lemma 5 in [1] Chapter 2.

Lemma 1.1 *Let M be a relatively dense subset of Γ and let M_1 be the closure of $M - M$ in Γ for the topology $\mathcal{T}$. Let $\Omega \subset \Gamma$ be an open set for the topology $\mathcal{T}$. Let us assume $0 \in \Omega$. Then $\Lambda = M_1 \cap \Omega$ is relatively dense in Γ.*

Corollary 1.1 *Let us assume $\Gamma = \mathbb{R}^n$ and let $\Lambda \subset \Gamma$ be a discrete set of points. Let us assume that for any positive ϵ there exists a finite subset F_ϵ of Λ such that the set $M_\epsilon = \{x; |\exp(2\pi i x \cdot y) - 1| \leq \epsilon, \ \forall y \in \Lambda \setminus F_\epsilon\}$ is relatively dense. Then Λ is harmonious.*

We first prove the corollary. It suffices to show that for any positive ϵ the set Λ_ϵ defined by

$$\Lambda_\epsilon = \{x; |\exp(2\pi i x \cdot y) - 1| \leq \epsilon,\ \forall y \in \Lambda\} \tag{3}$$

is relatively dense. We set

$$\Omega_\epsilon = \{x; |\exp(2\pi i x \cdot y) - 1| < \epsilon,\ \forall y \in F_\epsilon\}. \tag{4}$$

Then Ω_ϵ is open for the topology $\mathcal{T}$. On one hand $M_{\epsilon/2}$ is closed the topology $\mathcal{T}$. On the other hand $M_{\epsilon/2} - M_{\epsilon/2} \subset M_\epsilon$. It suffices to conclude to observe that $M_\epsilon \cap \Omega_\epsilon \subset \Lambda_\epsilon$ and to apply Lemma 1.1 to $M = M_{\epsilon/2}$ and $\Omega = \Omega_\epsilon$.

The proof of Lemma 1.1 follows the argument given in [1] for proving Lemma 5. Everything takes place on $\widetilde{\Gamma}$. Let us repeat that $\Gamma \subset \widetilde{\Gamma}$. For any $E \subset \Gamma$ we denote by $\overline{E}$ the closure of E in $\widetilde{\Gamma}$. Since $\widetilde{\Gamma}$ is compact we have $\overline{M - M} = \overline{M} - \overline{M}$. For any subset $E \subset M$ we have

$$(E + \Omega) \cap \overline{M} \subset E + \Omega \cap (\overline{M} - \overline{M}). \tag{5}$$

Indeed if $x + r = y$ with $x \in E \subset M$, $r \in \Omega$, and $y \in \overline{M}$ it implies

$$r = y - x \in \overline{M} - \overline{M} \tag{6}$$

and $y = x + r \in E + \Omega \cap (\overline{M} - \overline{M})$. Since Ω is a neighborhood of 0 we have

$$\overline{M} \subset M + \Omega. \tag{7}$$

Since $\overline{M}$ is compact and since Ω is open there exists a finite subset A of M such that

$$\overline{M} \subset A + \Omega. \tag{8}$$

Therefore $\overline{M} = (A + \Omega) \cap \overline{M}$ and (5) yields

$$\overline{M} \subset A + \Omega \cap (\overline{M} - \overline{M}). \tag{9}$$

Since M is relatively dense there exists a compact $K \subset \Gamma$ such that $M + K = \Gamma$. This together with (9) implies

$$\widetilde{\Gamma} = K + A + \Omega \cap (\overline{M} - \overline{M}) \tag{10}$$

and

$$\Gamma = K + A + \Gamma \cap \Omega \cap (\overline{M} - \overline{M}). \tag{11}$$

Therefore $\Lambda = \Gamma \cap \Omega \cap (\overline{M} - \overline{M})$ is relatively dense and the proof of Lemma 1.1 is completed.

References

[1] Y. Meyer. Algebraic numbers and harmonic analysis. Elsevier (1972)