# On the effect of symmetry requirement for rendezvous on the complete graph

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# On the effect of symmetry requirement for rendezvous on the complete graph*

Marthe Bonamy[†] Michał Pilipczuk[‡] Jean-Sébastien Sereni[§] Richard Weber[¶]

**Abstract**

We consider a classic rendezvous game where two players try to meet each other on a set of $n$ locations. In each round, every player visits one of the locations and the game finishes when the players meet at the same location. The goal is to devise strategies for both players that minimize the expected waiting time till the rendezvous.

In the asymmetric case, when the strategies of the players may differ, it is known that the optimum expected waiting time of $\frac{n+1}{2}$ is achieved by the *wait-for-mommy* pair of strategies, where one of the players stays at one location for $n$ rounds, while the other player searches through all the $n$ locations in a random order. However, if we insist that the players are symmetric — they are expected to follow the same strategy — then the best known strategy, proposed by Anderson and Weber [7], achieves an asymptotic expected waiting time of $0.829n$.

We show that the symmetry requirement indeed implies that the expected waiting time needs to be asymptotically larger than in the asymmetric case. Precisely, we prove that for every $n \geqslant 2$, if the players need to employ the same strategy, then the expected waiting time is at least $\frac{n+1}{2} + \varepsilon n$, where $\varepsilon = 2^{-36}$.

We propose in addition a different proof for one our key lemmas, which relies on a result by Ahlswede and Katona: the argument is slightly shorter and provides a constant larger than $2^{-36}$, namely $\frac{1}{3600}$. However, it requires that $n$ be at least 16. Both approaches seem conceptually interesting to us.

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# 1 Introduction

Rendezvous search questions fall within the long-established field of search games: instead of having a player searching for an otherwise indifferent treasure, there are now two players that want to meet as quickly as possible. This very natural problem lends itself to a number of very different, more or less formalised settings. It was first specified as an optimisation problem in 1976 by Alpern at the end of a talk (see [3]), in two different settings: the *astronaut problem* and the seemingly simpler *telephone problem*. In the former problem, two players are on a sphere, each with a given unit walking speed and no common orientation in space, and they want to minimise their expected meeting time. The telephone problem has since been rephrased as a *rendezvous game on discrete locations*, as follows. Two players wish to meet on a set of $n$ locations and they proceed in rounds. In each round, every player visits a location of her choice. The game finishes when both players meet at the same location. The goal of the players is to minimize the expected waiting time till a meet-up, also called a *rendezvous*. This formulation permits to easily impose extra constraints on how the players can move from one location to another by using different underlying space topologies. Here, we model such topologies as graphs: in the original telephone problem, the underlying graph is the complete graph on $n$ vertices, which means that in every round, each player can move from her current location to any other location. The most studied variants are when the graph representing the topology is either complete or a path.

Let us point out that, originally, no difference between the two players is assumed here, so that they must use the same strategy: this is called the *symmetric case*. It implies a level of randomness, as otherwise the players may well never meet. The case where the players are allowed to use different strategies, called *asymmetric*, was introduced in full generality in 1995 [2].

As pointed out for instance by Alpern [3], natural questions related to rendezvous games can be raised in a number of different contexts, such as that of migrating animals. There is a rich research literature on rendezvous games and its many variants, *e.g.* with more players [6], different rules of the game (for instance seeking to minimise the second meeting time [12]) or other topologies of the search space (including when the players know where they start from, that is, when they have a common labelling of the graph [4]). We invite an interested reader to the survey of Alpern [3] for a broader and formal introduction.

Coming back to the rendezvous game on the complete graph, the asymmetric case was solved by Anderson and Weber [7], using what is coined the *wait-for-mommy strategy*: one of the players stays for $n$ rounds in one location, while the other player searches through all the $n$ locations in a random order. Then the expected waiting time is equal to $\frac{n+1}{2}$, and it is known that this value is optimum [7]: every pair of strategies for the players yields expected waiting time not lower than $\frac{n+1}{2}$.

Apart from proving the aforementioned lower bound of $\frac{n+1}{2}$ in the asymmetric case, Anderson and Weber [7] also studied the symmetric variant of the problem, where the two players are required to use the same strategy. While always visiting a random location gives an expected waiting time of $n$, Anderson and Weber proposed a more clever symmetric strategy that achieves an asymptotic expected waiting time slightly smaller than $0.829n$, which we explain next.

The Anderson-Weber strategy works as follows. Let $\theta \in [0, 1]$ be a parameter, to be fixed later. On the first step, both players choose a location at random. If they do not meet, then the players divide the rest of the game into groups of $n-1$ consecutive rounds. At the beginning of each group of rounds, each player randomly decides her behavior during these rounds: with probability $\theta$ she will stay at her current location for all $n-1$ rounds, and with probability $1-\theta$ she will visit the $n-1$ locations different from her current one in a random order. Thus, intuitively, the Anderson-Weber strategy tries to break the symmetry by randomly assigning to each player either the role of the baby (who is passive), or the role of the mommy (who is active). However, there is a significant probability that both players get the same role, which results in an expected waiting time significantly higher than $\frac{n+1}{2}$. Indeed, while for different $n$, different values of $\theta$ optimize the expected waiting time, with $n$ tending to infinity one should pick $\theta$ tending to roughly $0.24749$, which results in an asymptotic expected waiting time slightly smaller than $0.829n$.

The Anderson-Weber strategy has been analyzed for small values of $n$. It is known that picking the right $\theta$ yields an optimum strategy for $n = 2$ [7] and for $n = 3$ [11], this latter result being much more difficult to prove. For $n = 4$, as proved by Weber [10], there is a slightly better strategy outside of the framework of Anderson and Weber. However, in general it is conjectured that the Anderson-Weber strategy is asymptotically optimum: there is no strategy for arbitrary $n$ that would yield an asymptotic expected waiting time smaller than (roughly) $0.829n$. However, to the best of our knowledge, no asymptotic lower bound higher than $\frac{n+1}{2}$, which holds even for the asymmetric variant, was known prior to this work.

**Our contribution.** We prove that for every $n \geqslant 2$, in the symmetric rendezvous game on $n$ locations the expected waiting time needs to be significantly larger than $\frac{n+1}{2}$. Precisely, if the players are requested to follow the same strategy, then whatever strategy they choose, the expected waiting time will be at least $\frac{n+1}{2} + \varepsilon n$ for $\varepsilon = \frac{1}{3600}$. See Theorem 1 in Section 2 for a formal statement. While this still leaves a large gap to the best known upper bound of $0.829n$, due to Anderson and Weber [7], this seems to be the first lower bound for arbitrary $n$ that significantly distinguishes the symmetric case from the asymmetric case, where $\frac{n+1}{2}$ is the optimum.

The idea behind our proof can be explained as follows. As in other works, *e.g.* [11], we restrict the game to the first $n$ rounds and prove a lower bound already for this simpler game. We classify deterministic strategies of the players (which we call *tactics*) into those that rather stay at few locations and those that seek through many locations. Formally, tactics of the first kind — the *passive tactics* — visit at most $n/2$ different locations, while tactics of the second kind — the *active tactics* — visit more than $n/2$ different locations.

