Insight into stability analysis of time-delay systems using Legendre polynomials

Mathieu Bajodek, Alexandre Seuret, Frédéric Gouaisbaut

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M. Bajodek * **A. Seuret** * **F. Gouaisbaut** *

* *LAAS-CNRS, Univ de Toulouse, UPS, 7 avenue du Colonel Roche, 31077 Toulouse, France. (e-mail: mbajodek,aseuret,fgouaisb@laas.fr).*

Abstract: In this paper, a numerical analysis to assess stability of time-delay systems is investigated. The proposed approach is based on the design of a finite-dimensional approximation of the infinite-dimensional space of solutions of the system. Indeed, based on the dynamical coefficients on the sequence made of the first Legendre polynomials, the original time-delay system is modelled by a finite-dimensional model interconnected to a modelling error.
Putting aside the interconnection, the resulting finite-dimensional system turns out to be a nice approximation of the time-delay system. Using Padé arguments, the eigenvalues of this finite-dimensional system are proven to converge towards a set of characteristic roots of the original time-delay system. Furthermore, considering now the whole interconnected system and having a deeper look at the interconnection, an enriched Lyapunov-Krasovskii functional is proposed to develop a sufficient condition expressed in terms of linear matrix inequalities for the stability of the time-delay system. Both results are illustrated on toys examples and compared with other existing methods.

Keywords: Time-delay, Numerical analysis, Eigenvalues, Lyapunov stability, LMI.

1. INTRODUCTION

In several fields, delay phenomena appear while processing information or connecting different networked systems. These transmission delays have a significant impact on the behaviour of the state of the complete system and can even destabilise it. In consequence, taking into consideration these lag times is crucial (see Richard [2003]). Furthermore, from a theoretical point of view, the analysis of such systems is a difficult task since they belong to the wide class of infinite-dimensional systems. Hence, characterising the stability of time-delay systems (TDSs) is a current research purpose.

Several ways have been proposed to analyse its stability. Some of them are relying on the design of Lyapunov-Krasovskii functionals (LKF). Indeed, some necessary and sufficient conditions can be established by using the so-called complete LKF. Nevertheless, these conditions reveal complicated to fulfil relying on a solution of a second order ordinary differential equation with boundary conditions. That is the reason why, in the literature, many works have been conducted to find only sufficient conditions often expressed in the linear matrix inequality (LMI) framework (see Fridman [2014] or Gu et al. [2003]). Recently, some methods based on augmented systems have shown its efficiently even for non small delays (see Ariba et al. [2018]). They are all based on some inequalities (Jensen, Wirtinger, Bessel as presented in Seuret and Gouaisbaut [2015]) and require to extend the state with a finite-dimensional system. A second approach is based on the inspection of the characteristic roots of the linear TDS. To assess stability in a direct manner, a determination of the root crossing points through Routh criterion (see Olgac and Sipahi [2002]) or a formulation based on matrix pencils (see Louisell [2015]) can be implemented. However, to evaluate each characteristic root, the infinite-dimensional system is often approximated, once again, by a finite-dimensional system. For example, a Padé approximant of the delay is largely implemented (see Golub and Van Loan [1989]). Otherwise, more recently, Breda et al. [2005] presents a method based on pseudospectral differentiation and different rough projections on Fourier, Chebychev or Legendre basis functions were also numerically investigated (see Pekar and Gao [2018]). All these numerical approaches can then characterise the root locus thanks to an approximate finite-dimensional model.

From comparative studies, both the best reduced LKF and the root approximation with the fastest convergence (see Vyasarayani et al. [2014]) are obtained using a decomposition on Legendre first polynomials. Based on these considerations, one proposes in this paper to get a deeper understanding of the equivalent model which includes the system satisfied by the first Legendre coefficients. The aim of this study is to highlight a link in between the reduced LKF and the finite-dimensional system, which approximates the characteristic roots of the TDS. Proving that the approximation is converging, this new link help to better understand the accurate underlying stability result using Legendre technique. First, the augmented system, which includes the dynamics satisfied by the $N+1$ first Legendre coefficients, is presented. This resulting augmented system is made of an interconnection between a finite-dimensional model and an infinite element. Focusing on the finite-dimensional part, it is equivalent to a Padé approximant of the original system, which consists in approximating the transfer function of the delay with a rational fraction which numerator of order N and denominator of order

$N+1$ are given by Padé table. From this equivalence, the convergence of the eigenvalues of the finite-dimensional model towards some of those of the infinite-dimensional system can be deduced and reinforces the choice of Legendre first polynomials. Based on the structure of the interconnection, the reduced LKF including Legendre coefficients provides a sufficient condition of stability thanks to Bessel inequality. This condition takes a very convenient form and is easy to express as an LMI. Knowing that the finite-dimensional part uniformly converges towards the original system, it gives now an understandable numerical stability condition with respect to the delay.

Notations : Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ the set of complex numbers, $\mathbb{R}^n$ the n -dimensional Euclidian space, $\mathbb{R}^{n\times m}$ the set of $n \times m$ real matrices, $\mathbb{S}^n$ the set of $n \times n$ symmetric matrices. Furthermore, $|.|$ is the modulus and $||.||$ denotes some matrix norm. Then, for any square matrix A, $\mathcal{H}(A) = A + A^T$, $\mathrm{adj}(A)$ the adjugate matrix (the transpose of its cofactor matrix), $\det(A)$ the determinant, $\det'(A)$ its derivatives given by Jacobi's formula, $\mathrm{tril}(A)$ the lower triangular part of the matrix A and $A > 0$ means that A is symmetric positive definite. Moreover, I is the identity matrix, $\mathrm{diag}(d_0, ..., d_N)$ is the diagonal matrix defined by its diagonal coefficients $(d_0, ..., d_N)$ and the operation $\otimes$ traduces a Kronecker product. The space $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$ represents the set of square-integrable functions from $(-h, 0)$ to $\mathbb{R}^n$. The space $\mathcal{H}^1(-h, 0; \mathbb{R}^n)$ refers to absolutely continuous functions from $[-h, 0]$ to $\mathbb{R}^n$ with derivative in $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$. Lastly, for any function $x : (-h, +\infty) \to \mathbb{R}^n$, the notation $x_t(\tau)$ stands for $x(t + \tau)$, for all $t \geq 0$ and all $\tau \in (-h, 0)$.

2. MODELLING OF A TIME-DELAY SYSTEM ON LEGENDRE POLYNOMIALS BASIS

2.1 Definition of a time-delay system

Consider the linear TDS given by

$$\begin{cases} \dot{x}(t) = Ax(t) + B_d C_d x(t-h), \ \forall t \in \mathbb{R}^+, \\ \big(x(0), x_0\big) = \big(f(0), f\big), \ f \in \mathcal{H}^1(-h, 0; \mathbb{R}^n), \end{cases} \tag{1}$$

where matrices $A \in \mathbb{R}^{n\times n}$, $B_d \in \mathbb{R}^{n\times m}$, $C_d \in \mathbb{R}^{m\times n}$ and the single delay $h > 0$ are assumed to be constant and known. In Laplace domain, with $s \in \mathbb{C}$, system (1) can be modelled by the block diagram of the Fig. 1.

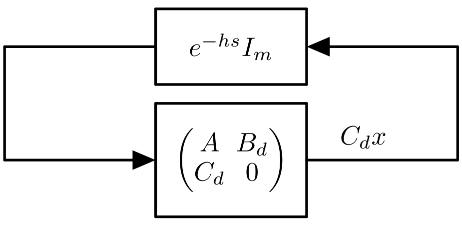

Fig. 1. Block diagram of time-delay system (1).

