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Fractional Interval Observers And Initialization Of Fractional Systems

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Abstract

In this paper an interval observer is synthesized for fractional linear systems with additive noise and disturbances. The contribution of system whole past to future output is taken into account as an initialization function. Provided the initialization function is upper and lower bounded, it is shown in this paper that the fractional interval observer (FIO) allows to bound pseudo-state free responses by an upper and a lower trajectory. In case interval observers cannot be synthesized straightforwardly, so as to obtain a stable and non-negative estimation error, it is shown that a change of coordinates allows to overcome this problem. The proposed methodology allows to bound fractional systems trajectories when the whole past is unknown but can be bounded. Finally, a numerical example is given to show the effectiveness of the proposed methods on the initialization of fractional linear systems.

Keywords: Fractional systems, Initialization, Interval observers; Stability of fractional linear systems.

1. Introduction

Fractional calculus has attracted increasing interests during the last decades [1, 2, 3] and the number of applications in widespread fields of science where fractional calculus is used is in rapid growth. This is due to the fact that real processes could be elegantly modeled by fractional differential equations. Diffusion processes [4, 5], biological systems [6], medicine [7], financial markets [8] and many other physical systems can be expressed in terms of fractional-order differential equations.

The state estimation of fractional systems is a widely explored field of investigation. Many theories and results regarding observer design of fractional linear and nonlinear systems exist. This field of research is motivated by the fact that, in many real cases, the problem of immeasurable pseudo-states can be solved by designing an observer that estimates the system behaviour. There exists many approaches dealing with pseudo-state

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estimation of fractional systems like fractional Kalman filters [9, 10] and Luenberger-based fractional observers [11]. Due to the presence of uncertainties and noises in some situations, many ordinary approaches for designing classical observers become obsolete. A technique based on the notion of interval observers allows to cope with uncertainties and noises affecting the system. By interval estimation we understand an observer that, using input-output measurements, evaluates the set of admissible values for the pseudo-states. There are several approaches to the construction of interval observers for rational systems [12, 13, 14].

Most of the results in observer design for fractional systems are based on Caputo's definition of fractional derivative. The considered initial conditions are fixed at initial time t_0 and the effect of past history, which has a profound impact on fractional system behavior, is neglected. However in [15, 16, 17], it was demonstrated that neither Caputo nor Rieman-Liouville definitions allow taking into account correctly initial conditions. In [18], the authors propose to use an initialization function which takes into account the whole past. In [17], a solution based on an equivalence principle between a fractional differential equations and an infinite dimensional ordinary differential equation is proposed.

The main contribution of this work is to construct fractional interval observers for fractional linear systems subject to bounded noises and disturbances. The whole past of the fractional system is used to estimate the pseudo-states. This technique allows to properly initialize a fractional system. In fact, an initialization function is used to encode the total information of system past. Bounding this initialization function in an interval allows to bound system response.

The paper is organized as follows. Section II recalls some preliminary definitions related to continuous-time fractional systems and their discretized version. A technique for constructing FIO in the discrete-time is presented in section III. In section IV, it is shown that an FIO could be constructed for fractional linear systems through change of coordinates. The effectiveness of the new results are illustrated numerically in section V. Finally, concluding remarks are given in section VI.

2. Preliminaries

2.1. Definitions related to fractional systems

The concept of differentiation to an arbitrary order (non-integer), was defined in the 19th century. One of the main contributions to the establishment of the definition is due to Grünwald and Letnikov. They extend differentiation by using not only integer but also non-integer (real or complex) orders on the basis of the backward difference method, generalized to fractional orders. The $\nu \in \mathbb{R}$ fractional order derivative of a continuous time function $x(t) \in \mathbb{R}$, when $\nu \in \mathbb{R}^+$ is defined as [19]:

$$\begin{aligned} D^\nu x(t) &= \lim_{h \to 0^+} \left(\frac{x(t) - x(t-h)}{h} \right)^\nu \\ &= \lim_{h \to 0^+} \frac{1}{h^\nu} \sum_{j=0}^{\infty} (-1)^j \binom{\nu}{j} x(t - jh) \end{aligned} \tag{1}$$

where the Newton binomial $\binom{\nu}{j}$ is defined either by

$$\binom{\nu}{j} = \begin{cases} 1 & \text{for} \quad j = 0 \\ \frac{\nu(\nu-1)\ldots(\nu-j+1)}{j!} & \text{for} \quad j > 0 \end{cases} \tag{2a}$$

$$\binom{\nu}{j} = \begin{cases} 1 & \text{for} \quad j = 0 \\ \frac{\Gamma(\nu+1)}{\Gamma(j)\Gamma(\nu-j+1)} & \text{for} \quad j > 0 \end{cases} \tag{2b}$$

where $\Gamma(j+1) = j!$ is the gamma function.

As shown in (1), a fractional derivative of a function depends on its whole past when ν is non integer.

A more concise algebraic tool can be used to represent such fractional systems: the Laplace transform. The Laplace transform of a ν^{th} order derivative ($\nu \in \mathbb{R}_+$) of $x(t)$ relaxed at $t = 0$ (i.e. $x(t)$ equals zero for all $t < 0$) is given by (see e.g. [20]):

$$\mathscr{L}\{D^\nu x(t)\} = s^\nu \mathscr{L}\{x(t)\} \tag{3}$$

where $s \in \mathbb{C}$ is the Laplace variable. The multivalued function $s \to s^\nu$ becomes holomorphic in the complement of its branch cut line of the complex plane, chosen to be along the negative real axis $\mathbb{R}_-$, including the branching point 0 and ∞.

2.2. Continuous-time fractional models

A continuous-time fractional linear system can be described by the pseudo-state space representation

$$\begin{cases} D^\nu x(t) = & Ax(t) + Bu(t) + Gw(t) \\ y(t) = & Cx(t) + v(t) \\ x(t) = & f(t) \quad t \leq 0 \end{cases} \tag{4}$$

with $0 < \nu < 2$. The results of this paper are restricted to differentiation orders $0 < \nu < 1$, for reasons presented later. $x(t) \in \mathbb{R}^n$ is the state vector and $y(t) \in \mathbb{R}^m$ is the output vector. A, B, C and G are constant matrices with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{m\times n}$. The input $u(t)$ is known, $w(t)$ and $v(t)$ are bounded disturbance and noise respectively. $f(t)$ corresponds to the initialization function which takes into account the whole past as pointed out in [15, 17, 21, 22, 16]. The pseudo-state space fractional system (4) has been utilized in (reference 1) (reference 2).

