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# ON THE CUSA-HUYGENS INEQUALITY 

YOGESH J. BAGUL, CHRISTOPHE CHESNEAU, AND MARKO KOSTIĆ


#### Abstract

Sharp bounds of various kinds for the famous unnormalized sinc function $\sin x / x$ are useful in mathematics, physics and engineering. In this paper, we reconsider the Cusa-Huygens inequality by solving the following problem: given real numbers $a, b, c \in \mathbb{R}$ and $T \in(0, \pi / 2]$, we find the necessary and sufficient conditions such that the inequalities


$$
\frac{\sin x}{x}>a+b \cos ^{c} x, \quad x \in(0, T)
$$

and

$$
\frac{\sin x}{x}<a+b \cos ^{c} x, \quad x \in(0, T)
$$

hold true. We use the elementary methods, only, improving several known results in the existing literature.

## 1. Introduction

The following inequality is the main inspiration of this paper:

$$
\begin{equation*}
\frac{\sin x}{x}<\frac{2+\cos x}{3}, \quad x \in(0, \pi / 2) . \tag{1.1}
\end{equation*}
$$

In the existing literature, it is known as the Cusa-Huygens inequality [13]; for more details, we refer the reader to $[6,8,9,11,13,14]$. This inequality has been extended and sharpened in many different ways [3,10,12,15-19]. For example, in $[9,16]$, it is obtained that

$$
\begin{equation*}
\left(\frac{2+\cos x}{3}\right)^{\alpha}<\frac{\sin x}{x}<\left(\frac{2+\cos x}{3}\right)^{\zeta}, \quad x \in(0, \pi / 2), \tag{1.2}
\end{equation*}
$$

where $\alpha=\ln (\pi / 2) / \ln (3 / 2) \approx 1.11374$ and $\zeta=1$ are the best possible constants. A very simple proof of (1.2) is offered in [2]. In [6], it is proved that,

[^0]for every $x \in(0, \pi / 2)$, we have
\[

$$
\begin{align*}
\frac{\alpha-1+\cos x}{\alpha} & <\frac{\sin x}{x}<\frac{\beta-1+\cos x}{\beta}  \tag{1.3}\\
\frac{2+\cos x}{3^{\alpha^{1}}} & <\frac{\sin x}{x}<\frac{2+\cos x}{3^{\beta_{1}}}  \tag{1.4}\\
\frac{2^{\alpha_{2}}+\cos x}{3} & <\frac{\sin x}{x}<\frac{2^{\beta_{2}}+\cos x}{3} \tag{1.5}
\end{align*}
$$
\]

with the best positive constants $\alpha \approx 2.75194, \beta=3 ; \alpha_{1} \approx 1.04198, \beta_{1}=1$ and $\alpha_{2} \approx 0.93345, \beta_{2}=1$.

The main aim of this paper is to consider the following problem. Let $a, b, c \in \mathbb{R}$ and $T \in(0, \pi / 2]$ be given real numbers; find the necessary and sufficient conditions such that the inequalities

$$
\begin{equation*}
\frac{\sin x}{x}>a+b \cos ^{c} x, \quad x \in(0, T) \tag{1.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\sin x}{x}<a+b \cos ^{c} x, \quad x \in(0, T) \tag{1.7}
\end{equation*}
$$

hold true. We completely solve this problem and thus provide generalizations to numerous known special results in the existing literature. We also consider the inequality

$$
\frac{2+\cos ^{a} x}{3}<\frac{\sin x}{x}<\frac{2+\cos ^{b} x}{3}, \quad x \in(0, T)
$$

and propose several auxiliary results of independent interest. For the sake of brevity and better exposition, we will not analyze the hyperbolic analogues of obtained results here.

## 2. PRELIMINARIES AND LEMMAS

We need to remind ourselves of the statement which is known in the existing literature as l'Hospital's rule of monotonicity; see, e.g., [1]:
Lemma 1. Let $f(x)$ and $g(x)$ be two real valued functions which are continuous on $[a, b]$ and differentiable on $(a, b)$, where $-\infty<a<b<\infty$ and $g^{\prime}(x) \neq 0$, for all $x \in(a, b)$. Let,

$$
A(x)=\frac{f(x)-f(a)}{g(x)-g(a)}, x \in(a, b)
$$

and

$$
B(x)=\frac{f(x)-f(b)}{g(x)-g(b)}, x \in(a, b) .
$$

Then, we have:
(i) $A(\cdot)$ and $B(\cdot)$ are increasing on $(a, b)$ if $f^{\prime}(\cdot) / g^{\prime}(\cdot)$ is increasing on ( $a, b$ ).
(ii) $A(\cdot)$ and $B(\cdot)$ are decreasing on $(a, b)$ if $f^{\prime}(\cdot) / g^{\prime}(\cdot)$ is decreasing on $(a, b)$.
The strictness of the monotonicity of $A(\cdot)$ and $B(\cdot)$ depends on the strictness of monotonicity of $f^{\prime}(\cdot) / g^{\prime}(\cdot)$.