The intuition drawn from the asymmetric case is that the expected waiting time is minimized when one player plays a passive tactic, while the other plays an active tactic. As now the players need to follow the same strategy (understood as a probability distribution over tactics), with probability at least $\frac{1}{2}$ they choose to use tactics of the same kind (activity level). Then it suffices to prove that when two tactics of the same kind are played against each other, the expected waiting time is significantly larger than $\frac{n+1}{2}$.

To this end, we show that if same-kind tactics are employed, the probability that no rendezvous happens at all is bounded from below by a positive constant. This easily implies a larger-than-$\frac{n+1}{2}$ lower bound on the expected waiting time. To analyze the probability of no rendezvous, we investigate a random variable $X$ that indicates the total number of rendezvous if the game is not stopped when the players meet for the first time. Then $X$ has mean (roughly) equal to 1, so to prove that $X = 0$ with significant probability, we show that $X$ is not well concentrated around its mean. This involves establishing a lower bound on the variance of $X$, which in turn follows from the assumption that the employed tactics are of the same kind.

## 2 The model and the problem

In this section we formalize the considered rendezvous search game and state the main result in precise terms. As in previous works, *e.g.* [11], we make the game finite by stopping it after $n$ rounds. Precisely, if the players did not meet after $n$ rounds, we stop the game and set $n + 1$ as the obtained time till rendezvous. Note that this may only decrease the expected waiting time as compared to allowing the players to play indefinitely.

We are given a set of $n$ locations and two players, $A$ and $B$. Each player has her own, private numbering of locations using numbers from $[n] \coloneqq \{1, \ldots, n\}$. A *tactic* for a player is a function $\tau\colon [n] \to [n]$, where $\tau(i)$ is interpreted as the index of the location that the player intends to visit at round $i$, in her own numbering. A *strategy* for a player is a probability distribution $\sigma$ over the tactics of this player. Note that the set of possible tactics is finite, hence we may use the discrete $\sigma$-field where every subset of tactics is measurable. The sets of tactics and strategies for the game on $n$ locations are denoted by $\Theta_n$ and $\Sigma_n$, respectively.

For two given strategies $\sigma_A$ and $\sigma_B$, the game is played as follows:

- Players $A$ and $B$ respectively draw their tactics $\tau_A$ and $\tau_B$ from the strategies $\sigma_A$ and $\sigma_B$ at random.
- A permutation $\pi\colon [n] \to [n]$ that matches the numberings of locations of $A$ and $B$ is drawn uniformly at random. This permutation $\pi$ will be called the *binding*.
- The waiting time till rendezvous is indicated by the random variable

$$T\langle \sigma_A, \sigma_B \rangle \coloneqq \min \left( \{\, i \colon \pi(\tau_A(i)) = \tau_B(i) \,\} \cup \{n+1\} \right).$$

Then the question of minimizing the waiting time till rendezvous for symmetric players corresponds to the problem of minimizing the expected value of $T\langle \sigma_A, \sigma_B \rangle$ over the strategies $\sigma_A$ and $\sigma_B$, subject to $\sigma_A = \sigma_B$.

Note that in this model, we assume that every player fixes her tactic at the beginning of the game and then follows this tactic. Observe that this does not restrict the players in any way, as throughout the play they receive no information that could influence the choice of the next moves. Indeed, when entering a location, the player only receives the information that the other player is not there, or otherwise the game immediately finishes. Hence, there is no point in considering adaptivity in strategies.

The main result of this work can be now phrased as follows.

**Theorem 1.** *There exists $\varepsilon > 0$ such that for every $n \geqslant 2$ and every strategy $\sigma \in \Sigma_n$, we have*

$$\mathbb{E}T\langle \sigma, \sigma \rangle \geqslant \frac{n+1}{2} + \varepsilon n.$$

A key step in the proof of Theorem 1 is Lemma 4, and two different arguments are provided for a central part of its proof: an elementary one using the Paley-Zygmund inequality [9], allowing to establish Theorem 1 with $\varepsilon = 2^{-36}$ (Sections 3.5 to 3.8), and a somewhat lighter argument relying on an extremal result of Ahlswede and Katona [1], allowing $\varepsilon$ to be as large as $\frac{1}{3600}$ at the expense of requiring that $n$ be at least 16 (Sections 3.3 and 3.4). Both arguments share a common ground (notably Section 3.2), but provide different perspective to the problem, making each of them interesting in its own right.

In the proof of Theorem 1 we will use the lower bound for the waiting time for asymmetric strategies of Anderson and Weber [7]. Note that the proof of this result also holds for the game stopped after round $n$.

**Theorem 2 (Anderson and Weber [7]).** *For every $n \in \mathbb{N}$ and pair of strategies $\sigma_A, \sigma_B \in \Sigma_n$, we have*

$$\mathbb{E}T\langle \sigma_A, \sigma_B \rangle \geqslant \frac{n+1}{2}.$$

As mentioned in Section 1, the lower bound provided by Theorem 2 is tight, as witnessed by the wait-for-mommy pair of strategies: $\sigma_A$ is the baby strategy that deterministically picks a tactic that maps all integers $i \in [n]$ to 1, while $\sigma_B$ is the mommy strategy that deterministically picks the identity function as the tactic.

## 3 Proof of Theorem 1

For the rest of the proof we fix the number of locations $n$ to be at least 2. For brevity we write $\Theta := \Theta_n$ and $\Sigma := \Sigma_n$.

### 3.1 Passive and active tactics

Let us start by taking a closer look at the mapping $\sigma_A, \sigma_B \mapsto \mathbb{E}T\langle\sigma_A, \sigma_B\rangle$, where $\sigma_A, \sigma_B \in \Sigma$. We shall try to understand this mapping from the point of view of linear algebra.

For tactics $\tau_A, \tau_B \in \Theta$, let

$$W(\tau_A, \tau_B) := \mathbb{E}\left[\min\left(\{\, i\colon \pi(\tau_A(i)) = \tau_B(i)\,\} \cup \{n+1\}\right)\right].$$

Note that here, the tactics $\tau_A, \tau_B$ are fixed and the expectation is taken only over the choice of the binding $\pi$. Let us define a bilinear operator

$$\Phi\colon \mathbb{R}^\Theta \times \mathbb{R}^\Theta \to \mathbb{R} \qquad \text{as} \qquad \Phi\langle x, y\rangle := \sum_{\tau_A, \tau_B \in \Theta} W(\tau_A, \tau_B) \cdot x_{\tau_A} y_{\tau_B},$$

where $x, y \in \mathbb{R}^\Theta$ are vectors indexed by the elements of $\Theta$. Then

$$\mathbb{E}T\langle\sigma_A, \sigma_B\rangle = \Phi\langle a, b\rangle,$$

where $a, b \in \mathbb{R}^\Theta$ are such that $a_\tau$ is the probability of drawing $\tau$ in the distribution $\sigma_A$, and similarly for $b_\tau$.