Stability of fractional systems was treated in different contexts (linear, non linear, commensurate, non commensurate, time-variant, time invariant, delayed, non delayed, analytical, numerical). A complete state of the art, with more than 20 references on the subject is proposed in [15] .

The bounded-input-bounded-output (BIBO) stability is defined as the $\mathscr{L}_\infty$ stability [23]. In [24], authors proved that the fractional system in the form of (4) is BIBO-stable if and only if the A-matrix has no eigenvalue outside the stability domain. The most well-known

BIBO-stability condition is [25, theorem 2.21], extended to take into account variations of $\nu \in (0, \infty)$ in [26].

Theorem 1 [25]: The system described by (4) is BIBO stable if and only if

$$0 < \nu < 2 \tag{5}$$

and

$$|\arg(\text{spec}(\text{A}))| > \nu\frac{\pi}{2} \tag{6}$$

$\square$

For the sake of observer design, the observability check of pseudo-state is necessary. The concept of observability of fractional linear systems is widely discussed in the literature [27, 28, 29]. The problem of observer design remains an open problem because the representation (4) does not correspond to a state-space representation as the knowledge of $x(0)$ is not sufficient to initialize fractional systems [30, 18]. That's why the representation (4) is named pseudo-state space.

Moreover, most of the results on the observability of fractional systems are based on Caputo's definition, and this definition does not allow to properly initialize a fractional system. A further discussion on the observability of such a class of systems is presented in [30].

2.3. *Discrete-time fractional models*

Consider the continuous-time linear system described in (4). This model can be expressed in the discrete time using forward difference approximation of the derivative of order ν of $x(t)$ [31]. Consider a sampling period h, then for $kh \leq t \leq (k+1)h$, the fractional order $D^\nu x(t)$ can be approximated by dropping of the limit in (1) which yields the Grünwald Letnikov approximation

$$D^\nu x(t) \approx \Delta^\nu x((k+1))h = \left(\frac{x((k+1)h) - x(kh)}{h}\right)^\nu \tag{7}$$

$$D^\nu x((k+1)h) \approx \Delta^\nu x_{k+1} = \frac{1}{h^\nu}\sum_{j=0}^{\infty}(-1)^j \binom{\nu}{j} x_{k+1-j} \tag{8}$$

In doing so, the error term is proportional to the sampling period [19]:

$$D^\nu x(t) = \Delta^\nu x((k+1)h) + O(h) \tag{9}$$

which consequently should be small enough for the approximation error to be negligible.

One can then write

$$\Delta^\nu x((k+1)h) \approx Ax(kh) + Bu(kh) + Gw(kh) \tag{10}$$

which yields the linear discrete-time fractional model

$$\Delta^{\nu} x_{k+1} = A x_k + B u_k + G w_k \tag{11}$$

Substituting (8) into (11) and isolating the term x_{k+1} yields

$$x_{k+1} + \sum_{j=1}^{\infty} (-1)^j \binom{\nu}{j} x_{k+1-j} = h^{\nu} (A x_k + B u_k + G w_k) \tag{12}$$

Then, by taking the term x_k, corresponding to $j = 1$, out of the sum:

$$x_{k+1} = \tilde{A} x_k + \tilde{B} u_k + \tilde{G} w_k + \sum_{j=2}^{\infty} (-1)^{j+1} \binom{\nu}{j} x_{k+1-j} \tag{13}$$

where

$$\begin{cases} \tilde{A} = h^{\nu} A + \nu I_n \\ \tilde{B} = h^{\nu} B \\ \tilde{G} = h^{\nu} G \end{cases}$$

Finally, the discretized pseudo-state system is reformulated as:

$$\begin{cases} x_{k+1} &= \tilde{A} x_k + \tilde{B} u_k + \tilde{G} w_k + \mathcal{R}^{+} + \mathcal{R}^{-} \\ y_k &= C x_k + v_k \\ x_k &= f_k \quad \text{for} \quad k \leq 0 \end{cases} \tag{14}$$

where

$$\begin{cases} \mathcal{R}^{+} &= \sum_{j=2}^{k} (-1)^{j+1} \binom{\nu}{j} x_{k+1-j} \\ \mathcal{R}^{-} &= \sum_{j=k+1}^{\infty} (-1)^{j+1} \binom{\nu}{j} f_{k+1-j} \end{cases} \tag{15}$$

correspond respectively to the influence of positive time samples and non-positive time samples (the latter are due to the initialization function).

Definition 1 **(Stability)**: A critical point is said to be stable if every solution which is initially close to it remains close to it for all times. It is said to be asymptotically stable, if it is stable and every solution which is initially close to it converges to it as $t \to \infty$. □

In the case of autonomous linear systems, the only critical point is the origin.

Definition 2 **(Asymptotic stability of linear systems)**: An autonomous initialized linear system (rational or fractional) is asymptotically stable if and only if the origin is an asymptotic stable point for that system (i.e. system free response converges asymptotically to zero). □

Next, condition for asymptotic stability of the system (13) is formulated in terms of eigenvalues of the $\tilde{A}$ matrix [32].

Theorem 2 [32]: The fractional system (13) is asymptotically stable if and only if all eigenvalues of the matrix $\tilde{A} = h^\nu A + \nu I$ are located inside the stability region $S(\nu)$ which boundary is defined by the parametric representation

$$\overline{S}(\nu) = \nu + (e^{jw} - 1)^\nu (e^{jw})^{1-\nu}, \quad \omega \in [0, 2\pi] \tag{16}$$

$\square$

Such a boundary is plotted in Fig.2 for $\nu = 0.5$.

Definition 3 **(Observability)**: A system with an initial state x_{k_0}, where k_0 is the initial iteration, is observable if and only if the value of the initial state can be determined from the system output $y(k)$ that has been observed through the interval $k_0 < k < k_f$. If the initial state cannot be so determined, the system is unobservable. $\square$

Definition 4 **(Full state observability)** : A system is said to be completely observable if all the possible initial states of the system can be observed. If this criterion is not satisfied, the system is said to be unobservable. $\square$

A useful criterion is now presented to check the observability of fractional discrete-time systems described by (13).