We proceed with some original lemmas and observations we will use later on.

Lemma 2. ([4]) The function

$$
F(x):=\frac{\sin x-x \cos x}{x^{2} \sin x}, \quad x \in(0, \pi / 2)
$$

is positive and strictly increasing on $(0, \pi / 2)$.
Remark 1. (i) The proof of [4, Lemma 2.2] contains a small mistake since the function $x \mapsto-x \tan x+3, x \in(0, \pi / 2)$ has a unique zero $\zeta \in$ $(0, \pi / 2)$ so that an application of l'Hospital's rule of monotonicity shows that the function $F(\cdot)$ is strictly increasing on $(0, \zeta)$, only. But, the result is actually true because in the former step of proof we have $H_{1}^{\prime}(x) / H_{2}^{\prime}(x)=\sin x /(2 \sin x+x \cos x)=1 /(2+x \cot x), x \in(0, \pi / 2)$, which is strictly increasing on $(0, \pi / 2)$.
(ii) It is clear that, due to Lemma 2, we have that for each number $\sigma \in$ $(-\infty, 2]$ the function

$$
F_{\sigma}(x)=\frac{\sin x-x \cos x}{x^{\sigma} \sin x}, \quad x \in(0, \pi / 2)
$$

is positive and strictly increasing on $(0, \pi / 2)$. Consider now the case that $\sigma>2$. Then, for each $x \in(0, \pi / 2)$, we have

$$
F_{\sigma}^{\prime}(x)=\left(x^{\sigma} \sin x\right)^{-2} x^{\sigma-1}\left[x^{2}-x \sin x \cos x-\sigma\left(\sin ^{2} x-x \sin x \cos x\right)\right] .
$$

So, if

$$
\sigma \geq \sigma_{0}:=\sup _{x \in(0, \pi / 2)} \frac{x^{2}-x \sin x \cos x}{\sin ^{2} x-x \sin x \cos x}:=\sup _{x \in(0, \pi / 2)} Q(x)=\frac{\pi^{2}}{4},
$$

then the function $F_{\sigma}(\cdot)$ is strictly decreasing on $(0, \pi / 2)$. In order to prove that $\sigma_{0}=\pi^{2} / 4$, we can use the facts that $\lim _{x \rightarrow 0} Q(x)=2$, established with the help of l'Hospital's rule, and $Q(\cdot)$ is strictly increasing on ( $0, \pi / 2$ ), which follows by applying l'Hospital's rule of monotonicity twice. Strictly speaking, by applying the usual l'Hospital's rule twice,
we get

$$
\begin{aligned}
\lim _{x \rightarrow 0} Q(x) & =\lim _{x \rightarrow 0} \frac{2 x-\sin x \cos x-x \cos 2 x}{\sin x \cos x-x \cos 2 x}=\lim _{x \rightarrow 0} \frac{1-\cos 2 x+x \sin 2 x}{x \sin 2 x} \\
& =\lim _{x \rightarrow 0}\left[1+\frac{1-\cos 2 x}{x \sin 2 x}\right]=\lim _{x \rightarrow 0}\left[1+\frac{\tan x}{x}\right]
\end{aligned}
$$

For the applications of l'Hospital's rule of monotonicity, we only need to observe yet that
$\sin x \cos x-x \cos 2 x=\cos 2 x[(\tan (2 x) / 2)-x]$

$$
=\cos 2 x\left[\tan x\left(1-\tan ^{2} x\right)^{-1}-x\right]>\cos 2 x[\tan x-x]>0, x \in(0, \pi / 2)
$$

and the function $\tan (\cdot) / \cdot$ is strictly increasing on $(0, \pi / 2)$. Finally, the above also implies that for each number $\sigma \in\left(2, \sigma_{0}\right)$ there exists a unique number $x_{\sigma} \in(0, \pi / 2)$ such that the function $F_{\sigma}(\cdot)$ is strictly decreasing on $\left(0, x_{\sigma}\right)$ and $F_{\sigma}(\cdot)$ is strictly increasing on $\left(x_{\sigma}, \pi / 2\right)$.