The main idea is as follows. As witnessed by the tightness example for Theorem 2, the operator $\Phi\langle\cdot,\cdot\rangle$ achieves its minimum possible value when the strategies $\sigma_A$ and $\sigma_B$ are sort of "orthogonal". Namely, one strategy should focus on baby-like tactics — being in a few locations and waiting for the other player — while the other strategy should focus on mommy-like tactics — seeking through a large number of location in search of the other player. Playing a baby-like tactic against a mommy-like tactic yields low waiting time, while the intuition is that playing two baby-like tactics against each other, or two mommy-like tactics against each other, should result in waiting time significantly larger than $\frac{n+1}{2}$. When the two players are forced to use the same strategy, there is a significant probability — at least $\frac{1}{2}$ — that they end up playing tactics of the same kind. This increases the expected waiting time significantly above $\frac{n+1}{2}$.

We now formalize this intuition, calling baby-like tactics *passive* and mommy-like tactics *active*.

**Definition 3.** A tactic $\tau \in \Theta$ is called *passive* if $|\tau([n])| \leqslant n/2$ and *active* otherwise. The sets of passive and active tactics are denoted by $\Theta^{\mathbf{P}}$ and $\Theta^{\mathbf{A}}$, respectively.

In the next sections we will focus on the following lemma.

**Lemma 4.** *There exists $\delta > 0$ such that for all $\tau_A, \tau_B \in \Theta$ satisfying either $\tau_A, \tau_B \in \Theta^{\mathbf{A}}$ or $\tau_A, \tau_B \in \Theta^{\mathbf{P}}$, we have*

$$W(\tau_A, \tau_B) \geqslant \frac{n+1}{2} + \delta n.$$

Before we proceed to prove Lemma 4, let us see how Theorem 1 follows from it.

Proof (of Theorem 1 assuming Lemma 4). We first note that from Theorem 2 applied to two deterministic strategies we may infer that

$$W(\tau_A, \tau_B) \geqslant \frac{n+1}{2} \qquad \text{for all } \tau_A, \tau_B \in \Theta. \tag{1}$$

Let $a \in \mathbb{R}^{\Theta}$ be such that $a_\tau$ is the probability that tactic $\tau$ is drawn by the strategy $\sigma$. Write

$$a = a^{\mathbf{P}} + a^{\mathbf{A}},$$

where the supports of $a^{\mathbf{P}}$ and $a^{\mathbf{A}}$ are passive and active tactics, respectively. As $W(\cdot,\cdot)$ is a symmetric function, we have

$$\mathbb{E}T\langle\sigma,\sigma\rangle = \Phi\langle a, a\rangle = \Phi\langle a^{\mathbf{P}}, a^{\mathbf{P}}\rangle + \Phi\langle a^{\mathbf{A}}, a^{\mathbf{A}}\rangle + 2\cdot\Phi\langle a^{\mathbf{P}}, a^{\mathbf{A}}\rangle. \tag{2}$$

Let $p \coloneqq \sum_{\tau\in\Theta^{\mathbf{P}}} a_\tau$ be the probability that $\sigma$ yields a passive tactic. Then, by Lemma 4, we have

$$\begin{aligned}\Phi\langle a^{\mathbf{P}}, a^{\mathbf{P}}\rangle &= \sum_{\tau_A,\tau_B\in\Theta^{\mathbf{P}}} W(\tau_A,\tau_B)\cdot a_{\tau_A}a_{\tau_B} \\ &\geqslant \left(\frac{n+1}{2}+\delta n\right)\cdot \sum_{\tau_A,\tau_B\in\Theta^{\mathbf{P}}} a_{\tau_A}a_{\tau_B} = p^2\cdot\left(\frac{n+1}{2}+\delta n\right). \end{aligned}\tag{3}$$

Using Lemma 4 again, we analogously infer that

$$\Phi\langle a^{\mathbf{A}}, a^{\mathbf{A}}\rangle \geqslant (1-p)^2\cdot\left(\frac{n+1}{2}+\delta n\right). \tag{4}$$

A similar computation using (1) yields that

$$\Phi\langle a^{\mathbf{P}}, a^{\mathbf{A}}\rangle \geqslant p(1-p)\cdot\frac{n+1}{2}. \tag{5}$$

Finally, letting $\varepsilon \coloneqq \delta/2$ we can combine (2), (3), (4), and (5) to conclude that

$$\begin{aligned}\mathbb{E}T\langle\sigma,\sigma\rangle &\geqslant (p^2+(1-p)^2+2p(1-p))\cdot\frac{n+1}{2} + p^2\cdot\delta\cdot n + (1-p)^2\cdot\delta\cdot n \\ &= \frac{n+1}{2} + 2\varepsilon\cdot(p^2+(1-p)^2)\cdot n \geqslant \frac{n+1}{2}+\varepsilon n,\end{aligned}$$

where the last inequality follows from the convexity of the function $x \mapsto x^2$. $\square$

It thus remains to prove Lemma 4.

### 3.2 High probability of no rendezvous gives high expected waiting time

We now start analyzing the game when played between a fixed pair of tactics, with the goal of establishing lower bounds for the expected waiting time till a rendezvous. The intuition is that this waiting time is significantly higher than $\frac{n+1}{2}$ under the following condition: the probability that during the $n$ rounds of the game there is no rendezvous at all is bounded from below by some positive constant. This is made formal in the following lemma.

**Lemma 5.** *Suppose $\tau_A, \tau_B \in \Theta$ are such that*

$$\mathbb{P}\left(\pi(\tau_A(i)) \neq \tau_B(i) \text{ for all } i\in[n]\right) \geqslant \beta$$

*for some constant $\beta > 0$. Then*

$$W(\tau_A,\tau_B) \geqslant \frac{n+1}{2} + \frac{\beta^2}{2}\cdot n.$$

Proof. Let $Z$ be the random variable defined as the waiting time till the first rendezvous, that is,

$$Z \coloneqq \min\left(\{\, i\colon \pi(\tau_A(i)) = \tau_B(i) \,\} \cup \{n+1\}\right).$$

Note that here $\tau_A$ and $\tau_B$ are fixed, so $Z$ depends only on the random choice of the binding $\pi$; formally, $Z$ is $\pi$-measurable. Then

$$W(\tau_A, \tau_B) = \mathbb{E}Z.$$

Observe that $Z$ is a random variable with values in $\{1, 2, \ldots, n+1\}$, hence we have

$$\mathbb{E}Z = \sum_{k=0}^{n} \mathbb{P}(Z > k).$$

Note that we have $Z > k$ if and only if during the first $k$ rounds the players did not meet. Clearly, during every fixed round, the players meet with probability $\frac{1}{n}$. Hence, by the union bound, the probability that they do not meet during the first $k$ rounds is at least $1 - \frac{k}{n}$. On the other hand, by the assumption of the lemma, this probability is also at least $\beta$. We conclude that

$$\mathbb{P}(Z > k) \geqslant \max\left(1 - \frac{k}{n}, \beta\right) \qquad \text{for all } k \in [n].$$