Remark 1. This linear time-invariant retarded differential equation satisfying the initial condition $\big(f(0), f\big)$ with $f \in \mathcal{H}^1(-h, 0, \mathbb{R}^n)$ is well defined in the Hilbert space $\mathbb{R}^n \times \mathcal{L}^2(-h, 0; \mathbb{R}^n)$. For each $t \in \mathbb{R}^+$, the unique analytic solution $\big(x(t), x_t\big)$ belongs therefore to $\mathbb{R}^n \times \mathcal{L}^2(-h, 0; \mathbb{R}^n)$. Since several years, one assists to a huge number of works dedicated to the stability analysis of TDSs based on an extended state space of a finite-dimensional system. These extension is related to the use of appropriate inequalities (Jensen, Wirtinger, Bessel) which needs extra-signals to be usefull. Usually, these extra-signals are based upon the projection of the state x_t on a basis of $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$ such as the one generated by Legendre polynomials, which definition is recalled in the next subsection.

2.2 Definition of the Legendre polynomials basis

By definition, for all $\tau \in [-h, 0]$ and $k \in \mathbb{N}$, each k-order Legendre polynomial is written as

$$L_k(\tau) = (-1)^k \sum_{l=0}^{k} (-1)^l \binom{k}{l} \binom{k+l}{l} \left(\frac{\tau + h}{h}\right)^l. \tag{2}$$

As noted in Lagrange [1939], these polynomials form an orthogonal basis of $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$. In addition, they have the following properties.

Lemma 2. For all $k \in \mathbb{N}$,

$$\begin{cases} \dfrac{d}{d\tau} L_k(\tau) = \displaystyle\sum_{l=0}^{k-1} \frac{2l+1}{h} \big(1 - (-1)^{k+l}\big) L_l(\tau) \quad k \geq 1, \\ \dfrac{d}{d\tau} L_0(\tau) = 0, \\ L_k(-h) = (-1)^k, \\ L_k(0) = 1. \end{cases} \tag{3}$$

Proof. The proof of (3), using Rodrigues formula, is given in Gautschi [2006].

2.3 Coefficients on the Legendre polynomials basis

Focusing on $C_d x_t$, which is the transported part of the state and can be seen as a function of $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$, its $N+1$ first components on Legendre polynomials orthogonal basis can be calculated. Let us define the vector X_N, which stores these Legendre coefficients.

$$X_N(t) = \begin{bmatrix} \displaystyle\int_{-h}^{0} C_d x_t(\tau) L_0(\tau) d\tau \\ \vdots \\ \displaystyle\int_{-h}^{0} C_d x_t(\tau) L_N(\tau) d\tau \end{bmatrix}, \ \forall t \in \mathbb{R}^+. \tag{4}$$

These first Legendre coefficients represent the projection on a finite-dimensional basis of the retarded state. Hence, increasing N adds information on the functional state and the behaviour of $C_d x_t$.

2.4 Dynamics of the coefficients

In order to analyse the behaviour of X_N, one has to compute its dynamics. This is formulated in the next proposition.

Proposition 3. The vector X_N is solution of the dynamical model

$$\begin{cases} \dot{X}_N(t) = \mathcal{A}_N X_N(t) + \mathcal{B}_N C_d x(t) - \mathcal{B}_N^* \epsilon_N(t) \\ \epsilon_N(t) = C_d x(t-h) - \mathcal{C}_N^* X_N(t) \end{cases}, \ \forall t \in \mathbb{R}^+, \tag{5}$$

with

$$\begin{cases} \mathbf{1}_N = [1 \ \dots \ 1]^T, \ \mathbf{1}_N^* = \left[(-1)^0 \ \dots \ (-1)^N\right]^T, \\ \mathbf{L}_N = \mathrm{tril}(\mathbf{1}_N \mathbf{1}_N^T - \mathbf{1}_N^* \mathbf{1}_N^{*T}), \ \mathcal{L}_N = -(\mathbf{L}_N + \mathbf{1}_N^* \mathbf{1}_N^{*T}), \\ \mathcal{I}_N = \dfrac{1}{h}\mathrm{diag}(1, \dots, 2N+1), \\ \mathcal{A}_N = (\mathcal{L}_N \mathcal{I}_N) \otimes I_m, \ \mathcal{B}_N = \mathbf{1}_N \otimes I_m, \ \mathcal{B}_N^* = \mathbf{1}_N^* \otimes I_m, \\ \mathcal{C}_N = (\mathbf{1}_N^T \mathcal{I}_N) \otimes I_m, \ \mathcal{C}_N^* = (\mathbf{1}_N^{*T} \mathcal{I}_N) \otimes I_m, \end{cases}$$

and satisfies an initial condition X_N^0 given by the coefficients of $C_d x_0$.

Proof. For all $k \in [\![0, N]\!]$, thanks to Legendre basis properties (3), to an integration by parts the derivation of each coefficient gives, for all $t \in \mathbb{R}^+$,

$$\frac{d}{dt}\int_{-h}^{0} C_d x(t+\tau) L_k(\tau) d\tau = C_d x(t) - (-1)^k C_d x(t-h) \\ - \sum_{l=0}^{k-1} \frac{2l+1}{h}\left(1 - (-1)^{k+l}\right) \int_{-h}^{0} C_d x(t+\tau) L_l(\tau) d\tau.$$

Gathering all the components, a compact expression is obtained

$$\dot{X}_N(t) = (\mathbf{1}_N \otimes I_m) C_d x(t) - (\mathbf{1}_N^* \otimes I_m) C_d x(t-h) \\ - (\mathbf{L}_N \mathcal{I}_N \otimes I_m) X_N(t).$$

Using the decomposition of $C_d x(t-h)$, for all $t \in \mathbb{R}^+$,

$$C_d x(t-h) = (\mathbf{1}_N^{*T} \mathcal{I}_N \otimes I_m) X_N(t) + \epsilon_N(t),$$

it gives

$$\begin{cases} \dot{X}_N(t) = (\mathbf{1}_N \otimes I_m) C_d x(t) - (\mathbf{1}_N^* \otimes I_m) \epsilon_N(t) \\ \qquad\qquad - \left((\mathbf{L}_N + \mathbf{1}_N^* \mathbf{1}_N^{*T}) \mathcal{I}_N \otimes I_m\right) X_N(t), \\ \epsilon_N(t) = C_d x(t-h) - \mathcal{C}_N^* X_N(t). \end{cases}$$

The resulting non-autonomous dynamical system (5) is finally driven by two inputs, the current transported solution ($C_d x$) and the remainder of Legendre serie evaluated at $-h$ (ϵ_N). Notice that the proposed procedure is equivalent to decomposing the block $e^{-hs} I_m$ into a finite-dimensional system to which is added a structured disturbance ϵ_N.

2.5 Augmented time-delay system

Gathering the dynamics of x and X_N, one can construct an augmented TDS as described in this subsection. The new system of state x and X_N is build up an augmented finite-dimensional system which state error is related to the remainder ϵ_N. This remainder includes the infinite-dimensional part. To sum up, this new augmented system is an interconnection between a finite-dimensional and an infinite-dimensional model as it is proposed in Theorem 4 and represented by the block diagram on Fig. 2.

Theorem 4. The system (1) takes the following form

$$\begin{cases} \dot{\xi}_N(t) = \underbrace{\begin{bmatrix} A & B_d \mathcal{C}_N^* \\ \mathcal{B}_N C_d & \mathcal{A}_N \end{bmatrix}}_{\mathbb{A}_N} \xi_N(t) + \underbrace{\begin{bmatrix} B_d \\ -\mathcal{B}_N^* \end{bmatrix}}_{\mathbb{B}_N} \epsilon_N(t) \\ \epsilon_N(t) = [C_d \ -\mathcal{C}_N^*] \begin{bmatrix} x(t-h) \\ X_N(t) \end{bmatrix} \end{cases}, \ \forall t \in \mathbb{R}^+, \tag{6}$$

with $\xi_N = \left[x^T \ X_N^T\right]^T$ satisfying $\xi_N(0) = \left[\, x(0)^T \ X_N^{0T} \,\right]^T$.