Theorem 3 [33]: The discretized fractional pseudo-state-space vector is observable if and only if there exists a finite time N such that $\text{rank}(O_N) = n$, where n is the dimension of the pseudo-state vector and

$$O_N = \begin{bmatrix} C \\ C\Phi_0 \\ \vdots \\ C\Phi_{N-1} \end{bmatrix} \tag{17}$$

where

$$\Phi_k = \begin{cases} I_n & \text{for} \quad k = 0 \\ \sum_{j=0}^{k-1} (-1)^j \binom{\nu}{j} c_j \Phi_{k-1-j} & \\ & \text{for} \quad k \geq 1 \end{cases} \tag{18}$$

and

$$\begin{cases} c_j & = (-1)^{j+1} \binom{\nu}{j} \\ c_0 & = (A - I_n) + c_1 I \end{cases} \tag{19}$$

$\square$

3. Problem formulation

Aoun and Raïssi [34] attempted to synthesize fractional observers on the basis of Caputo derivative. They derived two trajectories $\underline{x}(t)$ and $\overline{x}(t)$ which allow bounding the pseudo-states, starting from an unknown initial condition $x(t_0)$:

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t), \ \forall \ t > t_0 \tag{20}$$

Nonetheless, it is well known [17, 21, 22, 16] that the initialization of fractional systems does not obey to Caputo definition but must account on the whole past. On the other hand, the use of Grünwald Letnikov approximation provides the entire past of the system.

The main contribution of this paper is to implement fractional observers by considering properly the initial conditions. Discrete time approximation is employed in this paper to grant the use of the whole past of the system.

Starting from an initialization function $\underline{f}_k \leq f_k \leq \overline{f}_k \ \forall \ k \leq 0$, the goal is to bound the free response of FDE by an upper and a lower trajectory $\underline{x}_k$ and $\overline{x}_k$ (Fig.1) such that:

$$\underline{x}_k \leq x_k \leq \overline{x}_k, \ \forall \ k > 0 \tag{21}$$

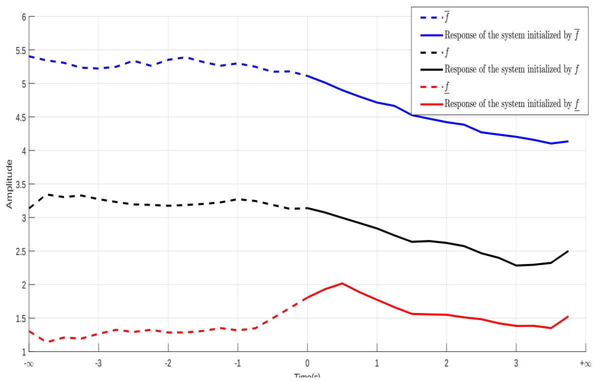

Figure 1: Initialization of fractional system with lower and upper trajectories

4. Synthesis of Fractional Interval Observers

Some useful lemmas are introduced before presenting the main result of the paper.

Lemma 2: If $0 < \nu < 1$, then

$$(-1)^{j+1} \binom{\nu}{j} > 0, \quad j = 1, 2, \ldots \tag{22}$$

Proof by induction: The hypothesis is true for $j = 1$, since

$$(-1)^{1+1} \binom{\nu}{1} = \nu > 0$$

Assuming that $(-1)^{j+1}\begin{pmatrix}\nu\\ j\end{pmatrix} > 0$ for $j > 1$, the hypothesis must remain valid for $j+1$. From (2), we have

$$\begin{aligned}(-1)^{j+2}\tbinom{\nu}{j} &= (-1)^{j+2}\ \frac{\nu(\nu-1)\dots(\nu-j+1)(\nu-j)}{(j+1)!}\\ &= (-1)^{j+1}\tbinom{\nu}{j}\frac{j-\nu}{j+1} > 0\end{aligned}$$

$\square$

Remark 1 : Lemma 2 is not valid when $1 < \nu < 2$. That's why the results of the paper are limited to the case when $0 < \nu < 1$. However, when $1 < \nu < 2$, differentiation similarity transformation can be applied [35] and an equivalent system of dimension $2n$ can be obtained with a commensurate order equal to $\frac{\nu}{2}$.

Lemma 3 [13]: Consider a vector $x \in \mathbb{R}^n$ such that $\underline{x} \leq x \leq \overline{x}$ for $\underline{x},\ \overline{x} \in \mathbb{R}^n$. If $M \in \mathbb{R}^{m\times n}$ is a constant matrix, then:

$$M^+\underline{x} - M^-\overline{x} \leq Mx \leq M^+\overline{x} - M^-\underline{x} \tag{23}$$

with

$$M^+ = \max(0, M) \tag{24}$$

$$M^- = M^+ - M \tag{25}$$

$$|M| = M^+ + M^- \tag{26}$$

$\square$

The main result is now presented.

Theorem 4: Consider the discretized fractional system (14), with a differentiation order $0 < \nu < 1$ and a bounded initialization function

$$\underline{f}_k \leq f_k \leq \overline{f}_k\ \forall\ k \leq 0 \tag{27}$$

Suppose that noises and disturbances are bounded, i.e $|v_k| \leq V$ and $|w_k| \leq W$. If there exists a gain L such that

(i) $\mathcal{A} = (h^\nu(A - LC) + \nu I_n)$ is non-negative and

(ii) all eigenvalues of $\mathcal{A}$ are located in the stability region $S(\nu)$

then a fractional interval observer for the fractional system (14) is

$$\overline{x}_{k+1} \quad = \mathcal{A}\overline{x}_k + \tilde{B}u_k + \overline{\Phi}_k \tag{28a}$$

$$\underline{x}_{k+1} \quad = \mathcal{A}\underline{x}_k + \tilde{B}u_k + \underline{\Phi}_k \tag{28b}$$

where

$$\begin{cases} \overline{\Phi}_k & = |\mathcal{G}|W + \mathcal{L}y_k + |\mathcal{L}|V + \overline{\mathcal{R}}^+ + \overline{\mathcal{R}}^- \\ \underline{\Phi}_k & = -|\mathcal{G}|W + \mathcal{L}y_k - |\mathcal{L}|V + \underline{\mathcal{R}}^+ + \underline{\mathcal{R}}^- \end{cases},$$

$$\begin{cases} \overline{\mathcal{R}}^+ = \sum_{j=2}^{k}(-1)^{j+1}\binom{\nu}{j}\overline{x}_{k+1-j} \\ \overline{\mathcal{R}}^- = \sum_{j=k+1}^{\infty}(-1)^{j+1}\binom{\nu}{j}\overline{f}_{k+1-j} \\ \underline{\mathcal{R}}^+ = \sum_{j=2}^{k}(-1)^{j+1}\binom{\nu}{j}\underline{x}_{k+1-j} \\ \underline{\mathcal{R}}^- = \sum_{j=k+1}^{\infty}(-1)^{j+1}\binom{\nu}{j}\underline{f}_{k+1-j} \end{cases}$$

and

$$\begin{cases} \tilde{B} = h^\nu B \\ \mathcal{L} = h^\nu L \\ |\mathcal{G}| = h^\nu |G| \\ |\mathcal{L}| = h^\nu |L| \end{cases}$$