By combining the above observations it follows that

$$\begin{aligned}
W(\tau_A, \tau_B) &= \sum_{k=0}^{n} \mathbb{P}(Z > k) \geqslant \sum_{k=0}^{n} \max\left(1 - \frac{k}{n}, \beta\right) \\
&= \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) + \sum_{k=0}^{n} \max\left(0, \beta - \left(1 - \frac{k}{n}\right)\right) \\
&= \frac{n+1}{2} + \sum_{k=\lceil(1-\beta)n\rceil}^{n} \left(\beta - \left(1 - \frac{k}{n}\right)\right) \\
&= \frac{n+1}{2} + (\beta - 1)\cdot(n - \lceil(1-\beta)n\rceil + 1) + \frac{1}{n}\cdot\frac{n + \lceil(1-\beta)n\rceil}{2}\cdot(n - \lceil(1-\beta)n\rceil + 1) \\
&= \frac{n+1}{2} + (n - \lceil(1-\beta)n\rceil + 1)\cdot\left(\beta + \frac{\lceil(1-\beta)n\rceil - n}{2n}\right) \\
&\geqslant \frac{n+1}{2} + \beta n \cdot \frac{\beta}{2} = \frac{n+1}{2} + \frac{\beta^2}{2}\cdot n.
\end{aligned}$$

This concludes the proof. $\square$

Thus, by Lemma 5, for the proof of Lemma 4 it suffices to show that the probability that no rendezvous occurs throughout the $n$ rounds of the game is bounded away from zero. We do so in two different ways: we start in Section 3.3 with an argument relying on a result by Ahlswede and Katona [1]. It allows $\delta$ to be as large as $\frac{1}{1800}$, however it is valid only when $n$ is at least 16. So it leaves some cases open, but due to its conceptual appeal it seems relevant for us to include here. After this, we provide a more elementary analysis starting in Section 3.5: it is slightly heavier and $\delta$ is reduced to $2^{-35}$, but it works for any value of $n$ larger than 1.

### 3.3 Same-kind tactics give non-trivial probability of no rendezvous

Our goal is to prove the following statement.

**Lemma 6.** *Assume that $n \geqslant 16$ and let $\tau_A$ and $\tau_B$ be tactics such that $\tau_A, \tau_B \in \Theta^{\mathbf{A}}$ or $\tau_A, \tau_B \in \Theta^{\mathbf{P}}$. Then*

$$\mathbb{P}\left(\pi(\tau_A(i)) \neq \tau_B(i) \textit{ for all } i \in [n]\right) \geqslant \frac{1}{30}.$$

We will need the following extremal result about the number of edges in bipartite graphs, established in 1978 by Ahlswede and Katona [1, Theorem 1]. Here a *simple* graph is a graph with no loops or multiple edges with same endpoints.

**Theorem 7 (Ahlswede and Katona [1]).** *Let $n$ and $k$ be positive integers, and write $n = q \cdot k + r$ where $r \in \{0, \ldots, k-1\}$. If $G$ is a bipartite simple graph with $n$ edges and at most $k$ vertices in each part of the bipartition, then the sum of the squares of degrees of the vertices of $G$ is at most $q(n + r + k^2) + r(r + 1)$.*

The original statement actually deals with bipartite simple graphs with not only a fixed number of edges but also a fixed number of vertices in each part, for which it provides a construction that maximises the sum of square of degrees of the vertices over all such graphs. The fact that we can write "at most" for the number of vertices in each part follows from adding degree-0 vertices to this optimal construction. Note also that the statement of Theorem 7 becomes false for multi-graphs.

Proof (of Lemma 6). Fix a pair of tactics $\tau_A, \tau_B \in \Theta$, both of the same kind. To be able to use Theorem 7, we rephrase our goal in graph-theoretic terms. We define a bipartite graph $G$ with bipartition $(L, R)$ where each part contains $n$ vertices numbered from 1 to $n$: we call them the *left vertices* and the *right vertices*, respectively. For each $i \in [n]$, an edge is added between the left vertex $\tau_A(i)$ and the right vertex $\tau_B(i)$. Note that if there are multiple indices $i$ giving rise to the same edge (i.e. for which the pair $(\tau_A(i), \tau_B(i))$ is the same), then the edge is added to $G$ only once; thus, $G$ has *at most* $n$ edges. Further, consider sampling a random injective colouring by colours from 1 to $n$ for each of the parts $L$ and $R$, independently. In other words, for each part we choose randomly and independently a permutation of $[n]$. An edge of $G$ is *monochromatic* if its left vertex and its right vertex have the same colour, and it readily follows from the construction that

$$\mathbb{P}\left(\pi(\tau_A(i)) \neq \tau_B(i) \text{ for all } i \in [n]\right) = \mathbb{P}\left(G \text{ has no monochromatic edge}\right). \tag{6}$$

We want to show that we can restrict to simple bipartite graphs with bipartition $(L, R)$ and *exactly* $n$ edges. First, recalling that $G$ has *at most* $n$ edges, if $\widehat{G}$ is any supergraph of $G$ with *exactly* $n$ edges that is still bipartite with bipartition $(L, R)$, then

$$\mathbb{P}\left(G \text{ has no monochromatic edge}\right) \leqslant \mathbb{P}\left(\widehat{G} \text{ has no monochromatic edge}\right). \tag{7}$$

However, in our context there is an additional constraint on $\widehat{G}$, namely that the properties of $G$ following from the assumption that $\tau_A$ and $\tau_B$ are of the same kind should be also satisfied in $\widehat{G}$. Precisely, we mean the following assertions:

- If $\tau_A$ and $\tau_B$ are active, then each of $L$ and $R$ contains more than $n/2$ vertices with positive degrees.
- If $\tau_A$ and $\tau_B$ are passive, then each of $L$ and $R$ contains at most $n/2$ vertices with positive degrees.

These assertions hold in $G$ and we would like to construct a supergraph $\widehat{G}$ of $G$ with exactly $n$ edges so that they are also satisfied in $\widehat{G}$. This is obvious in the case when $\tau_A$ and $\tau_B$ are active, because adding edges can only increase the number of vertices with positive degrees in $L$ and $R$. The case when $\tau_A$ and $\tau_B$ are passive is more problematic, as we need to make sure that the edges added in $\widehat{G}$ do not create too many vertices with positive degrees in $L$ or in $R$. For this, pick any $L' \subseteq L$ and $R' \subseteq R$ such that $|L'| = |R'| = \lfloor n/2 \rfloor$ and $L' \cup R'$ contains all vertices that have positive degrees in $G$. Then construct $\widehat{G}$ by adding any edges with one endpoint in $L'$ and the other in $R'$ so that $\widehat{G}$ remains simple and has exactly $n$ edges. This is always possible, because $\lfloor n/2 \rfloor^2 \geqslant n$ for all $n \geqslant 6$, and we assume that $n \geqslant 16$.