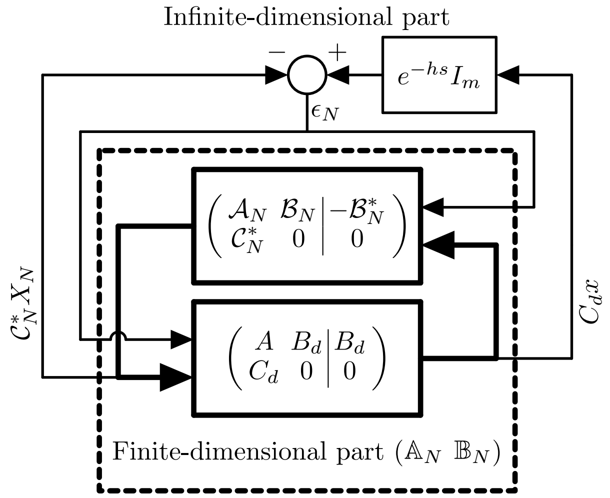

Fig. 2. Block diagram of augmented time-delay system (6).

Proof. First, Proposition 3 can be rewritten as

$$\begin{cases} \dot{X}_N(t) = [\mathcal{B}_N C_d \ \mathcal{A}_N]\, \xi_N(t) - \mathcal{B}_N^* \epsilon_N(t), \\ \epsilon_N(t) = C_d x(t-h) - \mathcal{C}_N^* X_N(t). \end{cases}$$

Then, equation (1) completes the dynamics. Using the previous equation, we have

$$\dot{x}(t) = A x(t) + B_d C_d x(t-h) \\ = [A \ B_d \mathcal{C}_N^*]\, \xi_N(t) + B_d \epsilon_N(t).$$

Since, intuitively the additional error ϵ_N is expected to become small enough increasing the size N, the finite-dimension part can be investigated as an approximation of the TDS, which is the aim of the next section.

3. STABILITY ANALYSIS OF THE APPROXIMATE FINITE-DIMENSIONAL MODEL

3.1 Approximation by a finite-dimensional model

This part is dedicated to the stability analysis of the finite-dimensional system getting rid of the effect of the error ϵ_N, which is expected to be small when N is sufficiently large. In that case, the resulting system corresponds to the finite-dimensional part of Fig. 2 and is depicted in Fig. 3.

The dynamical approximate model can be written as :

$$\dot{\hat{\xi}}_N(t) = \underbrace{\begin{bmatrix} A & B_d \mathcal{C}_N^* \\ \mathcal{B}_N C_d & \mathcal{A}_N \end{bmatrix}}_{\mathbb{A}_N} \hat{\xi}_N(t), \ \forall t \in \mathbb{R}^+, \tag{7}$$

with $\hat{\xi}_N = \left[\hat{x}^T \ \hat{X}_N^T\right]^T$ satisfying $\hat{\xi}_N(0) = \xi_N(0)$.

This model can then bring information on the locus of the eigenvalues and be used for the stability analysis of TDSs.

3.2 Link with the Padé approximant model

The aim of this subpart is to prove that system (7) described by the Fig. 3 can also be interpreted as an approximation of the original TDS, where the time-delay element e^{-hs} has been replaced by its Padé approximant which transfer function is $H_N(s)$.

Proposition 5. For each $N \in \mathbb{N}$, the state representation $\left(\begin{smallmatrix}\mathcal{A}_N & \mathcal{B}_N\\ \mathcal{C}_N^* & 0\end{smallmatrix}\right)$ is a realisation of $H_N = \frac{n_N(s)}{d_N(s)} I_m$, where

$$\begin{cases} n_N(s) = \displaystyle\sum_{j=0}^{N} \frac{N!(2N+1-j)!}{(N-j)!(2N+1)!}\frac{(-hs)^j}{j!}, \\ d_N(s) = \displaystyle\sum_{i=0}^{N+1} (-1)^i \frac{(N+1)!(2N+1-i)!}{(N+1-i)!(2N+1)!}\frac{(-hs)^i}{i!}, \end{cases} \tag{8}$$

are respectively the numerator and denominator of Padé approximant $(N, N+1)$ of the function e^{-hs} given in Baker [1975].

Proof. Consider, G_N the transfer function of the state space representation $\left(\begin{smallmatrix}\mathcal{A}_N & \mathcal{B}_N\\ \mathcal{C}_N^* & 0\end{smallmatrix}\right)$. The objective is to show that $G_N = H_N$, for any value of N. Let first note that

$$\begin{aligned} G_N(s) &= \mathcal{C}_N^*(sI_{m(N+1)} - \mathcal{A}_N)^{-1}\mathcal{B}_N, \\ &= \left(\mathbf{1}_N^{*T}(s\mathcal{I}_N^{-1} - \mathcal{L}_N)^{-1}\mathbf{1}_N\right) \otimes I_m, \\ &= \frac{\mathbf{1}_N^{*T}\mathrm{adj}(s\mathcal{I}_N^{-1} - \mathcal{L}_N)\mathbf{1}_N}{\det(s\mathcal{I}_N^{-1} - \mathcal{L}_N)} I_m. \end{aligned}$$

Hence, in order to prove this result, one needs to show that each numerator and denominator of G_N are equal to $2^N n_N$ and $2^N d_N$ respectively. For any $s \in \mathbb{C}$, this means

$$\begin{cases} \mathbf{1}_N^{*T}\mathrm{adj}(s\mathcal{I}_N^{-1} - \mathcal{L}_N)\mathbf{1}_N = 2^N n_N(s) \\ \det(s\mathcal{I}_N^{-1} - \mathcal{L}_N) = 2^N d_N(s) \end{cases}, \forall N \in \mathbb{N}.$$

This result is obtained recursively. The complete proof is given in Appendix A, but the initialization part is provided here to highlight the main features of this proof.

For $N = 0$, we easily find that

$$\begin{cases} \mathbf{1}_0^{*T}\mathrm{adj}(s\mathcal{I}_0^{-1} - \mathcal{L}_0)\mathbf{1}_0 = 1 = n_0(s), \\ \det(s\mathcal{I}_0^{-1} - \mathcal{L}_0) = sh + 1 = d_0(s). \end{cases}$$

For $N = 1$,

$$\begin{cases} \mathbf{1}_1^{*T}\mathrm{adj}(s\mathcal{I}_1^{-1} - \mathcal{L}_1)\mathbf{1}_1 = \left[\begin{smallmatrix}1 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}\frac{sh}{3}+1 & 1\\ -1 & sh+1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \\ \qquad = 2(1 - \dfrac{sh}{3}) = 2n_1(s), \\ \det(s\mathcal{I}_1^{-1} - \mathcal{L}_1) = \det\left[\begin{smallmatrix}sh+1 & -1\\ 1 & \frac{sh}{3}+1\end{smallmatrix}\right], \\ \qquad = 2\left(1 + \dfrac{2sh}{3} + \dfrac{(sh)^2}{6}\right) = 2d_1(s). \end{cases}$$

Then, to give an idea of the induction given in Appendix A, let express the result at the order $N = 2$ relying on the two previous ones $N \in \{0, 1\}$.