$h \in \mathbb{R}^{*+}$ is the sampling period ; $0 < \nu < 1$. $\square$

Proof: Consider the observer error $\overline{\varepsilon} = \overline{x} - x$. Starting with an initialization function $\underline{f}_k \leq f_k \leq \overline{f}_k \ \forall \ k \leq 0$ and based on (28a) and (14), the dynamics of $\overline{\varepsilon}$ is described by

$$\begin{aligned} \overline{\varepsilon}_{k+1} \;=\;& \mathcal{A}\overline{x}_k + \tilde{B}u_k + |\mathcal{G}|W + \mathcal{L}y_k + |\mathcal{L}|V + \overline{\mathcal{R}}^+ \\ +\;& \overline{\mathcal{R}}^- - (\tilde{A}x_k + \tilde{B}u_k + \tilde{G}w_k + \mathcal{R}^+ + \mathcal{R}^-) \end{aligned} \tag{29}$$

Finally, the observer error is expressed by

$$\overline{\varepsilon}_{k+1} \quad = \quad \mathcal{A}\overline{\varepsilon}_k + \mathcal{M}_k + \overline{\Upsilon}^+ + \overline{\Upsilon}^- \tag{30}$$

where

$$\overline{\varepsilon}_k = \overline{x}_k - x_k$$

$$\mathcal{M}_k = (|\mathcal{L}|V + \mathcal{L}v_k) + (|\mathcal{G}|W - \tilde{G}w_k) \tag{31}$$

$$\begin{cases} \overline{\Upsilon}^+ & = \sum_{j=2}^{k}(-1)^{j+1}\binom{\nu}{j}(\overline{f}_{k+1-j} - f_{k+1-j}) \\ \overline{\Upsilon}^- & = \sum_{j=k+1}^{\infty}(-1)^{j+1}\binom{\nu}{j}(\overline{f}_{k+1-j} - f_{k+1-j}) \end{cases} \tag{32}$$

The error dynamics (30) needs be non negative and stable.

Proof of the non-negativity of (30):
Since initial conditions are assumed to verify $\underline{f}_k \leq f_k \leq \overline{f}_k \ \forall \ k \leq 0$, and according to Lemma 2, the following inequality holds:

$$\overline{\Upsilon}^- \quad > 0 \tag{33}$$

As the gain L is designed such that $\mathcal{A} = h^\nu(A - LC) + \nu I_n \in \mathbb{R}_+^{n\times n}$ and by construction $|\mathcal{L}|V + \mathcal{L}v_k \geq 0$, $|\mathcal{G}|W - \tilde{G}w_k \geq 0$ and $\overline{\Upsilon}^+ > 0$, then the dynamics of $\bar{\varepsilon}$ is positive, i.e. $\overline{\varepsilon} = \overline{x}_k - x_k \geq 0$.

Proof of the asymptotic stability of (30):
The asymptotic stability follows straightforwardly from Condition (ii).

Similarly, the same methodology can be applied to prove that $\underline{e}_k = x - \underline{x} \geq 0$, and that $\underline{\varepsilon}$ is stable. ■

5. Synthesis of FIO with a change of coordinates

It is sometimes impossible to find a matrix L satisfying both conditions (i) and (ii) of *Theorem 4*. In such a case, a change of coordinates $z = Px$ is performed, where $P \in \mathbb{R}^{n\times n}$ is a nonsingular transformation matrix which allows satisfying both conditions (i) and (ii) of *Theorem 4*. In the new coordinates, L should be chosen such that all eigenvalues of $h^\nu(A-LC)+\nu I_n$ are located in the stability region $S(\nu)$ and $\mathcal{A}^* = P((h^\nu A+\nu I_n)-Lh^\nu C)P^{-1}$ is nonnegative. The existence condition of such a real transformation matrix P is given in the following Lemma:

Lemma 4 [12]: Given the matrices $A \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, if there exists a matrix $L \in \mathbb{R}^{n\times p}$ such that $\text{eig}(A - LC) = \text{eig}(R)$ and there exit vectors $e_1 \in \mathbb{R}^n$ and $e_2 \in \mathbb{R}^n$ such that the pairs $(A - LC, e_1)$ and (R, e_2) are observable, then there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that $R = P(A - LC)P^{-1}$. □

This Lemma is applied in [12, 13] to construct interval observers for continuous-time LTI systems, where R is a Metzler matrix. In [13], a discussion is proposed on the existence of a real and nonsingular matrix P and many techniques are recalled therein. In case where $A - LC$ has only real positive eigenvalues, then R is chosen as a diagonal or a Jordan representation of $A - LC$ and P is a constant matrix. In [36], a unified time-varying change of variable is proposed based on the Jordan decomposition of the matrix A in $\mathbb{C}$.

Next, given a gain L such that all eigenvalues of $h^\nu(A - LC) + \nu I_n$ are located in the stability region $S(\nu)$ and given the change of coordinates $z = Px$ such that $P((h^\nu A+\nu I_n) - Lh^\nu C)P^{-1}$ is nonnegative, an interval observer for (14) in the z-coordinates is given in the following theorem.

Theorem 5: Consider the discretized fractional system (14) initialized by f_k verifying (27). Suppose that noises and disturbances are bounded, i.e $|v(t)| \leq V$ and $|w| \leq W$. If there exist a matrix $P \in \mathbb{R}^{n\times n}$ and a gain L such that