We now focus on giving an upper bound on the probability that no edge of $\widehat{G}$ is monochromatic. Let us arbitrarily enumerate the edges of $\widehat{G}$ as $e_1, \ldots, e_n$. For each $i \in [n]$, let $A_i$ be the event that the edge $e_i$ is monochromatic. In particular, $\mathbb{P}(A_i) = \frac{1}{n}$ for each $i \in [n]$. By definition,

$$\mathbb{P}\left(\widehat{G} \text{ has no monochromatic edge}\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right). \tag{8}$$

By one of Bonferroni's inequalities (see, e.g., the book by Comtet [8, p. 193–194]),

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leqslant \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1 \leqslant i < j \leqslant n} \mathbb{P}(A_i \cap A_j) + \sum_{1 \leqslant i < j < k \leqslant n} \mathbb{P}(A_i \cap A_j \cap A_k). \tag{9}$$

Clearly, we have $\sum_{i=1}^{n} \mathbb{P}(A_i) = 1$. Further, for all $1 \leqslant i < j < k \leqslant n$ we have

$$\mathbb{P}(A_i \cap A_j \cap A_k) = \begin{cases} \frac{1}{n(n-1)(n-2)} & \text{if } e_i, e_j, e_k \text{ are pairwise disjoint;} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Here, edges are *disjoint* if they do not share any endpoint. Therefore,

$$\sum_{1 \leqslant i < j < k \leqslant n} \mathbb{P}(A_i \cap A_j \cap A_k) \leqslant \binom{n}{3} \cdot \frac{1}{n(n-1)(n-2)} = \frac{1}{6}. \tag{11}$$

As a result, it suffices to show that if $\tau_A, \tau_B \in \Theta^{\mathbf{A}}$ or $\tau_A, \tau_B \in \Theta^{\mathbf{P}}$, then

$$\sum_{1 \leqslant i < j \leqslant n} \mathbb{P}(A_i \cap A_j) \geqslant \frac{1}{5}.$$

This would indeed yield by (9) and (11) that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leqslant 1 - \frac{1}{5} + \frac{1}{6} = \frac{29}{30},$$

which by (6), (7), and (8) would finish the proof.

Similarly as in (10), for all $1 \leqslant i < j \leqslant n$ we have

$$\mathbb{P}(A_i \cap A_j) = \begin{cases} \frac{1}{n(n-1)} & \text{if } e_i, e_j \text{ are disjoint;} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, letting $c_{i,j}$ be 1 if $e_i$ and $e_j$ are disjoint and 0 otherwise,

$$\sum_{1\leqslant i<j\leqslant n} \mathbb{P}(A_i \cap A_j) = \frac{1}{n(n-1)} \cdot \sum_{1\leqslant i<j\leqslant n} c_{i,j}. \tag{12}$$

To evaluate the sum on the right side of (12), fix $i \in [n]$ and let $a_i$ and $\bar{a}_i$ be the degrees of the left and right vertices of the edge $e_i$, respectively. The number $n_i$ of edges disjoint from $e_i$ is then

$$n_i = (n-1) - (a_i - 1) - (\bar{a}_i - 1) = n + 1 - a_i - \bar{a}_i. \tag{13}$$

For each $t \in [n]$, let $d_t$ be the degree of the left vertex $t$ and $\bar{d}_t$ be the degree of the right vertex $t$. Observe that $\sum_{i=1}^n a_i = \sum_{i=1}^n d_i^2$ and $\sum_{i=1}^n \bar{a}_i = \sum_{i=1}^n \bar{d}_i^2$. Summing (13) over all values $i \in [n]$ thus yields that

$$2 \sum_{1\leqslant i<j\leqslant n} c_{i,j} = n(n+1) - \sum_{i=1}^n d_i^2 - \sum_{i=1}^n \bar{d}_i^2. \tag{14}$$

We now consider two cases depending on the common kind of $\tau_A$ and $\tau_B$. Suppose first that both $\tau_A$ and $\tau_B$ are passive tactics. In this case, at most $\lfloor n/2 \rfloor$ vertices in each part of $\widehat{G}$ have positive degree, and consequently we infer from Theorem 7 applied with $k = \lfloor n/2 \rfloor$ that $\sum_{i=1}^n (d_i^2 + \bar{d}_i^2)$ is at most $n^2/2 + 2n$ when $n$ is even and at most $n^2/2 + n + \frac{9}{2}$ when $n$ is odd. Since $n \geqslant 5$, we deduce that $n^2/2 + 2n$ is always an upper bound on the sum of the squares of degrees of the vertices in $\widehat{G}$. As a result, we deduce from (12) and (14) that

$$\begin{aligned} \sum_{1\leqslant i<j\leqslant n} \mathbb{P}(A_i \cap A_j) &\geqslant \frac{1}{2n(n-1)} \cdot \left( n(n+1) - \frac{n^2}{2} - 2n \right) \\ &= \frac{n-2}{4(n-1)} \geqslant \frac{1}{5} \qquad\qquad \text{as } n \geqslant 6. \end{aligned}$$

This concludes the proof in this case.

Assume now that both $\tau_A$ and $\tau_B$ are active tactics. In this case, at least $\lceil n/2 \rceil$ vertices in each part of $\widehat{G}$ have a positive degree. Consequently, the sum of the squares of the degrees of the vertices in each of the parts cannot exceed

$$(\lceil n/2 \rceil - 1) \cdot 1^2 + 1 \cdot (\lfloor n/2 \rfloor + 1)^2,$$

which corresponds to the case where one vertex has degree $\lfloor n/2 \rfloor + 1$ and exactly $\lceil n/2 \rceil - 1$ vertices have degree 1. Considering the cases when $n$ is even and odd, we conclude that this sum is upper bounded by $n^2/4 + \max(3n/2, n - 1/4) = n^2/4 + 3n/2$. Therefore, we deduce from (12) and (14) that

$$\begin{aligned} \sum_{1\leqslant i<j\leqslant n} \mathbb{P}(A_i \cap A_j) &\geqslant \frac{1}{2n(n-1)} \cdot \left( n(n+1) - \frac{n^2}{2} - 3n \right) \\ &= \frac{n-4}{4(n-1)} \geqslant \frac{1}{5} \qquad\qquad \text{as } n \geqslant 16. \end{aligned}$$

This concludes the proof. $\square$

### 3.4 Wrapping up the proof, I

With all the tools prepared, we are now in a position to prove Lemma 4.