To begin with, we know that

$$\begin{cases} \mathbf{1}_2^{*T}\mathrm{adj}(s\mathcal{I}_2^{-1} - \mathcal{L}_2)\mathbf{1}_2 = \mathbf{1}_2^{*T}E_0\mathrm{adj}\big(F_0(s\mathcal{I}_2^{-1} - \mathcal{L}_2)E_0\big)F_0\mathbf{1}_2, \\ \det(s\mathcal{I}_2^{-1} - \mathcal{L}_2) = \det\big(F_0(s\mathcal{I}_2^{-1} - \mathcal{L}_2)E_0\big), \end{cases}$$

where E_0 and F_0 are nonsingular matrices given by

$$E_0 = \begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}, \quad F_0 = \begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}.$$

Indeed, we have

$$\begin{cases} \mathbf{1}_2^{*T}\mathrm{adj}(s\mathcal{I}_2^{-1} - \mathcal{L}_2)\mathbf{1}_2 = \left[\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right]^T \mathrm{adj}\left[\begin{smallmatrix}sh+1 & -1 & 0\\ 1 & \frac{sh}{3} & \frac{sh}{3}\\ 0 & -\frac{sh}{3} & 2\left(1-\frac{sh}{15}\right)\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right], \\ \det(s\mathcal{I}_2^{-1} - \mathcal{L}_2) = \det\left[\begin{smallmatrix}sh+1 & -1 & 0\\ 1 & \frac{sh}{3} & \frac{sh}{3}\\ 0 & -\frac{sh}{3} & 2\left(1-\frac{sh}{15}\right)\end{smallmatrix}\right]. \end{cases}$$

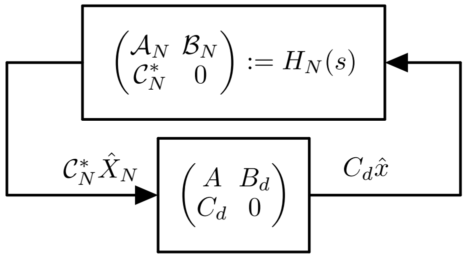

Fig. 3. Block diagram of approximate model (7).

From one side, we note that

$$\begin{aligned} \mathbf{1}_2^{*T}\mathrm{adj}(s\mathcal{I}_2^{-1} - \mathcal{L}_2)\mathbf{1}_2 =& 2(1 - \frac{sh}{15})\mathbf{1}_1^{*T}\mathrm{adj}(s\mathcal{I}_1^{-1} - \mathcal{L}_1)\mathbf{1}_1, \\ &+ (\frac{sh}{3})^2\mathbf{1}_0^{*T}\mathrm{adj}(s\mathcal{I}_0^{-1} - \mathcal{L}_0)\mathbf{1}_0, \\ =& 4\left((1 - \frac{sh}{15})n_1(s) + (\frac{sh}{6})^2 n_0(s)\right), \\ =& 4\left(1 - \frac{2sh}{5} + \frac{(sh)^2}{20}\right) = 4n_2(s). \end{aligned}$$

From the other side, we obtain

$$\begin{aligned} \det(s\mathcal{I}_2^{-1} - \mathcal{L}_2) =& 2(1 - \frac{sh}{15})\det(s\mathcal{I}_1^{-1} - \mathcal{L}_1), \\ &+ 4(\frac{sh}{6})^2\det(s\mathcal{I}_0^{-1} - \mathcal{L}_0), \\ =& 4\left((1 - \frac{sh}{15})d_1(s) + (\frac{sh}{6})^2 d_0(s)\right), \\ =& 4\left(1 + \frac{3sh}{5} + \frac{3(sh)^2}{20} + \frac{(sh)^3}{60}\right) = 4d_2(s), \end{aligned}$$

which completes the proof for $N = 2$. The proof for any $N \geq 3$ is given in appendix A.

From this result given by induction, we obtain the following transfer function

$$G_N(s) = \frac{2^N n_N(s)}{2^N d_N(s)} I_m = \frac{n_N(s)}{d_N(s)} I_m = H_N(s), \ \forall N \in \mathbb{N}.$$

The previous calculations and statement allow us to state the main result of this paper.

Theorem 6. Approximate model (7) is a Padé approximant of time-delay system (1).

Proof. Identifying the transfer function H_N given in Proposition 5, one can recognise a Padé approximant of the exponential function e^{-hs} repeated m times. That directly gives Theorem 6.

Then, the uniform convergence result on open ball of the Padé approximant towards the exponential function e^{-hs} could be used on our finite-dimensional model.

3.3 Convergence of the characteristic roots of the model towards some of those of the time-delay system

The finite-dimensional model studied is equivalent to a Padé approximant. Hence, the convergence results issued from Padé approximant theory (see Baker [1975]) can be used to link the characteristic roots of the TDS (1) and the eigenvalues of $\mathbb{A}_N$, state matrix of approximate model (7). More precisely, one proposes Theorem 8. But, before, a first technical lemma is recalled, showing that, on a compact set, the Padé approximant converges to the delay transfer function e^{-hs}.

Lemma 7. Let $R > 0$. On a compact set $\mathcal{B}(0, R)$, $n_N(s)$ and $d_N(s)$ uniformly converge when $N \to \infty$ towards $n(s) = e^{\frac{-hs}{2}}$ and $d(s) = e^{\frac{hs}{2}}$ respectively. In other words,

$$\forall \epsilon > 0\ \exists N^*;\ \forall N \geq N^*,\ \forall s \in \mathcal{B}(0,R),\ \begin{cases} |n_N(s) - n(s)| \leq \epsilon \\ |d_N(s) - d(s)| \leq \epsilon \end{cases}.$$

Proof. The proof of this convergence result is given in Baker [1975].

For all $s \in \mathbb{C}$, let matrices $\Delta_N(s)$ and $\Delta(s)$ in $\mathbb{R}^{n \times n}$ be

$$\begin{cases} \Delta_N(s) = (sI_n - A)d_N(s) - A_d n_N(s), \\ \Delta(s) = (sI_n - A)d(s) - A_d n(s), \end{cases}$$

with $A_d = B_d C_d$, $d(s) = e^{\frac{hs}{2}}$ and $n(s) = e^{-\frac{hs}{2}}$.

Now, the aim is to prove that, for N sufficiently large, the characteristic roots of model (7), i.e. zeros of $\chi_N(s) = \det\big(\Delta_N(s)\big)$, are close enough to some of those of the TDS (1), i.e. zeros of $\chi(s) = \det\big(\Delta(s)\big)$.

Theorem 8. For all $R > 0$, if the time-delay system (1) contains K characteristic roots with multiplicities $\nu^*_{k \in [\![1,K]\!]}$ into the open ball $\mathcal{B}(0, R)$, then $\sum\limits_{k=1}^{K} \nu_k^*$ eigenvalues of $\mathbb{A}_N$ converges towards them. More precisely,

$$\forall r \in (0, r^*),\ \exists N^*;\ \forall N \geq N^*,\ \max_{\substack{k \in [\![1,K]\!] \\ i = [\![1,\nu_k^*]\!]}} |s_{k,i}^N - s_k^*| \leq r. \quad (9)$$

Proof. This proof follows the one provided by Breda et al. [2015] in the case of the uniform convergence of the eigenvalues given by the pseudospectral differentiation method towards the characteristic roots directly.

Step 1 : Uniform convergence of Δ_N towards Δ.

$$\begin{aligned} &||[\Delta_N - \Delta](s)|| \\ &\quad = ||(sI_n - A)\big(d_N(s) - d(s)\big) - A_d\big(n_N(s) - n(s)\big)||, \\ &\quad \leq ||(sI_n - A)\big(d_N(s) - d(s)\big)|| + ||A_d\big(n_N(s) - n(s)\big)||. \end{aligned}$$

Let $\epsilon > 0$, $R > 0$. According to Lemma 7, there exists N^* such that,

$$\forall N \geq N^*,\ \forall s \in \mathcal{B}(0,R),\ ||\Delta_N(s) - \Delta(s)|| \leq \epsilon.$$

Then, the matrix Δ_N converges uniformly to the matrix Δ.