(i) $\mathcal{A}^* = P(h^\nu(A - LC) + \nu I_n)P^{-1}$ is non-negative

(ii) all eigenvalues of $\mathcal{A}^*$ are located in the stability region $S(\nu)$

then a fractional interval observer for the fractional system (14), in the $z = Px$ coordinates, is

$$\overline{z}_{k+1} = \mathcal{A}^*\overline{z}_k + P\tilde{B}u_k + \overline{\Psi}_k \tag{34a}$$

$$\underline{z}_{k+1} = \mathcal{A}^*\underline{z}_k + P\tilde{B}u_k + \underline{\Psi}_k \tag{34b}$$

where

$$\begin{cases} \overline{\Psi}_k &= |P\mathcal{G}|W + P\mathcal{L}y_k + |P\mathcal{L}|V + \overline{\mathcal{R}}^+ + \overline{\mathcal{R}}^- \\ \underline{\Psi}_k &= -|P\mathcal{G}|W + P\mathcal{L}y_k - |P\mathcal{L}|V + \underline{\mathcal{R}}^+ + \underline{\mathcal{R}}^- \end{cases}$$

$$\begin{cases} \overline{\mathcal{R}}^+ &= \sum_{j=2}^{k}(-1)^{j+1}\binom{\nu}{j}\overline{z}_{k+1-j} \\ \overline{\mathcal{R}}^- &= \sum_{j=k+1}^{\infty}(-1)^{j+1}\binom{\nu}{j}\overline{z}_{k+1-j} \\ \underline{\mathcal{R}}^+ &= \sum_{j=2}^{k}(-1)^{j+1}\binom{\nu}{j}\underline{z}_{k+1-j} \\ \underline{\mathcal{R}}^- &= \sum_{j=k+1}^{\infty}(-1)^{j+1}\binom{\nu}{j}\underline{z}_{k+1-j} \end{cases}$$

with

$$\begin{cases} \overline{z}_k &= P^+\overline{f}_k - P^-\underline{f}_k \quad ,\forall\ k \leq 0 \\ \underline{z}_k &= P^+\underline{f}_k - P^-\overline{f}_k \quad ,\forall\ k \leq 0 \end{cases} \tag{35}$$

and

$$\begin{cases} \tilde{B} = h^\nu B \\ \mathcal{L} = h^\nu L \\ |\mathcal{G}| = h^\nu|G| \\ |\mathcal{L}| = h^\nu|L| \end{cases}$$

$h \in \mathbb{R}^{*+}$ is the sampling period ; $0 < \nu < 1$. P^+ and P^- are obtained by applying (24) and (13).

Back in the x-coordinates, the interval observer of (14) is given by:

$$\begin{cases} \overline{x}_k &= Q^+\overline{z}_k - Q^-\underline{z}_k \\ \underline{x}_k &= Q^+\underline{z}_k - Q^-\overline{z}_k \end{cases} \tag{36}$$

where $Q = P^{-1}$ and Q^{+} and Q^{-} are obtained by applying (24) and (13) $\square$

Proof: The system (14) can be written as

$$\left\{\begin{array}{lcl} z_{k+1} & = & P\tilde{A}Qz_k + P\tilde{B}u_k + P\tilde{G}w_k + \mathcal{R}^{+} + \mathcal{R}^{-} \\ y_k & = & CQz_k + v_k \\ z_k & = & Qf_k \;\text{ for }\; k \leq 0 \end{array}\right. \tag{37}$$

Consider the observer error $\overline{\varepsilon}^z = \overline{z} - z$. Based on (37) and (34a), the dynamics of $\overline{\varepsilon}^z$ is given by

$$\begin{array}{rcl} \overline{\varepsilon}^z_{k+1} & = & \mathcal{A}^*\overline{z}_k + P\tilde{B}u_k + |P\mathcal{G}|W + P\mathcal{L}y_k + |P\mathcal{L}|V + \overline{\mathcal{R}}^{+} + \overline{\mathcal{R}}^{-} \\ & & -(P\tilde{A}Qz_k + P\tilde{B}u_k + P\tilde{G}w_k + \mathcal{R}^{+} + \mathcal{R}^{-}) \\ & = & \mathcal{A}^*\overline{z}_k + |P\mathcal{G}|W - P\tilde{G}w_k + |P\mathcal{L}|V + PLv_k + \overline{\mathcal{R}}^{+} - \mathcal{R}^{+} \\ & & +\overline{\mathcal{R}}^{-} - \mathcal{R}^{-} \end{array} \tag{38}$$

The matrix $\mathcal{A}^* = P(h^\nu(A - LC) + \nu I_n)Q$ is nonnegative and by construction $|P\mathcal{L}|V + PLv_k \geq 0$, $|P\mathcal{G}|W - P\tilde{G}w_k \geq 0$, therefore the dynamics of $\overline{\varepsilon}^z$ is positive, i.e $\overline{\varepsilon}^z_k = \overline{z}_k - z_k \geq 0 \;\; \forall\, k$.

Also, the gain L is constructed such that all eigenvalues of $\mathcal{A}^* = P(h^\nu(A-LC)+\nu I_n)Q$ are located in the stability region $S(\nu)$, so that the upper error $\overline{\varepsilon}^z$ is stable.

The same methodology is applied to prove the non negativity and the stability of the lower error bound $\underline{\varepsilon}^z$.

Based on Lemma 3, it can be shown that

$$\overbrace{M^{+}\underline{z} - M^{-}\overline{z}}^{\underline{x}} \leq \overbrace{Mz}^{x} \leq \overbrace{M^{+}\overline{z} - M^{-}\underline{z}}^{\overline{x}} \tag{39}$$

The stability of the upper error $\overline{\varepsilon} = \overline{x} - x$ and the lower error $\underline{\varepsilon} = x - \underline{x}$ is simply deduced from that of $\overline{\varepsilon}^z$ and $\underline{\varepsilon}^z$ which is preserved under the change of coordinates. $\blacksquare$

6. Simulation results

6.1. Example 1

Consider the fractional continuous-time linear system described by the equations

$$\left\{\begin{array}{l} D^\nu x(t) = Ax(t) + Bu(t) + Gw(t) \\ y(t) = Cx(t) + Du(t) + v(t) \end{array}\right. \tag{40}$$

with $\alpha = 0.5$, and

$$A = \begin{bmatrix} -5 & 1.5 \\ 0.7 & -4 \end{bmatrix}; \;\; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}; \;\; D = \; 0$$

$$G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \ | \, w(t) \, | \quad \leq \ 0.1 \qquad ; \ | \, v(t) \, | \quad \leq \ 0.1$$

Consider an unknown initialization function bounded as follows

$$\begin{cases} \overline{f}(t) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} & \text{if} \quad -30 \leq t \leq 0 \\ \overline{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if} \qquad t \leq -30 \\ \underline{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if} \qquad t \leq 0 \end{cases} \tag{41}$$

By choosing $h = 5 \times 10^{-3} s$, the discretized system corresponding to (40) is:

$$x_{k+1} \quad = \tilde{A}x(k) + \tilde{B}u(k) + \tilde{G}w(k) + \mathcal{R}^{+} + \mathcal{R}^{-} \tag{42}$$

with

$$\tilde{A} = \begin{bmatrix} 0.1464 & 0.1061 \\ 0.0495 & 0.2172 \end{bmatrix} ; \ \tilde{B} = \begin{bmatrix} 0.0707 \\ 0.0707 \end{bmatrix} \ \tilde{G} = \begin{bmatrix} 0.0707 & 0 \end{bmatrix}^{T}$$

and $\mathcal{R}^{+}$and $\mathcal{R}^{-}$ are defined in (15).