Lemma 3. For $x \in(0, \pi / 2)$, it is true that

$$
\frac{2 x^{2}}{\pi}<\frac{x}{\sin x}-\cos x<\frac{2 x^{2}}{3}
$$

which is equivalent to

$$
\frac{3}{2 x^{2}+3 \cos x}<\frac{\sin x}{x}<\frac{\pi}{2 x^{2}+\pi \cos x}
$$

Proof. Let us consider the function

$$
f(x)=\frac{x^{2} \sin x}{x-\sin x \cos x}, x \in(0, \pi / 2)
$$

We have

$$
\begin{aligned}
(x & -\sin x \cos x)^{2} f^{\prime}(x) \\
\quad & =(x-\sin x \cos x)\left(x^{2} \cos x+2 x \sin x\right)-2 \sin ^{2} x\left(x^{2} \sin x\right) \\
& =x^{3} \cos x+2 x^{2} \sin x-x^{2} \sin x \cos ^{2} x-2 x \sin ^{2} x \cos x-2 x^{2} \sin ^{3} x \\
& =x^{3} \cos x+x^{2} \sin x \cos x-2 x \sin ^{2} x \cos x \\
& =x \Delta(x) \cos x
\end{aligned}
$$

where $\Delta(x)=x^{2}+x \sin x \cos x-2 \sin ^{2} x(x \in(0, \pi / 2))$. Now, let us notice that

$$
\begin{aligned}
\Delta^{\prime}(x) & =2 x+x \cos ^{2} x-x \sin ^{2} x+\sin x \cos x-4 \sin x \cos x \\
& =x+2 x \cos ^{2} x-3 \sin x \cos x, \quad x \in(0, \pi / 2)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta^{\prime \prime}(x) & =1+2 \cos ^{2} x-4 x \sin x \cos x-3 \cos ^{2} x+3 \sin ^{2} x \\
& =4 \sin ^{2} x-4 x \sin x \cos x=4 \sin x(\sin x-x \cos x), \quad x \in(0, \pi / 2)
\end{aligned}
$$

Since $\sin x-x \cos x>0$ for $x \in(0, \pi / 2)$, we get $\Delta^{\prime \prime}(x)>0$, implying that $\Delta^{\prime}(\cdot)$ is strictly increasing on $(0, \pi / 2)$ and, a fortiori, $\Delta(x)>\Delta(0+)=0$, implying that $f^{\prime}(x)>0$. Hence, $f(\cdot)$ is strictly increasing on $(0, \pi / 2)$ and $f(0+)<$ $f(x)<f(\pi / 2)$, with $f(0+)=3 / 2$ by l'Hospital's rule and $f(\pi / 2)=\pi / 2$. This ends the proof of Lemma 3.

In the remainder of paper, it will be always assumed that $T \in(0, \pi / 2]$. Observe also that the equation $\sin x / x=2 / 3$ has a unique zero $\nu$ belonging to the interval $(0, \pi / 2)$; furthermore, we have $\nu \approx 1.49578$, as well as $\sin x / x>$ $2 / 3$ and $x \in(0, \pi / 2)$ iff $x \in(0, \nu)$.

Lemma 4. For each number $b \in(1 / 3, F(T))$, there exists a unique number $\sigma_{b, T} \in(0, T)$ such that

$$
\frac{x \cos x-\sin x}{x^{2} \sin x}+b=0
$$

holds with $x=\sigma_{b, T}$.
Proof. Follows immediately from Lemma 2 as well as the limit equalities

$$
\lim _{x \rightarrow 0+} \frac{x \cos x-\sin x}{x^{2} \sin x}=-\frac{1}{3} \text { and } \lim _{x \rightarrow T-} \frac{x \cos x-\sin x}{x^{2} \sin x}=-F(T),
$$

which can be easily verified.
The function

$$
W(x):=\cos x \frac{x^{2}+x \sin x \cos x-2 \sin ^{2} x}{(\sin x-x \cos x) x \sin ^{2} x}, \quad x \in(0, \pi / 2)
$$

has an important role in our study, as well. This function is positive (see the proof of Lemma 3, showing that $x^{2}+x \sin x \cos x-2 \sin ^{2} x>0, x \in(0, \pi / 2)$ ). Moreover, we have $\lim _{x \rightarrow \pi / 2-} W(x)=0$ and $\lim _{x \rightarrow 0+} W(x)=2 / 15$ because $(\sin x-x \cos x) x \sin ^{2} x \sim x^{6} / 3, x \rightarrow 0+$ and $x^{2}+x \sin x \cos x-2 \sin ^{2} x \sim$ $2 x^{6} / 45, x \rightarrow 0+$.