Step 2 : Uniform convergence of the characteristic polynomial χ_N towards χ.

$$\begin{aligned} |\chi_N(s) - \chi(s)| &= \left|\det\big(\Delta_N(s)\big) - \det\big(\Delta(s)\big)\right|, \\ &= \left|\int_0^1 \det{}'\big(\Delta(s) - \sigma[\Delta - \Delta_N](s)\big).[\Delta - \Delta_N](s) d\sigma\right|, \\ &\leq \max_{\sigma \in [0,1]} ||\det{}'\big(\Delta(s) - \sigma[\Delta - \Delta_N](s)\big)||.||[\Delta - \Delta_N](s)||. \end{aligned}$$

For all $s \in \mathcal{B}(0, R)$ and considering now $N > N^*$,

$$|\chi_N(s) - \chi(s)| \leq \max_{\substack{z \in \mathcal{B}(0,R) \\ \Gamma \in \mathbb{C}^{n \times n}; ||\Gamma|| < \epsilon}} ||\det{}'\big(\Delta(z) + \Gamma\big)||\epsilon.$$

With this bound, the uniform convergence of χ_N towards χ on any open ball $\mathcal{B}(0, R)$ is verified.

Step 3 : Application of Rouché's theorem.

Using Rouché's theorem, the aim is to proove that χ and χ_N have the same number of zeros on open balls $\mathcal{B}(s^*, r)$ where s^* is a zero of χ.

First, the fact that χ and χ_N are holomorphic functions on $\mathcal{B}(0, R)$ enables to use Rouché's theorem. Then, thanks to Taylor's expansion of χ around a root s^* by multiplicity ν^*,

$$\forall s \in \mathcal{B}(s^*, r^*) \backslash \{s^*\},\ |\chi(s)| > \frac{1}{2} \frac{|\chi^{(\nu^*)}(s^*)|}{\nu^*!} |s - s^*|^{\nu^*},$$

with r_0 the smallest radius in between s^* and other zeros of χ and

$$r^* = \min\left(r_0, \frac{1}{2} \frac{(\nu^* + 1)|\chi^{(\nu^*)}(s^*)|}{\max\limits_{z \in \mathcal{B}(s^*, r_0)} |\chi^{(\nu^*+1)}(z)|}\right).$$

Then, for $r = \left(\frac{\max ||\det'\left(\Delta(z) + \Gamma\right)||\epsilon}{\frac{1}{2}\frac{|\chi^{(\nu^*)}(s^*)|}{\nu^*!}}\right)^{\frac{1}{\nu^*}} \in (0, r^*)$, thanks to the uniform convergence of Step 2, it exists N^* such that, for all $N \geq N^*$,

$$\forall |s - s^*| = r,\ |\chi_N(s) - \chi(s)| \leq \frac{1}{2} \frac{|\chi^{(\nu^*)}(s^*)|}{\nu^*!} |s - s^*|^{\nu^*} < |\chi(s)|.$$

Applying Rouché's theorem, the characteristic equation $\chi_N(s) = 0$ has ν^* roots in $\mathcal{B}(s^*, r)$ each counted with its multiplicities. This involves,

$$\forall r \in (0, r^*),\ \exists N^*;\ \forall N \geq N^*,\ \forall i \in [\![1, \nu^*]\!],\ |s_i^N - s^*| \leq r.$$

Step 4 : Convergence of some zeros of χ_N towards those of χ.

Assume that the open ball $\mathcal{B}(0, R)$ contains K zeros of χ with multiplicities $\nu^*_{k \in [\![1,K]\!]}$. Repeating the previous Step 3 on each ball $\mathcal{B}(s_k^*, r_k^*)$ for each root s_k^* and taking $r^* = \min\limits_{k \in [\![1,K]\!]} r_k^*$, $N^* = \max\limits_{k \in [\![1,K]\!]} (N_k^*)$, condition (9) is obtained.

Thus, a certain number of the $n + m(N+1)$ eigenvalues of $\mathbb{A}_N$ can approximate as close as desired the characteristic roots of system (1) increasing N. Especially, for unstable TDSs, which have at least one unstable root on a compact set, one can find a value N^* such that matrix $\mathbb{A}_N$ has at least one eigenvalue with positive real parts for each $N \geq N^*$. From these promising properties of this finite-dimensional model, a Lyapunov-Krasovskii stability analysis is proposed going back on the interconnected system (6) to take in account the infinite-dimensional part which have been neglected in this section.

4. STABILITY ANALYSIS OF THE INTERCONNECTED SYSTEM

The aim of this part is to analyse the stability of the whole system depicted in Fig. 2. One proposes to design an LKF, highly related to system (6).

4.1 A Lyapunov-Krasovskii functional

To be consistent with augmented TDS (6), let define the LKF enriched by Legendre coefficients,

$$V_N\big(x(t), x_t\big) = V_{\mathbb{P}_N}\big(\xi_N(t)\big) + V_S(x_t) + V_R(x_t), \quad (10)$$

with

$$\begin{cases} V_{\mathbb{P}_N}\big(\xi_N(t)\big) = \xi_N^T(t)\mathbb{P}_N\xi_N(t), \\ V_S(x_t) = \displaystyle\int_{-h}^{0} \big(C_d x_t(\tau)\big)^T S \big(C_d x_t(\tau)\big) d\tau - X_N^T(t)\mathcal{S}_N X_N(t), \\ V_R(x_t) = \displaystyle\int_{-h}^{0} (h + \tau)\big(C_d x_t(\tau)\big)^T R \big(C_d x_t(\tau)\big) d\tau. \end{cases}$$

Matrices $\mathbb{P}_N \in \mathbb{S}^{n+m(N+1)}$ and $S, R \in \mathbb{S}^m$ are assumed to be symmetric positive definite and $\mathcal{S}_N$ stands for $\mathcal{I}_N \otimes S$.

4.2 Bessel-Legendre inequality

Bessel inequality, applied to $C_d x_t$ and its $N+1$ first components X_N on Legendre polynomial basis, is a tool allowing to bound the integral terms which appear in V_S or in the derivative of V_R.

Lemma 9. For any positive definite matrix $M \in \mathbb{S}^m$, Bessel-Legendre inequality provides

$$\int_{-h}^{0} \big(C_d x_t(\tau)\big)^T M \big(C_d x_t(\tau)\big) d\tau \geq X_N^T(t) \mathcal{M}_N X_N(t),$$

where $\mathcal{M}_N = \mathcal{I}_N \otimes M$.

This inequality leads to the following stability condition.

4.3 Sufficient condition of stability

The LKF defined previously combined with Lemma 9 provides Theorem 10, a rewrite of the LMI condition given by Seuret and Gouaisbaut [2015].

Theorem 10. If it exists symmetric positive definite matrices $\mathbb{P}_N > 0$, $S > 0$ and $R > 0$ such that

$$\begin{bmatrix} \mathcal{H}(\mathbb{P}_N \mathbb{A}_N) + \mathbb{C}_N^T S \mathbb{C}_N + \begin{bmatrix} hC_d^T R C_d & 0 \\ * & -\mathcal{R}_N \end{bmatrix} & \mathbb{P}_N \mathbb{B}_N \\ * & -S \end{bmatrix} < 0, \tag{11}$$

where $\mathcal{R}_N = \mathcal{I}_N \otimes R$ and with

$$\mathbb{A}_N = \begin{bmatrix} A & B_d \mathcal{C}_N^* \\ \mathcal{B}_N C_d & \mathcal{A}_N \end{bmatrix},\ \mathbb{B}_N = \begin{bmatrix} B_d \\ -\mathcal{B}_N^* \end{bmatrix},\ \mathbb{C}_N = \begin{bmatrix} C_d & -\mathcal{C}_N \end{bmatrix},$$

then system (1) is asymptotically stable, for the delay h.

Proof. Consider the LKF candidate V_N given by (10). Indeed, the positivity of V_N is ensured by the positive definiteness of $\mathbb{P}_N$, S and R. More precisely, we show that V_S is positive by application of Lemma 9.