The gain $L = \begin{bmatrix} 0.8056 & 0.1944 \end{bmatrix}^{T}$ is chosen so that it satisfies both conditions of *Theorem 4*. Indeed

(i) $h^{0.5}(A - LC) + 0.5I$ is a non-negative matrix:

$$h^{0.5}(A - LC) + 0.5I = \begin{bmatrix} 0.0895 & 0.0491 \\ 0.0357 & 0.2034 \end{bmatrix}$$

(ii) all eigenvalues of $h^{0.5}(A - LC) + 0.5I$ are in the stability region $S(0.5)$ as shown in Fig.2.

Case 1: Free response

Using (28) and supposing that the true initialization function is inside the set bound by (41):

$$f(t) \quad = \begin{cases} \begin{bmatrix} 3 \\ 4 \end{bmatrix} & \text{if} \quad -30 \leq t \leq 0 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if} \qquad t \leq -30 \end{cases} \tag{43}$$

then the interval estimation of the free response of x_1 and x_2 (with $u = 0$) are shown in Fig.3.

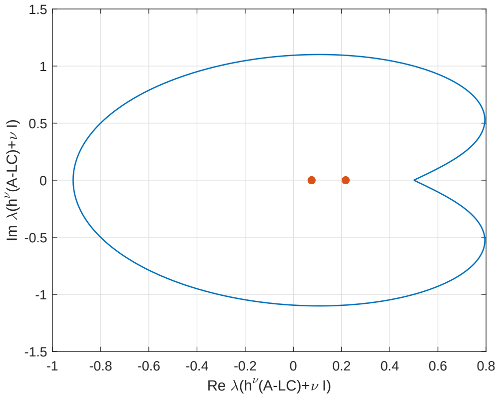

Figure 2: Stability region S(0.5). Eigenvalues(•) of $h^{0.5}(A - LC) + 0.5I$ in the simulation example 1.

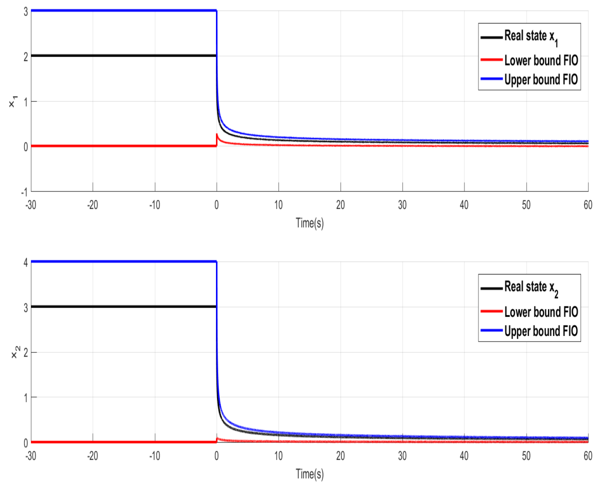

Figure 3: States x_1, x_2 and their interval observers with $u = 0$ and f_k constant $\forall k < 0$.

Case 2: Forced response

The initialization function $f(t)$ is now chosen to be non constant but bounded by constant functions $\underline{f}(t)$ and $\overline{f}(t)$ defined in (41). The input signal is $u_k = \sin(k)$, $k > 0$. Interval estimation for x_1 and x_2 are plotted in Fig.4. The pseudo-states are always inside the estimated bounds. This technique may be regarded as a new alternative for the initialization of fractional systems.

6.2. Example 2

Consider the fractional electrical circuit given in Fig.5. R is the resistance, C and L are fractors. C is a fractional order supercapacitor and L is a fractional order inductance [37].

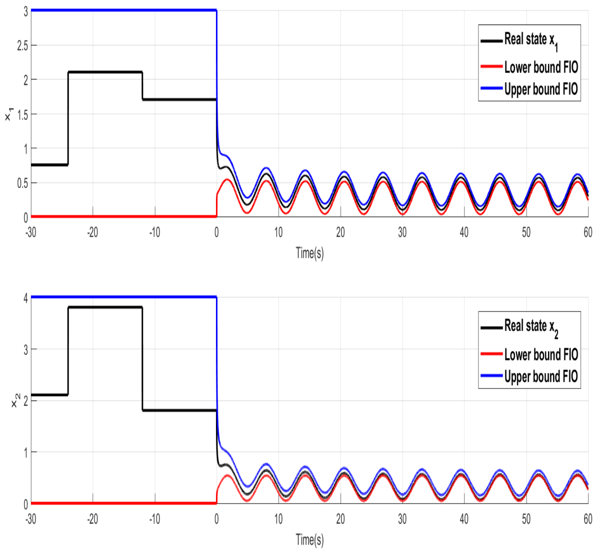

Figure 4: States x_1, x_2 and their lower and upper bounds with $u > 0$

Using Kirchhoff's laws, the obtained fractional differential equations are the following:

$$i(t) = \mathsf{C}\frac{d^{\alpha}u_c(t)}{dt} \tag{44}$$

$$u(t) = \mathsf{R}i(t) + u_c(t) + \mathsf{L}\frac{d^{\beta}i(t)}{dt} \tag{45}$$

Assuming $\nu = \alpha = \beta$ and by measuring only $u_c(t)$, the resulted fractional state-space

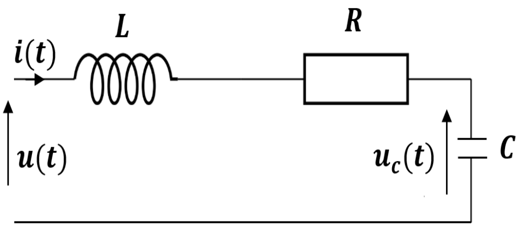

Figure 5: Fractional electrical circuit

representation is:

$$\begin{cases} \begin{bmatrix} u_c(t) \\ i(t) \end{bmatrix}^{\nu} = \begin{bmatrix} 0 & \frac{1}{\mathsf{C}} \\ -\frac{1}{\mathsf{L}} & -\frac{\mathsf{R}}{\mathsf{L}} \end{bmatrix} \begin{bmatrix} u_c(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\mathsf{L}} \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \omega(t) \\ y(t) = u_c(t) + \upsilon(t) \end{cases} \tag{46}$$