Lemma 5. The function $W(\cdot)$ is strictly decreasing on $(0, \pi / 2)$.
Proof. Set

$$
W_{0}(x):=\cos x \frac{x^{2}+x \sin x \cos x-2 \sin ^{2} x}{x^{3} \sin ^{3} x}, \quad x \in(0, \pi / 2) .
$$

Then, we have $W_{0}(x)>0$ for all $x \in(0, \pi / 2)$. Therefore, due to Lemma 2 and the decomposition

$$
W(x)=\frac{x^{2} \sin x}{\sin x-x \cos x} W_{0}(x), \quad x \in(0, \pi / 2),
$$

it suffices to show that the function $W_{0}(\cdot)$ is strictly decreasing on $(0, \pi / 2)$. Towards this end, we note that the calculus established with the use of symbolab.com package shows that, for every $x \in(0, \pi / 2)$, we have:

$$
\begin{aligned}
& W^{\prime}(x)=-\frac{2 x^{2}+x \sin 2 x-4 \sin ^{2} x}{2 x^{3} \sin ^{2} x} \\
& +\frac{\cot x \times\left[-6 x^{3} \sin 2 x-4 x^{2} \sin ^{2} x+4 x^{2} \sin ^{2} \cos 2 x+24 \sin ^{4} x-3 x^{2} \sin ^{2} x\right]}{4 x^{4} \sin ^{4} x} .
\end{aligned}
$$

After multiplying the first addend with $2 x \sin ^{2} x$ and grouping the terms, we need to show that, for every $x \in(0, \pi / 2)$, we have

$$
\begin{aligned}
& 4 x^{2} \cos x \sin x \cos 2 x+24 \cos x \sin ^{3} x+8 x \sin ^{4} x-12 x^{3} \cos ^{2} x \\
& <4 x^{2} \cos x \sin x+12 x^{2} \cos ^{3} x \sin x+4 x^{3} \sin ^{2} x+2 x^{2} \sin 2 x \sin ^{2} x .
\end{aligned}
$$

Since $1-\cos 2 x=2 \sin ^{2} x$ and $\sin ^{2} x+\cos ^{2} x=1$, after collecting similar terms and dividing with 4 , the above is equivalent with

$$
\begin{equation*}
2 x^{3} \cos ^{2} x+x^{3}+3 x^{2} \sin x \cos x>6 \cos x \sin ^{3} x+2 x \sin ^{4} x, \quad x \in(0, \pi / 2) \tag{2.1}
\end{equation*}
$$

Suppose first that $x \in(0, \nu)$. Then, (1.1) implies $\cos x>3(\sin x / x)-2>0$ so that

$$
\begin{aligned}
& 2 x^{3} \cos ^{2} x+x^{3}+3 x^{2} \sin x \cos x \\
& \quad>2 x^{3}\left(3 \frac{\sin x}{x}-2\right)^{2}+x^{3}+3 x^{2} \sin x\left(3 \frac{\sin x}{x}-2\right) \\
& \quad=27 x \sin ^{2} x+9 x^{3}-18 x^{2} \sin x .
\end{aligned}
$$

Therefore, it suffices to show that

$$
27 x \sin ^{2} x+9 x^{3}>6 \cos x \sin ^{3} x+2 x \sin ^{4} x+18 x^{2} \sin x
$$

This follows from the inequalities $9 x \sin ^{2} x+9 x^{3} \geq 2 \sqrt{9^{2} x^{4} \sin ^{2} x}=18 x^{2} \sin x$, $18 x \sin ^{2} x<18 x^{3}$ and $6 \cos x \sin ^{3} x+2 x \sin ^{4} x<6 x^{3}+2 x^{5}<8 x^{3}$. If $x \in$ $[\nu, \pi / 2)$, then the proof is much simpler and follows from the fact that, for such values of variable $x$, we have $x^{3}>2 x \geq 2 x \sin ^{4} x$ and $x^{2}>2 \geq 2 \sin ^{2} x$, implying that $3 x^{2} \sin x \cos x>6 \cos x \sin ^{3} x$ (see (2.1)).

Hence, $\inf \{W(x): x \in(0, \pi / 2)\}=0$ and $\min \{W(x): x \in(0, \pi / 2)\}$ does not exist as well as $\sup \{W(x): x \in(0, \pi / 2)\}=2 / 15$ and $\max \{W(x): x \in$ $(0, \pi / 2)\}$ does not exist.