Proof (of Lemma 4 when $n \geqslant 16$). By Lemma 6,

$$\mathbb{P}\left(\pi(\tau_A(i)) \neq \tau_B(i) \text{ for all } i \in [n]\right) \geqslant \frac{1}{30}.$$

Consequently, Lemma 5 ensures that

$$W(\tau_A, \tau_B) \geqslant \frac{n+1}{2} + \frac{1}{1800} \cdot n.$$

Hence, Lemma 4 holds for $\delta = \frac{1}{1800}$. □

Recalling that the proof of Theorem 1 sets $\varepsilon$ to be $\delta/2$, we conclude that Theorem 1 holds for $\varepsilon = \frac{1}{3600}$ if $n \geqslant 16$.

We now proceed with the more elementary proof of Lemma 4, which covers all values of $n$ and yields $\varepsilon = 2^{-36}$ in Theorem 1. Recall that, after Lemma 5, our goal is to show that the probability that no rendezvous occurs throughout the $n$ rounds of the game is bounded away from zero.

### 3.5 High variance gives high probability of no rendezvous

Fix a pair of tactics $\tau_A, \tau_B \in \Theta$. Let

$$F := \{(\tau_A(i), \tau_B(i)) \colon i \in [n]\} \subseteq [n] \times [n].$$

Set $m := |F|$ and note that $m \leqslant n$. Similarly, for the random binding $\pi$, let

$$E(\pi) := \{(i, \pi(i)) \colon i \in [n]\} \subseteq [n] \times [n].$$

For $f \in F$, let $X_f$ be the indicator random variable taking value 1 if $f \in E(\pi)$ and 0 otherwise. Further, let

$$X := |F \cap E(\pi)| = \sum_{f \in F} X_f.$$

Note that here $\tau_A, \tau_B$ are considered fixed and $\pi$ is drawn at random, hence $(X_f)_{f \in F}$ and therefore $X$ depend only on the choice of the random binding $\pi$; formally, these variables are $\pi$-measurable. Observe that the probability that no rendezvous occurs can be understood in terms of the random variable $X$ as follows:

$$\mathbb{P}\left(\pi(\tau_A(i)) \neq \tau_B(i) \text{ for all } i \in [n]\right) = \mathbb{P}(X = 0). \tag{15}$$

From now on, we adopt the above notation whenever the pair of tactics $\tau_A, \tau_B$ is clear from the context.

The next lemma is the key conceptual step in the proof. We show that in order to give a lower bound on the probability that no rendezvous occurs, it suffices to give a lower bound on the variance of $X$.

**Lemma 8.** *Suppose $\tau_A, \tau_B \in \Theta$ are such that*

$$m \geqslant \left(1 - \sqrt{\alpha/2}\right) \cdot n \qquad \text{and} \qquad \mathrm{Var} X \geqslant \alpha,$$

*for some constant $\alpha > 0$. Then*

$$\mathbb{P}(X = 0) \geqslant \frac{\alpha^2}{128}.$$

The proof of Lemma 8 spans the rest of this section. The intuition is that high variance of $X$ means that $X$ is not well concentrated around its mean, which in turns implies that the probability of it being below the mean — equivalently equal to $0$ — is high. Hence, we need to understand the mean of $X$ as well as estimate its higher moments.

Observe that if $f = (i, j) \in F$, then the probability that $\pi(i) = j$ is equal to $\frac{1}{n}$. Hence, $X_f$ takes value $1$ with probability $\frac{1}{n}$ and $0$ with probability $1 - \frac{1}{n}$. Consequently, we have

$$\mathbb{E}X_f = \frac{1}{n} \qquad \text{for each } f \in F.$$

By linearity of expectation,

$$\mathbb{E}X = \frac{m}{n} \leqslant 1.$$

In the sequel we will also need an upper bound on the fourth central moment of $X$, that is, on $\mathbb{E}|X - \mathbb{E}X|^4$. To this end, we first establish, in the next two assertions, an upper bound on the fourth moment of $X$, that is, on $\mathbb{E}X^4$.

**Assertion 1.** *For pairwise different pairs $e, f, g, h \in F$, we have*

$$\begin{aligned} \mathbb{E}X_e X_f &\leqslant \frac{1}{n(n-1)} \\ \mathbb{E}X_e X_f X_g &\leqslant \frac{1}{n(n-1)(n-2)} \\ \mathbb{E}X_e X_f X_g X_h &\leqslant \frac{1}{n(n-1)(n-2)(n-3)}. \end{aligned}$$

Proof. Let us focus on the first inequality. Write $e = (i, j)$ and $f = (i', j')$. Observe that if $i = i'$ or $j = j'$, then $X_e$ and $X_f$ cannot simultaneously be equal to $1$ since $e \neq f$, and hence $X_e X_f = 0$ surely. Otherwise, the probability that for $\pi$ chosen uniformly at random we have $\pi(i) = j$ and $\pi(i') = j'$ is $\frac{1}{n(n-1)}$. Consequently $\mathbb{P}(X_e X_f = 1) = \frac{1}{n(n-1)}$. This implies the first inequality. The proofs of the remaining two inequalities are analogous. □

**Assertion 2.** *It holds that*

$$\mathbb{E}X^4 \leqslant 15.$$

Proof. For each $e \in F$, since $X_e \in \{0, 1\}$ we have $X_e = X_e^2 = X_e^3 = X_e^4$. By Assertion 1 and the fact that $m \leqslant n$, we have

$$\begin{aligned}
\mathbb{E}X^4 &= \mathbb{E}\left(\sum_{e\in F} X_e\right)^4 \\
&= \sum_{e\in F} \mathbb{E}X_e^4 + \sum_{\{e,f\}\subseteq F} \mathbb{E}(4X_e^3X_f + 6X_e^2X_f^2 + 4X_eX_f^3) \\
&\quad + \sum_{\{e,f,g\}\subseteq F} \mathbb{E}(12X_e^2X_fX_g + 12X_eX_f^2X_g + 12X_eX_fX_g^2) \\
&\quad + \sum_{\{e,f,g,h\}\subseteq F} \mathbb{E}(24X_eX_fX_gX_h) \\
&= \sum_{e\in F} \mathbb{E}X_e + 14 \sum_{\{e,f\}\subseteq F} \mathbb{E}X_eX_f \\
&\quad + 36 \sum_{\{e,f,g\}\subseteq F} \mathbb{E}X_eX_fX_g + 24 \sum_{\{e,f,g,h\}\subseteq F} \mathbb{E}X_eX_fX_gX_h \\
&\leqslant \frac{m}{n} + 14\cdot\frac{\binom{m}{2}}{n(n-1)} + 36\cdot\frac{\binom{m}{3}}{n(n-1)(n-2)} + 24\cdot\frac{\binom{m}{4}}{n(n-1)(n-2)(n-3)} \\
&\leqslant 1+7+6+1 = 15.
\end{aligned}$$

This concludes the proof. $\square$

We will also use the following well-known anti-concentration inequality.