$w(t)$ and $v(t)$ are unknown additive disturbance and noise. The unknown upper and lower initialization functions are chosen to be constant and equal to:

$$\begin{cases} \overline{f}(t) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} & \text{if} \quad -10^{-4} \leq t \leq 0 \\ \overline{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if} \qquad t \leq -10^{-4} \end{cases}$$
$$\begin{cases} \underline{f}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \text{if} \quad -10^{-4} \leq t \leq 0 \\ \overline{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if} \qquad t \leq -10^{-4} \end{cases} \tag{47}$$

The following values of components are used in this simulation:

$$\begin{array}{ll} \mathsf{R} = 20\Omega & \mathsf{C} = 600\mu\text{F} \\ \mathsf{L} = 30\text{mH} & \nu = 0.5 \end{array}$$

The following gain L is used.

$$L = \begin{bmatrix} 0.2866 & 0.8196 \end{bmatrix}^T$$

Note that $h^{0.5}(A - LC) + 0.5I_n$ is a negative matrix. The fractional interval observer of *Theorem 5* cannot be applied. A change of coordinates $z = Px$ permits to design fractional interval observer in the z-coordinates and then to properly initialize the fractional system by applying *Theorem 5*. By choosing the following P,

$$P = \begin{bmatrix} 0.2526 & 0.7861 \\ 0.5812 & 0.6027 \end{bmatrix}, \tag{48}$$

both conditions of *Theorem 5* are satisfied:

(i) $P(h^{0.5}(A - LC) + 0.5I)P^{-1} = \begin{bmatrix} 0.4679 & 0.0036 \\ 0.2510 & 0.3829 \end{bmatrix}$

is a non-negative matrix;

(ii) The eigenvalues of $(P(h^{0.5}(A - LC) + 0.5I)P^{-1})$ are in the stability region $S(0.5)$ as shown in Fig.6.

Noises $w(t)$ and $v(t)$ are supposed to be bounded:

$$|W| = |V| = 0.1$$

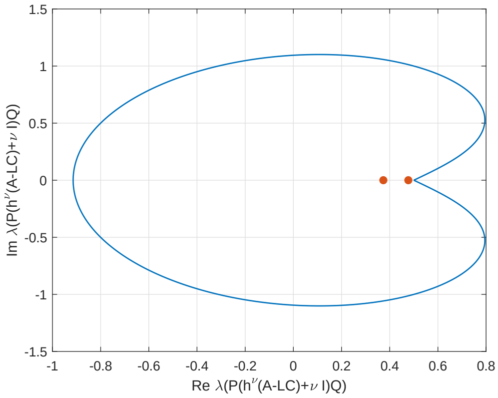

Figure 6: Stability region S(0.5). Eigenvalues(•) of $P(h^{0.5}(A - LC) + 0.5I)P^{-1}$ in the simulation example 2

By choosing an initialization function constant and equal to

$$\begin{cases} f(t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} & \text{if} \quad -10^{-4} \leq t \leq 0 \\ f(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{if} \qquad t \leq -10^{-4} \end{cases} \tag{49}$$

, interval estimation of the free response of z_1 and z_2 (with $u = 0$) in z coordinates are displayed in Fig.7. The bounds of x_1 and x_2 are exposed in Fig.8. It is shown that the measurements evolve inside the estimated bounds in both bases.

7. Conclusion

This paper is devoted to the design of fractional interval observers and initialization of fractional linear systems. The proposed technique is based on the construction of two reliable bounds of the pseudo-state free responses considering the contribution of system whole past as an initialization function. When the observer matrix cannot be found, a change of coordinates is proposed to transform the observation errors into a cooperative form. The proposed methodology allows to initialize fractional systems when the past is unknown but can be bounded. Simulation results show the effectiveness of the proposed methodology.

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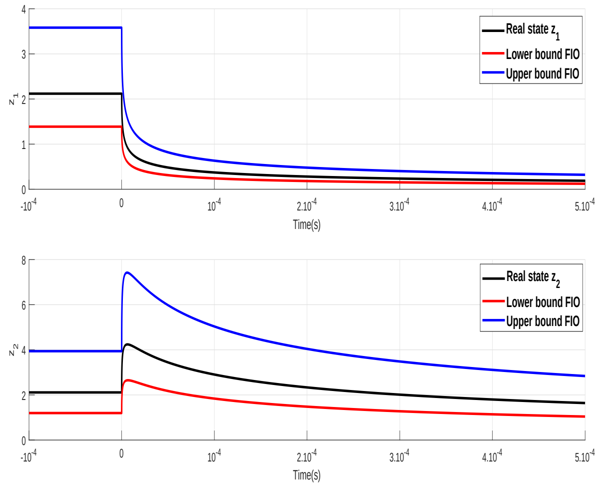

Figure 7: Bounds of the states z_1 and z_2 in z coordinates.

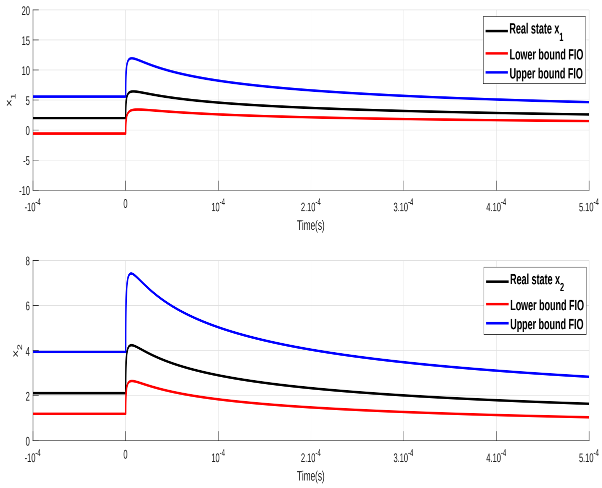

Figure 8: Bounds of the states x_1 and x_2 in x coordinates.

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Review Reply Concerning submission – CNSNS-D-19-01015 – Fractional Interval Observers And Initialization Of Fractional Systems

General

We would like to thank the associate editor and the reviewers for their careful reading during the review process. There comments helped us improving clarity and readability of the manuscript.