## 3. Main results

Now we are in a position to prove the main results of the paper. We start by stating the following

Theorem 1. Let $a, b \in \mathbb{R}$. Then, the inequality

$$
\begin{equation*}
\frac{\sin x}{x}>a+b \cos x, \quad x \in(0, T) \tag{3.1}
\end{equation*}
$$

holds iff

1. $b \leq 1 / 3$ and $a \leq(\sin T / T)-b \cos T$, or
2. $b \geq F(T)$ and $a \leq 1-b$, or
3. $b \in(1 / 3, F(T))$ and $a \leq \min (1-b,(\sin T / T)-b \cos T)$.

Proof. It is clear that (3.1) holds iff $a<\inf \{(\sin x / x)-b \cos x: x \in(0, T)\}$ or $a \leq \inf \{(\sin x / x)-b \cos x: x \in(0, T)\}$ if there is no $x \in(0, T)$ such that $(\sin x / x)-b \cos x=a$. Consider the function

$$
M(x):=\frac{\sin x}{x}-b \cos x, \quad x \in(0, T)
$$

Then, a simple computation shows that

$$
M^{\prime}(x)=\sin x\left[\frac{x \cos x-\sin x}{x^{2} \sin x}+b\right], \quad x \in(0, T)
$$

Note that $\left\{(x \cos x-\sin x) /\left(x^{2} \sin x\right): x \in(0, T)\right\}=(-F(T),-1 / 3)$ (see Lemma 2 and Lemma 4) and $F(0+)=1-b$. Hence, if $b \leq 1 / 3$, then $M^{\prime}(x) \leq 0$ for all $x \in(0, T)$ and the zeroes of function $M^{\prime}(\cdot)$ do not form an interval in $(0, T)$ so that the function $M(\cdot)$ is strictly decreasing on $(0, T)$ and therefore (3.1) holds iff $a \leq(\sin T / T)-b \cos T$. Similarly, if $b \geq F(T)$, then the function $M(\cdot)$ is strictly increasing on $(0, T)$ and (3.1) holds iff $a \leq 1-b$. Finally, if $b \in(1 / 3, F(T))$, then the function $M(\cdot)$ strictly increases on $\left(0, \sigma_{b, T}\right)$ and the function $M(\cdot)$ strictly decreases on $\left(\sigma_{b, T}, T\right)$ (see Lemma 4) which implies that $\inf \{(\sin x / x)-b \cos x: x \in(0, T)\}=\min (1-b,(\sin T / T)-b \cos T)$ and (3.1) holds iff $a \leq \min (1-b,(\sin T / T)-b \cos T)$.

Remark 2. Theorem 1 improves the well-known Baricz's inequality

$$
\frac{1+\cos x}{2} \leq \frac{\sin x}{x}, \quad x \in(0, \pi / 2)
$$

see $[5, \mathrm{p} .111]$.
It is clear that Theorem 1 substantially improves the left hand sides of equations (1.3)-(1.5) with $T=\pi / 2$. Strictly speaking, Theorem 1 implies that $\beta=\pi / \pi-2$ is the smallest positive constant strictly greater than 1 such that the left hand side of (1.3) holds, which can be simply inspected.

We can similarly prove the following extension of the Cusa-Huygens inequality, which taken together with Theorem 1 provides a proper generalization of [6, Theorem 1.6, Theorem 3.9]:

Theorem 2. Let $a, b \in \mathbb{R}$. Then, the inequality

$$
\frac{\sin x}{x}<a+b \cos x, \quad x \in(0, T)
$$

holds iff

1. $b \leq 1 / 3$ and $a \geq 1-b$, or
2. $b \geq F(T)$ and $a \geq(\sin T / T)-b \cos T$, or
3. $b \in(1 / 3, F(T))$ and $a>\left(\sin \sigma_{b, T} / \sigma_{b, T}\right)-b \cos \sigma_{b, T}$.

Remark 3. Taken together, Theorem 1 and Theorem 2 improve the well-known results of J. Sándor and R. Oláh-Gal [18, Theorem 1, Theorem 2] and the wellknown Oppenheim's double inequality

$$
\frac{2+(\pi-2) \cos x}{\pi}<\frac{\sin x}{x}<\frac{2+(4 / \pi) \cos x}{\pi}
$$

(see, e.g., [7]).
Suppose now that $a, b, c \in \mathbb{R}$ and $c>1$. Consider the function

$$
M_{1}(x):=\frac{\sin x}{x}-b \cos ^{c} x, \quad x \in(0, T),
$$

whose first derivative is given by

$$
M_{1}^{\prime}(x)=\cos ^{c-1} x \sin x\left[\frac{x \cos x-\sin x}{x^{2} \sin x \cos ^{c-1} x}+b c\right], \quad x \in(0, T) .
$$