Then, the derivative of V_N along the trajectories of system (6) is composed of the derivative of the finite-dimensional part,

$$\dot{V}_{\mathbb{P}_N}(\xi) = \xi_N^T \mathcal{H}(\mathbb{P}_N \mathbb{A}_N) \xi_N + \xi_N^T \mathbb{P}_N \mathbb{B}_N \epsilon_N + \epsilon_N^T \mathbb{B}_N^T \mathbb{P}_N \xi_N,$$

to which is added the derivative of V_S,

$$\begin{aligned} \dot{V}_S(x_t) &= \big(C_d x(t)\big)^T S \big(C_d x(t)\big) - \big(C_d x(t-h)\big)^T S \big(C_d x(t-h)\big) \\ &\quad -2X_N^T(t) \mathcal{S}_N \big(\mathbf{L}_N \mathcal{I}_N X_N(t) + \mathbf{1}_N C_d x(t) - \mathbf{1}_N^* C_d x(t-h)\big), \end{aligned}$$

which, reorganising the terms, is equal to

$$\begin{aligned} \dot{V}_S(x_t) &= \big(C_d x(t) - \mathcal{C}_N X_N(t)\big)^T S \big(C_d x(t) - \mathcal{C}_N X_N(t)\big) \\ &\quad - \big(C_d x(t-h) - \mathcal{C}_N^* X_N(t)\big)^T S \big(C_d x(t-h) - \mathcal{C}_N^* X_N(t)\big), \\ &= \xi_N^T(t) \mathbb{C}_N^T S \mathbb{C}_N \xi_N(t) - \epsilon_N^T(t) S \epsilon_N(t), \end{aligned}$$

and, lastly, the derivative of V_R,

$$\begin{aligned} \dot{V}_R(x_t) &= h\big(C_d x(t)\big)^T R \big(C_d x(t)\big) \\ &\quad - \int_{-h}^{0} \big(C_d x_t(\tau)\big)^T R \big(C_d x_t(\tau)\big) d\tau. \end{aligned}$$

Putting all the terms together, according to Lemma 9,

$$\begin{aligned} \dot{V}_N(x(t), x_t) &\leq \xi_N^T \mathcal{H}(\mathbb{P}_N \mathbb{A}_N) \xi_N + \begin{bmatrix} \xi_N \\ \epsilon_N \end{bmatrix}^T \begin{bmatrix} \mathbb{C}_N^T S \mathbb{C}_N & \mathbb{P}_N \mathbb{B}_N \\ * & -S \end{bmatrix} \begin{bmatrix} \xi_N \\ \epsilon_N \end{bmatrix} \\ &\quad + \xi_N^T \begin{bmatrix} hC_d^T R C_d & 0 \\ * & -\mathcal{R}_N \end{bmatrix} \xi_N. \end{aligned}$$

Therefore, if the LMI (11) is satisfied, system (1) is asymptoticaly stable by application of the Lyapunov-Krasovskii theorem.

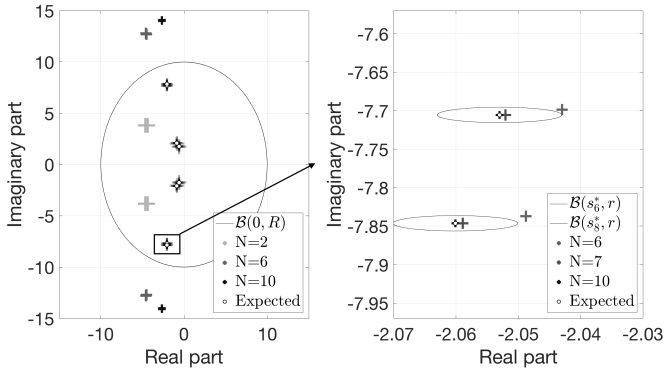

Fig. 4. Example 11 for $h = 1$: Convergence of the eigenvalues with $(R, r) = (10, 10^{-2})$.

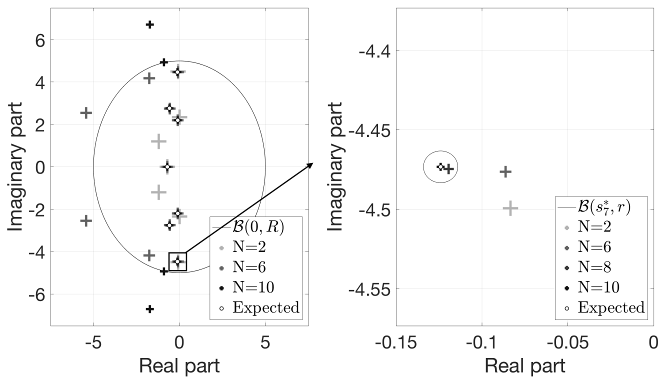

Fig. 5. Example 12 for $h = 3$: Convergence of the eigenvalues with $(R, r) = (5, 10^{-2})$.

As the eigenvalues of $\mathbb{A}_N$ approximate the characteristic roots of the original TDS, one expects that the stability condition proposes in Theorem 10 can approximate the entire stability chart with respect to h.

5. EXAMPLES

Two examples were studied to illustrate our results.

Example 11. $A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}$, $B_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $C_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$.

Example 12. $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -11 & 10 & 0 & 0 \\ 5 & -15 & 0 & -\frac{1}{4} \end{bmatrix}$, $B_d = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $C_d = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$.

5.1 Analysis of the eigenvalues

For two given values of h, the eigenvalues of $\mathbb{A}_N$ are depicted in Figures 4 and 5, for respectively Examples 11 and 12, where there are materialised, increasing N, by increasingly dark and small crosses. Theorem 8 ensures the convergence of some of them towards the expected ones contained in a ball $\mathcal{B}(0, R)$. These first expected eigenvalues were calculated with a precision 10^{-4} following Breda et al. [2005], materialised by white points on Figures 4 and 5 and recalled on Table 1.

The convergence of some of the eigenvalues is confirmed by zooming on expected characteristic roots s^* contained in $\mathcal{B}(0, R)$ and finding a value $N^* = 7$ for Example 11 and $N^* = 8$ for Example 12 from which the computed ones are inside a ball $\mathcal{B}(s^*, r)$ with $r = 10^{-2}$.

Table 1. First eigenvalues expected.

Examples	Example 11 with $h = 1$	Example 12 with $h = 3$
Eigenvalues	$-0.5777 \pm 1.7526j$	-0.7026
	$-0.8610 \pm 2.0732j$	$-0.1007 \pm 2.1919j$
	$-2.0530 \pm 7.7054j$	$-0.5712 \pm 2.7559j$
	$-2.0601 \pm 7.8463j$	$-0.1241 \pm 4.4733j$

Table 2. Example 11 : Eigenvalues for $h = 1$.

Method	Order	
	$N = 2$	$N = 6$
Legendre	$-0.5761 \pm 1.7487j$	$-0.5777 \pm 1.7526j$
	$-0.8538 \pm 2.0615j$	$-0.8610 \pm 2.0732j$
Theorem 8	$-4.3739 \pm 3.8079j$	$-2.0430 \pm 7.6987j$
	$-4.6462 \pm 3.8166j$	$-2.0488 \pm 7.8373j$
Collocation	$-0.5503 \pm 1.7598j$	$-0.5777 \pm 1.7526j$
	$-0.7876 \pm 2.0799j$	$-0.8610 \pm 2.0732j$
Breda et al. [2005]	$-3.0663 \pm 2.9122j$	$-2.1342 \pm 7.6534j$
	$-3.3791 \pm 2.8267j$	$-2.1563 \pm 7.8002j$
Least-Square	$-0.5658 \pm 1.7617j$	$-0.5777 \pm 1.7526j$
	$-0.8288 \pm 2.0922j$	$-0.8610 \pm 2.0732j$
Vyasarayani [2012]	$-3.5092 \pm 4.1011j$	$-2.1106 \pm 7.6126j$
	$-3.7962 \pm 4.0853j$	$-2.1343 \pm 7.7501j$
Legendre-Tau	$-0.5780 \pm 1.7522j$	$-0.5777 \pm 1.7526j$
	$-0.8614 \pm 2.0713j$	$-0.8610 \pm 2.0732j$
Ito and Teglas [1986]	$-5.3720 \pm 6.1739j$	$-2.0521 \pm 7.7027j$
	$-5.6386 \pm 6.3044j$	$-2.0593 \pm 7.8429j$

Table 3. Example 12 : Eigenvalues for $h = 3$.