**Theorem 9 (Paley-Zygmund inequality, [9]).** *Let $Z$ be a non-negative random variable with finite variance and let $\lambda \in [0,1]$. Then*

$$\mathbb{P}(Z \geqslant \lambda\mathbb{E}Z) \geqslant (1-\lambda)^2 \cdot \frac{(\mathbb{E}Z)^2}{\mathbb{E}Z^2}.$$

With all the tools prepared, we proceed with the proof of Lemma 8. We use Theorem 9 with $\lambda = \frac{1}{2}$ for the random variable

$$Z \coloneqq |X - \mathbb{E}X|^2.$$

By Assertion 2 and the fact that $\mathbb{E}X \leqslant 1$, we have

$$\mathbb{E}Z^2 = \mathbb{E}|X-\mathbb{E}X|^4 \leqslant 1 + \mathbb{E}X^4 \leqslant 16.$$

As $\mathbb{E}Z = \mathrm{Var}X \geqslant \alpha$, from Theorem 9 we infer that

$$\mathbb{P}\left(Z \geqslant \alpha/2\right) \geqslant \mathbb{P}\left(Z \geqslant \mathbb{E}Z/2\right) \geqslant \frac{1}{4}\cdot\frac{(\mathbb{E}Z)^2}{16} \geqslant \frac{1}{4}\cdot\frac{\alpha^2}{16} = \frac{\alpha^2}{64}. \tag{16}$$

Observe now that the assumption that $m > \left(1-\sqrt{\alpha/2}\right)\cdot n$ implies that

$$1 - \mathbb{E}X = 1 - \frac{m}{n} < \sqrt{\alpha/2}.$$

This, in turns, implies that the event

$$\left\{|X-\mathbb{E}X|^2 \geqslant \alpha/2\right\}$$

is disjoint with the event $\{X = 1\}$. By combining this with (16), we conclude that

$$\mathbb{P}(X \neq 1) \geqslant \mathbb{P}\left(|X - \mathbb{E}X|^2 \geqslant \alpha/2\right) = \mathbb{P}\left(Z \geqslant \alpha/2\right) \geqslant \frac{\alpha^2}{64}.$$

Since $X$ is a non-negative integer-valued random variable with mean not larger than 1, we have

$$\mathbb{P}(X \neq 1) = \mathbb{P}(X = 0) + \mathbb{P}(X \geqslant 2) \qquad \text{and} \qquad \mathbb{P}(X = 0) \geqslant \mathbb{P}(X \geqslant 2).$$

By combining the two inequalities above we conclude that

$$\mathbb{P}(X = 0) \geqslant \frac{1}{2} \cdot \mathbb{P}(X \neq 1) \geqslant \frac{\alpha^2}{128}.$$

This concludes the proof of Lemma 8.

### 3.6 Many disjoint pairs give high variance

Two pairs $(i, j)$ and $(i', j')$, each in $[n] \times [n]$, are *disjoint* if $i \neq i'$ and $j \neq j'$. We now prove that to ensure that for a pair of tactics $\tau_A, \tau_B$, the variance of $X$ is high, it suffices to show that among pairs in $F$, there is a quadratic number of pairs of pairs that are disjoint.

**Lemma 10.** *Suppose $\tau_A, \tau_B \in \Theta$ are such that there are at least $\alpha\binom{n}{2}$ disjoint pairs in $F$, for some positive constant $\alpha$. Then* $\mathrm{Var}X \geqslant \alpha$*.*

Proof. As in the proof of Assertion 1, we observe that for every pair of different elements $e, f \in F$, we have

$$\mathbb{E}X_eX_f = \begin{cases} \frac{1}{n(n-1)} & \text{if } e \text{ and } f \text{ are disjoint;} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for all different $e, f \in F$ we have

$$\begin{aligned} \mathrm{Var}X_e &= \mathbb{E}X_e^2 - (\mathbb{E}X_e)^2 = \frac{n-1}{n^2}, \quad \text{and} \\ \mathrm{Cov}(X_e, X_f) &= \mathbb{E}X_eX_f - \mathbb{E}X_e\mathbb{E}X_f = [e \cap f = \varnothing] \cdot \frac{1}{n(n-1)} - \frac{1}{n^2}, \end{aligned}$$

where the expression $[e \cap f = \varnothing]$ takes value 1 if $e$ and $f$ are disjoint, and 0 otherwise. Consequently,

$$\begin{aligned} \mathrm{Var}X &= \sum_{e \in F} \mathrm{Var}X_e + 2 \cdot \sum_{\{e,f\} \subseteq F} \mathrm{Cov}(X_e, X_f) \\ &\geqslant m \cdot \frac{n-1}{n^2} - 2 \cdot \binom{m}{2} \cdot \frac{1}{n^2} + 2 \cdot \alpha \binom{n}{2} \cdot \frac{1}{n(n-1)} \\ &= \frac{m(n-1) - m(m-1)}{n^2} + \alpha, \end{aligned}$$

which is at least $\alpha$ because $m \leqslant n$. This concludes the proof. $\square$

### 3.7 Finding many disjoint pairs

Finally, we prove that if $\tau_A$ and $\tau_B$ are two tactics of the same kind, then the set of pairs $F$ defined for $\tau_A$ and $\tau_B$ contains many pairs of disjoint pairs. For this, it will be convenient to interpret $F$ as the edge set of a bipartite graph, with each side of the bipartition consisting of a copy of the set $[n]$. In this view, a pair of disjoint pairs corresponds to a pair of disjoint edges: two edges in a graph being *disjoint* if all the four endpoints of these edges are pairwise different.

We first prove the following graph-theoretic lemma. The degree $\deg(u)$ of a vertex $u$ in a graph $G$ is the number of edges of $G$ incident to $u$.

**Lemma 11.** *Let $G = (A, B, E)$ be a bipartite graph such that $A$ and $B$ — the sides of the bipartition — have size $n$ each, $\frac{11}{12}n \leqslant |E| \leqslant n$, and the degree of each vertex in $G$ is at most $\frac{2}{3}n$. Then there are two disjoint subsets of edges $E_1, E_2 \subseteq E$, each of size at least $n/8$, such that every edge from $E_1$ is disjoint with every edge in $E_2$.*

Proof. For $X \subseteq A \cup B$, we let $\deg(X) \coloneqq \sum_{u \in X} \deg(u)$.