Using Lemma 2 and the assumption $c>1$, it follows that the function $x \mapsto(x \cos x-\sin x) /\left(x^{2} \sin x \cos ^{c-1} x\right), x \in(0, T)$ is strictly decreasing; moreover, the range of this function is equal to $(-\infty,-1 / 3)$, if $T=\pi / 2$, resp. $\left(-F(T) \cos ^{1-c} T,-1 / 3\right)$, if $T<\pi / 2$. Therefore, arguing as above, we may conclude that the following holds:

Theorem 3. Suppose that $a, b, c \in \mathbb{R}$ and $c>1$. Then, we have the following:
(i) The inequality (1.6) holds iff:

1. $b c \leq 1 / 3$ and $a \leq 2 / \pi$, or bc $>1 / 3$ and $a \leq \min (1-b, 2 / \pi)$, provided that $T=\pi / 2$.
2. $b c \leq 1 / 3$ and $a \leq(\sin T / T)-b \cos ^{c} T$, or $b c \geq F(T) \cos ^{1-c} T$ and $a \leq 1-b$, or $b c \in(1 / 3, F(T))$ and $a \leq \min (1-b,(\sin T / T)-$ $b \cos ^{c} T$ ), provided that $T<\pi / 2$.
(ii) The inequality (1.7) holds iff:
3. $b c \leq 1 / 3$ and $a \geq 1-b$, or $b c>1 / 3$ and $a>\left(\sin \sigma_{b, c} / \sigma_{b, c}\right)-$ $b \cos ^{c} \sigma_{b, c}$, where $\sigma_{b, c}$ denotes the unique solution of equation

$$
\frac{x \cos x-\sin x}{x^{2} \sin x \cos ^{c-1} x}+b c=0
$$

on $(0, \pi / 2)$, provided that $T=\pi / 2$.
2. $b c \leq 1 / 3$ and $a \geq 1-b$, or $b c \geq F(T) \cos ^{1-c} T$ and $a \geq(\sin T / T)-$ $b \cos ^{c} T$, or $b c \in(1 / 3, F(T))$ and $a>\left(\sin \zeta_{b, c, T} / \zeta_{b, c, T}\right)-b \cos ^{c} \zeta_{b, c, T}$, where $\zeta_{b, c, T}$ denotes the unique solution of equation

$$
\frac{x \cos x-\sin x}{x^{2} \sin x \cos ^{c-1} x}+b c=0, \quad x \in(0, T)
$$

provided that $T<\pi / 2$.
Therefore, it remains to consider the case in which $a, b, c \in \mathbb{R}$ and $c<1$. In this case, we have

$$
\begin{aligned}
& \left(\frac{x \cos x-\sin x}{x^{2} \sin x \cos ^{c-1} x}\right)^{\prime}=\cos ^{-c} x \frac{x \cos x-\sin x}{x^{2}} \\
& \times\left[\cos x \frac{x^{5} \sin ^{2} x-x^{2}(\sin x-x \cos x)\left(2 x \sin x+x^{2} \cos x\right)}{(\sin x-x \cos x) x^{4} \sin ^{2} x}+c-1\right] \\
& =\cos ^{-c} x \frac{x \cos x-\sin x}{x^{2}}\left[\cos x \frac{x^{2}+x \sin x \cos x-2 \sin ^{2} x}{(\sin x-x \cos x) x \sin ^{2} x}+c-1\right] \\
& =\cos ^{-c} x \frac{x \cos x-\sin x}{x^{2}}[W(x)+c-1], x \in(0, T) .
\end{aligned}
$$

Based on this computation, Lemma 5 and the analysis preceding it, we can clarify the following extension of Lemma 2 :

Proposition 1. Let $d \in \mathbb{R}$. Then, the function

$$
Q_{d}(x):=\frac{x \cos x-\sin x}{x^{2} \sin x \cos ^{d-1} x}, \quad x \in(0, T)
$$

has the following properties:
(i) $Q_{d}(\cdot)$ is strictly increasing on $(0, T)$ iff $d \geq 1-W(T)$.
(ii) $Q_{d}(\cdot)$ is strictly decreasing on $(0, T)$ iff $d \leq 13 / 15$.
(iii) If $d \in(13 / 15,1-W(T))$, then there exists a unique number $\theta_{d, T} \in$ $(0, T)$ such that the function $Q_{d}(\cdot)$ is strictly increasing on $\left(0, \theta_{d, T}\right)$ and $Q_{d}(\cdot)$ is strictly decreasing on $\left(\theta_{d, T}, T\right)$.