Method	Order	
	$N = 2$	$N = 6$
Legendre	-0.6955	-0.7026
	$+0.0058 \pm 2.3377j$	$-0.1007 \pm 2.1920j$
Theorem 8	$-1.1997 \pm 1.2033j$	$-0.5656 \pm 2.7489j$
	$-0.0834 \pm 4.4994j$	$-0.0863 \pm 4.4764j$
Collocation	-0.6542	-0.7026
	$-0.0249 \pm 2.3553j$	$-0.1007 \pm 2.1938j$
Breda et al. [2005]	$-0.5895 \pm 0.9612j$	$-0.6155 \pm 2.7404j$
	$-0.0810 \pm 4.5049j$	$-0.0951 \pm 4.4905j$
Least-Square	-0.6768	-0.7026
	$-0.0367 \pm 2.3150j$	$-0.1013 \pm 2.1940j$
Vyasarayani [2012]	$-0.7097 \pm 1.1383j$	$-0.6089 \pm 2.7134j$
	$-0.0915 \pm 4.5078j$	$-0.1447 \pm 4.5007j$
Legendre-Tau	-0.7018	-0.7026
	$+0.0185 \pm 2.2573j$	$-0.1007 \pm 2.1919j$
Ito and Teglas [1986]	$-1.2259 \pm 1.1696j$	$-0.5712 \pm 2.7535j$
	$-0.1210 \pm 4.5012j$	$-0.0987 \pm 4.4641j$

To see how fast the proposed computation is converging, a comparison with collocation (pseudospectral discretization given by Breda et al. [2005] here), least-square (Vyasarayani [2012] on Legendre basis) and Tau (Ito and Teglas [1986] on Legendre basis too) methods are performed. The closest calculated eigenvalues of those expected are given in Tables 2 and 3 for Examples 11 and 12, respectively.

From these tables, one can conclude that the proposed approximate model seems to give a better approximation than collocation or least-square techniques. Even though the Legendre-Tau method seems to converge faster, the Legendre method has the advantage to bring, in addition, sufficient stability results.

5.2 Lyapunov-Krasovskii stability analysis

The sufficient stability condition given by the LMI (11) can be easily implemented on Matlab and ensures pointwise stability with respect to the delay. On each example, a numerical test was done varying h step by step with a precision of 10^{-3} and for $N \in [\![0, 10]\!]$. The first analytical bound of stability $h = 6.172$ and $h = 1.142$ for Examples 11 and 12, respectively, are reached with a precision of 10^{-3} from $N = 3$. As expected, these numerical results are equivalent and as efficient as those presented in Seuret and Gouaisbaut [2015].

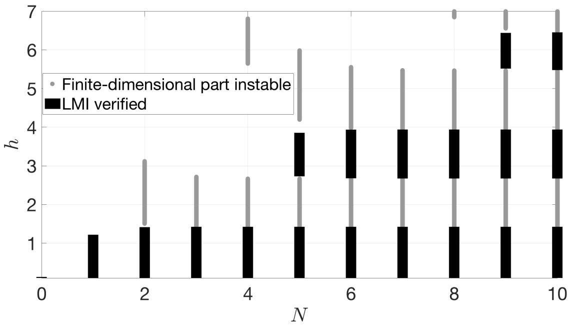

Fig. 6. Example 12 : Instability of $\mathbb{A}_N$ versus Lyapunov-Krasovskii stability

On Figure 6, for Example 12, the intervals of stability with respect to the delay given by Theorem 10 are represented with thick dark lines and the instability of $\mathbb{A}_N$ with respect to the delay with thin gray lines.

First, by increasing N, the set of instability of $\mathbb{A}_N$ with respect to h converges as expected towards the entire set of instability of the original TDS. Likewise, the intervals of stability given by Theorem 10 appear to slightly grow until to fill in the set of stability of the TDS. For example, from $N = 5$, a second interval of stability is found. Then, as suggested before, LMI (11) based on the finite-dimensional model also seems to converge to the entire stability region with respect to h. Lastly, the intervals of stability of the LMI at order N and those of instability of $\mathbb{A}_N$ are disjoints. In other words, the stability of $\mathbb{A}_N$ could be a necessary condition for the LMI at order N.

6. CONCLUSIONS

This work proposes some new insights for the stability analysis of TDSs using the first projections on Legendre polynomials. Taking into account these coefficients and its dynamics, an interconnection scheme between a finite-dimensional part and an infinite-dimensional error part was designed to model such systems. By getting rid of the error, the finite-dimensional system turns out to be a Padé approximant which eigenvalues converges therefore towards the expected characteristic roots. From the whole augmented system, a sufficient stability condition of TDSs expressed in terms of LMIs is also proposed. Thus, the new model proposed in this paper seems to be really usefull to yield numerical accurate stability conditions. Therefore, keeping this same framework, future work focused on control and observation of TDSs can provide interesting new numerical solutions.

REFERENCES

Y. Ariba, F. Gouaisbaut, A. Seuret, and D. Peaucelle. Stability Analysis of time-delay Systems via Bessel

Inequality : A quadratic separation approach. *Int. J. Robust Nonlin.*, 28(5):1507–1527, 2018.
G. A. Baker. *Essentials of Padé Approximants.* Academic Press, 1975.
D. Breda, S. Maset, and R. Vermiglio. Pseudospectral differencing methods for characteristic roots of delay differential equations. *SIAM J SCI Comput*, 27(2), 2005.
D. Breda, S. Maset, and R. Vermiglio. *Stability of linear delay differential equations : A numerical approach with Matlab.* Springer, 2015.
E. Fridman. *Introduction to Time-Delay Systems.* Analysis and Control. Birkhäuser, 2014.
W. Gautschi. Orthogonal polynomials, quadrature, and approximation: Computational methods and software. *Lecture Notes in Mathematics*, 1883:1–77, 2006.
G.H. Golub and C.F. Van Loan. *Matrix Computations.* Johns Hopkins University Press, 1989.
K. Gu, J. Chen, and V. Kharitonov. *Stability of Time-Delay Systems.* Birkhäuser, 2003.
K. Ito and R. Teglas. Legendre-Tau approximations for functional differential equations. *SIAM Journal on Control and Optimization*, 24(4):737–759, 1986.
R. Lagrange. *Polynômes et fonctions de Legendre.* Gauthier-Villars, 1939.
J. Louisell. Matrix polynomials, similar operators, and the imaginary axis eigenvalues of a matrix delay equation. *SIAM J. Control Optim.*, 53(1):399–413, 2015.
N. Olgac and R. Sipahi. An exact method for the stability analysis of time-delayed linear time-invariant (LTI) systems. *IEEE Trans. on Automatic Control*, 47 (5):793–797, 2002.
L. Pekar and Q. Gao. Spectrum analysis of LTI continuous-time systems with constant delays: A literature overview of some recent results. *IEEE Access*, 6: 35457–35491, 2018.
J-P. Richard. Time delay systems: an overview of some recent advances and open problems. *Automatica*, 39: 1667–1694, 10 2003.
A. Seuret and F. Gouaisbaut. Hierarchy of LMI conditions for the stability analysis of time-delay systems. *Systems and Control Letters*, 81:1–7, 2015.
C.P. Vyasarayani. Galerkin approximations for higher order delay differential equations. *ASME J Comput Nonlin Dyn*, 7(3), 2012.
C.P. Vyasarayani, S. Subhash, and T. Kalmár-Nagy. Spectral approximations for characteristic roots of delay differential equations. *International Journal of Dynamics and Control*, 2(2):126–132, 2014.