Let $a_1, \ldots, a_n$ be the vertices of $A$ in non-increasing order with respect to their degrees. Let $t \in \{0, 1, \ldots, n\}$ be the largest index such that $A_1 \coloneqq \{a_1, \ldots, a_t\}$ satisfies $\deg(A_1) \leqslant \frac{2}{3}n$. Since the degree of every vertex is at most $\frac{2}{3}n$ and $|E| > \frac{2}{3}n$, we know that neither $A_1$ nor $A_2 \coloneqq A \setminus A_1$ is empty. In other words, $t \in \{1, \ldots, n-1\}$. Further, since $\deg(A_1) \leqslant \frac{2}{3}n$, $\deg(A_1 \cup \{a_{t+1}\}) > \frac{2}{3}n$, and $\deg(a_{t+1}) \leqslant \deg(v)$ for every $v \in A_1$, it follows that $\deg(A_1) > n/3$. Since $\deg(A_1) \leqslant \frac{2}{3}n$, and $\deg(A) = |E| \geqslant \frac{11}{12}n$, we also have $\deg(A_2) \geqslant n/4$. We conclude that we have found a partition $A_1 \uplus A_2$ of $A$ such that

$$\deg(A_1) \geqslant n/4 \qquad \text{and} \qquad \deg(A_2) \geqslant n/4.$$

Symmetrically, we can find a partition $B_1 \uplus B_2$ of $B$ such that

$$\deg(B_1) \geqslant n/4 \qquad \text{and} \qquad \deg(B_2) \geqslant n/4.$$

For all $s, t \in \{1, 2\}$, let $F_{st}$ be the set of all edges from $E$ with one endpoint in $A_s$ and the other in $B_t$, and set $m_{st} \coloneqq |F_{st}|$. The above lower bounds on the degrees of $A_1, A_2, B_1, B_2$ imply that

$$m_{11} + m_{12} \geqslant n/4, \qquad m_{21} + m_{22} \geqslant n/4, \qquad m_{11} + m_{21} \geqslant n/4, \qquad m_{12} + m_{22} \geqslant n/4. \tag{17}$$

Observe that if $m_{11} \geqslant n/8$ and $m_{22} \geqslant n/8$, then $E_1 = F_{11}$ and $E_2 = F_{22}$ satisfy the condition from the lemma statement. Similarly, if $m_{12} \geqslant n/8$ and $m_{21} \geqslant n/8$, then taking $E_1 = F_{12}$ and $E_2 = F_{21}$ concludes the proof. We are thus left with the case when there is $st \in \{11, 22\}$ such that $m_{st} < n/8$ and there is $s't' \in \{12, 21\}$ such that $m_{s't'} < n/8$. But then $m_{st} + m_{s't'} < n/4$, which contradicts one of the inequalities (17). □

From Lemma 11 we immediately infer the following result.

**Lemma 12.** *Suppose that $\tau_A, \tau_B \in \Theta$ is a pair of tactics such that $\tau_A, \tau_B \in \Theta^{\mathbf{A}}$ or $\tau_A, \tau_B \in \Theta^{\mathbf{P}}$, and that $|F| \geqslant \frac{11}{12}n$. Then $\operatorname{Var} X \geqslant \frac{1}{32}$.*

Proof. Let $G = (A, B, F)$ be the bipartite graph constructed by taking $A$ and $B$ to be two disjoint copies of the set $[n]$, and interpreting each pair $(i, j) \in F$ as an edge that connects the copy of $i$ in $A$ with the copy of $j$ in $B$. Our next goal is to show that there exist two disjoint subsets of pairs $F_1, F_2 \subseteq F$, each of size at least $n/8$, such that every pair from $F1$ is disjoint from every pair of $F_2$. This is the conclusion of Lemma 11, so it is enough to show that $G$ satisfies the prerequisites of this lemma, which we will do if $n \geqslant 3$. If $n = 2$, note that the sought conclusion amounts to finding two disjoint edges, since $n/8 < 2$. It then suffices to notice that $G$ has at least 2 edges, since $\frac{11}{12}n > 1$, and consequently the two tactics cannot both be passive. It follows that both are active and hence $G$ indeed contains two disjoint edges. We now suppose that $n \geqslant 3$ and verify that $G$ satisfies the prerequisites of Lemma 11. We have $|F| \geqslant \frac{11}{12}n$ by assumption, so we are left with checking the requirements on degrees.

Suppose first that $\tau_A, \tau_B \in \Theta^{\mathbf{P}}$. Then $|\tau_A([n])| \leqslant n/2$, so there are only at most $n/2$ indices $i \in [n]$ that may be the first coordinates of pairs from $F$. Hence in $G$, the degree of every vertex in $B$ is at most $n/2$. A symmetric reasoning shows that the degree of every vertex in $A$ is at most $n/2$.

Suppose now that $\tau_A, \tau_B \in \Theta^{\mathbf{A}}$. Then $|\tau_A([n])| > n/2$, hence there are at least $\frac{n+1}{2}$ indices $i \in [n]$ that are the first coordinates of pairs from $F$. Every $i \in [n]$ is the first coordinate of at most $\frac{n+1}{2}$ pairs from $F$. Indeed, otherwise it would not be possible that each of the at least $\frac{n-1}{2}$ indices $i' \in \tau_A([n]) \setminus \{i\}$ would be the first coordinate of one of the remaining less than $\frac{n-1}{2}$ pairs from $F$. This means that in $G$, the degree of each vertex from $A$ is at most $\frac{n+1}{2} \leqslant \frac{2}{3}n$ since $n \geqslant 3$. A symmetric reasoning shows that the degree of each vertex from $B$ is at most $\frac{2}{3}n$.

Having verified the prerequisites of Lemma 11, we can conclude that there exist disjoint subsets of pairs $F_1, F_2 \subseteq F$, each of size at least $n/8$, such that every pair from $F_1$ is disjoint with every pair from $F_2$. This implies that in $F$ there are at least $\frac{n^2}{64} \geqslant \frac{1}{32} \cdot \binom{n}{2}$ pairs of pairs that are disjoint. By Lemma 10, this implies that $\mathrm{Var}X \geqslant \frac{1}{32}$. □

### 3.8 Wrapping up the proof, II

With all the tools prepared, we are now in a position to prove Lemma 4 for all values of $n \geqslant 2$.

Proof (of Lemma 4). Let $F$, $m$, and $X$ be defined for $\tau_A, \tau_B$ as in Section 3.5.

We first consider the corner case when $m \leqslant \frac{11}{12}n$. Then

$$\mathbb{E}X = \frac{m}{n} \leqslant \frac{11}{12}.$$

Therefore, by Markov's inequality we infer that

$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(X \geqslant 1) \geqslant 1 - \frac{11}{12} = \frac{1}{12}.$$

Now consider the case when $m > \frac{11}{12}n$. By Lemma 12 we infer that $\mathrm{Var}X \geqslant \frac{1}{32}$. Applying Lemma 8 for $\alpha = \frac{1}{32}$, we conclude that in this case

$$\mathbb{P}(X = 0) \geqslant \frac{1}{128 \cdot 32^2} = 2^{-17}.$$

Note here that the assumption $m \geqslant \left(1 - \sqrt{\alpha/2}\right) \cdot n$ is satisfied, because $1 - \sqrt{\alpha/2} = \frac{7}{8} < \frac{11}{12}$.

Hence, we have $\mathbb{P}(X = 0) \geqslant 2^{-17}$ in both cases. By Lemma 5 we now conclude that

$$W(\tau_A, \tau_B) \geqslant \frac{n+1}{2} + 2^{-35} \cdot n.$$

Hence, Lemma 4 holds for $\delta = 2^{-35}$. □

Recalling that the proof of Theorem 1 sets $\varepsilon$ to be $\delta/2$, we conclude that Theorem 1 holds for $\varepsilon = 2^{-36}$.

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