We continue with the analysis of case $a, b, c \in \mathbb{R}$ and $c<1$. In actual fact, the following theorem holds true:

Theorem 4. (i) Let $c \leq 13 / 15$. Then, (1.6), resp. (1.7), holds iff:

1. $b c \geq 1 / 3$ and $a \leq 1-b$, resp. $b c \geq 1 / 3$ and $a \geq 2 / \pi$ if $c>0$ and $T=\pi / 2 \quad\left[b c \geq 1 / 3\right.$ and $a \geq M_{1}(T)$ if $\left.T<\pi / 2\right]$, or
2. $b c \leq F(T) \cos ^{1-c} T$ and $a \leq(\sin T / T)-b \cos ^{c} T$ if $T<\pi / 2$ or $T=\pi / 2$ and $c \geq 0$, or $a \in \mathbb{R}, b<0, c<0, T=\pi / 2$ or $b c \leq 0$, $a \leq 2 / \pi, b=0, c<0, T=\pi / 2, r e s p . b c \leq F(T) \cos ^{1-c} T$ and $a \geq 1-b$, or
3. $b c \in\left(F(T) \cos ^{1-c} T, 1 / 3\right)$. Then, there exists a unique number $\sigma_{b ; c ; T} \in(0, T)$ such that the equation

$$
\frac{\sin x-x \cos x}{x^{2} \sin x} \cos ^{1-c} x=b c
$$

holds with $x=\sigma_{b ; c ; T}$; in this case, (1.6), resp. (1.7), holds iff $a<\left(\sin \sigma_{b ; c ; T} / \sigma_{b ; c ; T}\right)-b \cos ^{c} \sigma_{b ; c ; T}$, resp. $c>0$ and $a \geq \max (1-$ $\left.b, M_{1}(T-)\right)$.
(ii) Let $c \geq 1-W(T)$ and $T<\pi / 2$. Then, (1.6), resp. (1.7), holds iff:

1. $b c \leq F(T) \cos ^{1-c} T$ and $a \leq \sin T / T$, resp. $a \geq 1-b$, or
2. $b c \geq 1 / 3$ and $a \leq 1-b$, resp. $b c \geq 1 / 3$ and $a \geq M_{1}(T)$, or
3. $b c \in\left(F(T) \cos ^{1-c} T, 1 / 3\right)$ and $a<M_{1}\left(\eta_{b ; c ; T}\right)$, where $\eta_{b ; c ; T}$ denotes the unique solution of equation $Q_{c}(x)+b c=0$ on $(0, T)$, resp. $b c \in\left(F(T) \cos ^{1-c} T, 1 / 3\right)$ and $a \geq \max \left(1-b, M_{1}(T)\right)$.
(iii) Let $c \in(13 / 15,1-W(T))$. Then, (1.6), resp. (1.7), holds iff:
4. $b c \leq-Q_{c}\left(\theta_{c, T}\right)$ and $a \leq M_{1}(T-)$, resp. bc $\leq-Q_{c}\left(\theta_{c, T}\right)$ and $a \geq 1-b$, or
5. $b c \geq-\min \left(-1 / 3, Q_{c}(T)\right)$ and $a \leq 1-b$, resp. $b c \geq-\min \left(-1 / 3, Q_{c}(T)\right)$ and $a \geq M_{1}(T-)$, or
6. $-b c \in\left(\min \left(-1 / 3, Q_{c}(T)\right), Q_{c}\left(\theta_{c, T}\right)\right)$ and the equation $Q_{c}(x)+b c=$ 0 has exactly one zero $\zeta_{b, c, T}$ in $(0, T)$. Then, (1.6), resp. (1.7), holds iff $a<M_{1}\left(\zeta_{b, c, T}\right)$, resp. $a \geq \max \left(1-b, M_{1}(T-)\right)$.
7. $-b c \in\left(\min \left(-1 / 3, Q_{c}(T)\right), Q_{c}\left(\theta_{c, T}\right)\right)$ and the equation $Q_{c}(x)+b c=$ 0 has exactly two zeroes $\zeta_{b, c, T}^{1}$ and $\zeta_{b, c, T}^{2}$ in $(0, T)$ such that, say, $\zeta_{b, c, T}^{1}<\zeta_{b, c, T}^{2}$. Then, (1.6), resp. (1.7), holds iff $a \leq M_{1}(T-)<$ $M_{1}\left(\zeta_{b, c, T}^{1}\right)$ or $a<M_{1}\left(\zeta_{b, c, T}^{1}\right) \leq M_{1}(T-)$, resp. $a \geq 1-b>$ $M_{1}\left(\zeta_{b, c, T}^{2}\right)$ or $a>M_{1}\left(\zeta_{b, c, T}^{2}\right) \geq 1-b$.
Proof. The proof of theorem is very similar to those of Theorem 1 and Theorem 2 , and we will only outline the main details for the statement (iii). If $c \in$ $(13 / 15,1-W(t))$ and $b c \leq-Q_{c}\left(\theta_{c, T}\right)$, we have that the function $M_{1}(\cdot)$ is strictly decreasing on $(0, T)$ since its first derivative is non-negative, which simply implies that the statement 1 . holds true. Similarly, if we have $b c \geq$ $-\min \left(-1 / 3, Q_{c}(T)\right)$, then the function $M_{1}(\cdot)$ is strictly increasing on $(0, T)$ and the statement 2. follows as above. If $-b c \in\left(\min \left(-1 / 3, Q_{c}(T)\right), Q_{c}\left(\theta_{c, T}\right)\right)$, the equation can have exactly one or exactly two zeroes belonging to the
interval $(0, T)$. In the first case, the function $M_{1}(\cdot)$ is strictly decreasing on $\left(0, \zeta_{b, c, T}\right)$ and the function $M_{1}(\cdot)$ is strictly decreasing on $\left(\zeta_{b, c, T}, T\right)$, while in the second case, the function $M_{1}(\cdot)$ is strictly decreasing on the intervals $\left(0, \zeta_{b, c, T}^{1}\right)$ and $\left(\zeta_{b, c, T}^{2}, T\right)$ and the function $M_{1}(\cdot)$ is strictly increasing on $\left(\zeta_{b, c, T}^{1}, \zeta_{b, c, T}^{2}\right)$.