Appendix A. PROOF OF THE INDUCTION

Assume that there exists $N \in \mathbb{N}$ such that functions

$$\begin{cases} g_N(s) = \mathbf{1}_N^{*T}(s\mathcal{I}_N^{-1} - \mathcal{L}_N)^{-1}\mathbf{1}_N, \\ g_{N+1}(s) = \mathbf{1}_{N+1}^{*T}(s\mathcal{I}_{N+1}^{-1} - \mathcal{L}_{N+1})^{-1}\mathbf{1}_{N+1}, \end{cases}$$

are, by induction hypothesis, given by Padé approximant (8) to a factor 2^N. That means, at the numerator,

$$\begin{cases} \mathbf{1}_N^{*T}\mathrm{adj}(s\mathcal{I}_N^{-1} - \mathcal{L}_N)\mathbf{1}_N = 2^N n_N(s), \\ \mathbf{1}_{N+1}^{*T}\mathrm{adj}(s\mathcal{I}_{N+1}^{-1} - \mathcal{L}_{N+1})\mathbf{1}_{N+1} = 2^{N+1} n_{N+1}(s), \end{cases}$$

and, at the denominator,

$$\begin{cases} \det(s\mathcal{I}_N^{-1} - \mathcal{L}_N) = 2^N d_N(s), \\ \det(s\mathcal{I}_{N+1}^{-1} - \mathcal{L}_{N+1}) = 2^{N+1} d_{N+1}(s). \end{cases}$$

Let now focus on g_{N+2} the approximate transfer function of a delay given by the $N+3$ first Legendre polynomials.
Step 1 : Elementary operations.
To begin with, two operations have been conduced.
The first one consists in adding to the last column the previous column. The second one consists in removing the previous row to the last row. These operations have been chosen to fill in with zeros matrix $(s\mathcal{I}_{N+2}^{-1} - \mathcal{L}_{N+2})$ and simplify the inversion. As elementary operations, it is equivalent to multiply on the right by E_N and on the left by F_N where

$$E_N = \begin{bmatrix} I_{N+1} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_N = \begin{bmatrix} I_{N+1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

are two invertible matrices of determinant equal to 1. Hence, from the structure of $\mathcal{L}_{N+2}$,

$$\begin{aligned} g_{N+2}(s) &= \mathbf{1}_{N+2}^{*T} E_N \left(F_N(s\mathcal{I}_{N+2}^{-1} - \mathcal{L}_{N+2})E_N\right)^{-1} F_N \mathbf{1}_{N+2}, \\ &= \begin{bmatrix}\mathbf{1}_{N+1}^{*T} & 0\end{bmatrix} \begin{bmatrix} (s\mathcal{I}_{N+1}^{-1} - \mathcal{L}_{N+1}) & \begin{matrix} 0 \\ 2\alpha_N^{1/2}(s) \end{matrix} \\ 0 \ \ -2\alpha_N^{1/2}(s) & 2\beta_N(s) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{1}_{N+1} \\ 0 \end{bmatrix}, \end{aligned}$$

with $\alpha_N(s) = \left(\frac{sh}{2(2N+3)}\right)^2$ and $\beta_N(s) = 1 - \frac{sh}{(2N+5)(2N+3)}$.
Step 2 : Expression of the denominator of g_{N+2}.
Developing the determinant yields

$$\det \begin{bmatrix} (s\mathcal{I}_{N+1}^{-1} - \mathcal{L}_{N+1}) & \begin{matrix} 0 \\ 2\alpha_N^{1/2}(s) \end{matrix} \\ 0 \ \ -2\alpha_N^{1/2}(s) & 2\beta_N(s) \end{bmatrix} = \begin{matrix} 4\alpha_N(s)\det(s\mathcal{I}_N^{-1} - \mathcal{L}_N) \\ +2\beta_N(s)\det(s\mathcal{I}_{N+1}^{-1} - \mathcal{L}_{N+1}) \end{matrix}.$$

Then, by induction hypothesis,
$\det(s\mathcal{I}_{N+2}^{-1} - \mathcal{L}_{N+2}) = 2^{N+2}\big(\alpha_N(s)d_N(s) + \beta_N(s)d_{N+1}(s)\big)$,
and recognising the Padé $(N, N+1)$ three-term identities given in Baker [1975],

$$\det(s\mathcal{I}_{N+2}^{-1} - \mathcal{L}_{N+2}) = 2^{N+2} d_{N+2}(s).$$

Step 3 : Expression of the numerator of g_{N+2}.
The numerator is given by

$$\begin{bmatrix}\mathbf{1}_{N+1}^{*T} & 0\end{bmatrix} \mathrm{adj} \begin{bmatrix} (s\mathcal{I}_{N+1}^{-1} - \mathcal{L}_{N+1}) & \begin{matrix} 0 \\ 2\alpha_N^{1/2}(s) \end{matrix} \\ 0 \ \ -2\alpha_N^{1/2}(s) & 2\beta_N(s) \end{bmatrix} \begin{bmatrix} \mathbf{1}_{N+1} \\ 0 \end{bmatrix}.$$

Focusing on the upper left part of the adjugate matrix to get rid of terms multiplied by 0, the numerator is equal to

$$\mathbf{1}_{N+1}^{*T} \begin{pmatrix} 4\alpha_N(s) \begin{bmatrix} \mathrm{adj}(s\mathcal{I}_N^{-1} - \mathcal{L}_N) & 0 \\ 0 & 0 \end{bmatrix} \\ +2\beta_N(s)\mathrm{adj}(s\mathcal{I}_{N+1}^{-1} - \mathcal{L}_{N+1}) \end{pmatrix} \mathbf{1}_{N+1}.$$

Hence, developing and by induction hypothesis,

$$\begin{aligned} &\mathbf{1}_{N+2}^{*T}\mathrm{adj}(s\mathcal{I}_{N+2}^{-1} - \mathcal{L}_{N+2})\mathbf{1}_{N+2} \\ &\qquad = \begin{bmatrix} 4\alpha_N(s)\mathbf{1}_N^{*T}\mathrm{adj}(s\mathcal{I}_N^{-1} - \mathcal{L}_N)\mathbf{1}_N \\ +2\beta_N(s)\mathbf{1}_{N+1}^{*T}\mathrm{adj}(s\mathcal{I}_{N+1}^{-1} - \mathcal{L}_{N+1})\mathbf{1}_{N+1} \end{bmatrix}, \\ &\qquad = 2^{N+2}\big(\alpha_N(s)n_N(s) + \beta_N(s)n_{N+1}(s)\big), \\ &\qquad = 2^{N+2} n_{N+2}(s). \end{aligned}$$

To sum up, if the the numerator and denominator of g_N are respectively equal to those of the Padé approximant fraction to a factor 2^N at order N and $N+1$, for any $N \in \mathbb{N}$, then it is also true at order $N+2$. To conclude, if the equations hold for $N = 0$ and $N = 1$ as shown in the main text, by mathematical induction, for any $s \in \mathbb{C}$,

$$\begin{cases} \mathbf{1}_N^{*T}\mathrm{adj}(s\mathcal{I}_N^{-1} - \mathcal{A}_N)\mathbf{1}_N = 2^N n_N(s) \\ \det(s\mathcal{I}_N^{-1} - \mathcal{A}_N) = 2^N d_N(s) \end{cases}, \quad \forall N \in \mathbb{N}.$$