Remark 4. Theorem 4 improves the well-known inequality

$$
\sqrt[3]{\cos x}<\frac{\sin x}{x}, \quad x \in(0, \pi / 2)
$$

due to by D. D. Adamović and D. S. Mitrinović (see, e.g., [13, p. 238]).
Theorem 4 is an abstract result and the direct use of l'Hospital's rule of monotonicity is sometimes a much better choice for examing the best possible constants for which certain concrete inequalities hold true. The theorem below presents the sharp bounds of the form $\left(2+\cos ^{u} x\right) / 3$ for $\sin x / x$. In order to formulate this theorem, let us recall that $\nu \approx 1.49578$ denotes the unique solution of equation $\sin x / x=2 / 3$ on $(0, \pi / 2)$.

Theorem 5. Let $\lambda \in(0, \alpha)$ with $\alpha \approx 1.49578$. Then, the best possible constants $a$ and $b$ such that

$$
\frac{2+\cos ^{a} x}{3}<\frac{\sin x}{x}<\frac{2+\cos ^{b} x}{3}, \quad x \in(0, \lambda)
$$

are $\log (3 \sin \lambda / \lambda-2) / \log (\cos \lambda)$ and 1, respectively.
Proof. Let us consider the function

$$
f(x)=\frac{\log (3 \sin x / x-2)}{\log (\cos x)}=\frac{f_{1}(x)}{f_{2}(x)}, \quad x \in(0, \lambda) .
$$

For such values of $x$, we have

$$
\frac{f_{1}^{\prime}(x)}{f_{2}^{\prime}(x)}=\frac{3(\sin x-x \cos x)}{x^{2} \sin x} \frac{x \cos x}{3 \sin x-2 x}=f_{3}(x) f_{4}(x),
$$

where $f_{3}(x)=3(\sin x-x \cos x) /\left(x^{2} \sin x\right)$ and $f_{4}(x)=x \cos x /(3 \sin x-2 x)$. It follows by Lemma 2 that $f_{3}(\cdot)$ is positive and strictly increasing. Now, remark that

$$
(3 \sin x-2 x)^{2} f_{4}^{\prime}(x)=-3 x+3 \sin x \cos x+2 x^{2} \sin x,
$$

which is positive by Lemma 3. Hence, $f_{4}(\cdot)$ is also positive and strictly increasing. Therefore, $f_{1}^{\prime}(\cdot) / f_{2}^{\prime}(\cdot)$ is strictly increasing and, by l'Hospital's rule of monotonicity (see Lemma 1 ), $f(\cdot)$ is strictly increasing on $(0, \lambda)$ with $f(0+)<f(x)<f(\lambda)$. We end the proof of Theorem 5 by noticing that $f(0+)=1$ by l'Hospital's rule and $f(\lambda)=\log (3 \sin \lambda / \lambda-2) / \log (\cos \lambda)$.

Figure 1 displays the functions involved in the inequalities of Theorem 5.


Figure 1. Illustration of the inequalities in Theorem 5.

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