All the points raised by the associate editor and the reviewers, have been addressed in the revised version of the manuscript. We give more specific answers (in black) to the comments and the questions (in blue). Descriptions on how the changes are implemented are provided in red in the paper to facilitate the revision process.

We hope the highlighted modifications will satisfy the reviewers and the associate editor and we look forward to their decision.

Yours sincerely,

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Reply written on September 5, 2019

Authors' reply to the associate editor

Q: stands for AE's Question or Comment.
R: stands for authors' response.

Q: Reviewers have now commented on your paper. You will see that they are advising that you revise your manuscript. If you are prepared to undertake the work required, I will send your revised manuscript back to the reviewers for further advice.

R: The AE is gratefully acknowledged for providing us the opportunity to reconsider the paper for publication after revision. Please find below answers to all questions and comments raised by the reviewers.

Author's reply to Reviewer 1

Q: stands for reviewer's Question or Comment.
R: stands for authors' response.

Q: The authors dicuss the discretized pseudo-state system of a continuous-time fractional linear system with bounded disturbance and noise, taking into account the whole past to future output as a bounded initialization function. Their main result is the establishment of a fractional interval observer for fractional differentiation order $0 < \nu < 1$ and the existence of a gain matrix L such that the estimation error stays nonnegative and stable. In case the latter conditions cannot be combined, the authors give an additional theorem which resolves this issue via an appropriate change of coordinates. The two theorems are then illustrated with reference to two corresponding examples. The paper has a clear structure and clear novel results whose usefulness is well exemplified numerically.

R: The reviewer is gratefully acknowledged for his careful reading, for pointing out the main results, the clarity and the novelty of the paper.

Q1. The preliminaries seem very sloppy at certain points. The fractional order derivative is defined for some $x(t)$ withouth making clear what kind of object it is (for example a continuous function). In the definition of $D^{\nu}x(t)$ (1) there is a discretization which is not explained. Additionally, when the Laplace transform is introduced (3), the dependence on s is not made clear and just before there is the nonsensical explanation "t equals zero for all t < 0". On a similar note, it is not explained what the role of $y(t)$ is in system (4). I ask for a mathematically consistent description of these objects and a careful introduction of the concepts.

R1: The reviewer is completely right regarding the definition of $x(t)$, which is now defined as a continuous time function $\in \mathbb{R}$ (check text in red before (1)). Regarding the definition of $D^{\nu}x(t)$ in (1), the discretization was not meant to be present. It was an error of the authors. Please check the definition of (1). Regarding the s variable, it is now clearly defined as the Laplace variable (check the text after equation (3)). Moreover, the quoted sentence has been corrected as '$x(t)$ equals zero for all $t < 0$'. The role of $y(t)$ in system (4) is now explained after equation (4). We hope that the objects are consistently described and that the objects are better introduced.

Q2. Model (4), and thereby also Model (14), are not motivated at all. What is the physical meaning? Why the fractional derivative? What are the roles of disturbance and noise, in particular in relation to similar models?

R2: Model (4) and (14) and now motivated in the last sentence of paragraph after equation (4). All sensors provide measurements with a certain level of noise, which is usually assessed by a signal to noise ratio. Hence, this noise is external to the system (due to the sensor). Disturbances however are noises which may affect the states internal. Our objective is to implement estimation methods robust to measurements noise and disturbances.

Regarding the use of fractional derivative, the authors pointed out some applications of fractional calculus in the first paragaph of the introduction (see the text in red). As shown in Fig. 9 below, the number of contributions related to fractional calculus is very high. It becomes impossible to mention them all. The authors tried to select the most significant ones.

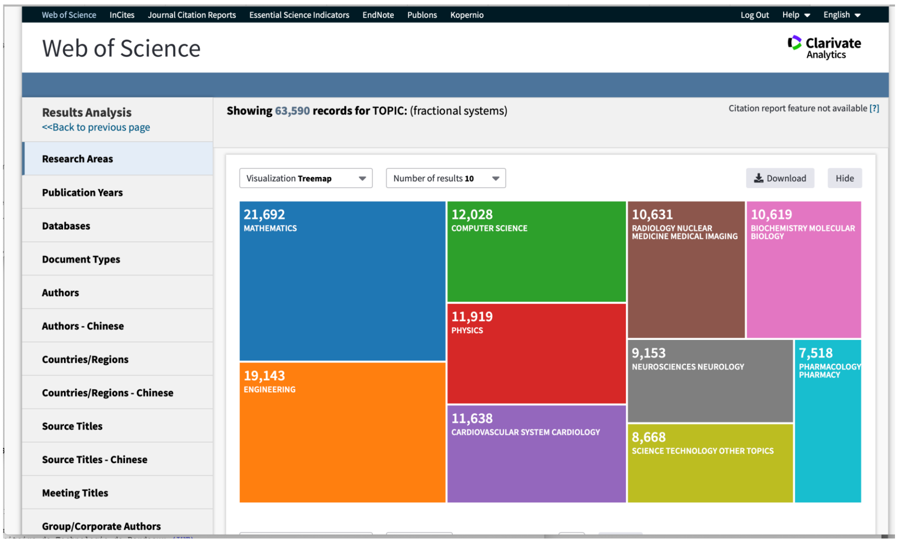

Figure 9: Search on Clarivate Analytics (ex-Thomson-Reuters) with the key words “Fractional systems” from year 1999 to 2019. For example in orange: there were 19143 contributions with the key words "fractional systems" in the field of Engineering from year 1999 to 2019.

Q3: In Theorems 2 and 3 the concepts of asymptotic stability and observability of fractional discrete-time systems are taken for granted. It would improve the readability of the manuscript enormously if such concepts were properly introduced.

R3: As suggested by the reviewer, the asymptotic stability and the observabiliy of linear systems are defined in def 1,2,3,4 (pages 5-6).

Q4: In 2.3, when the discretization is introduced, it would be interesting to have some comment on the accuracy of the Grünwald Letnikov approximation compared to the continuous time fractional derivative.

R4: When the discretization is introduced, the error is proportional to the sampling period as it is now explained after equation (8).

Q5: The problem formulation, Section 3, compares to [34] which considered continuous-time fractional observers on the basis of the Caputo derivative. Then there is a direct jump to the discrete-time problem as object of the subsequent analysis. It would help the reader to be a little more precise about the efforts in [34] and make clear why the Grünwald Letnikov derivative is taken here.

R5: As previously explained by the authors, the use of the Caputo does not allow to account the whole past of the system. However, Grünwald Letnikov approximation allows considering the whole past. In order to consider the whole past, the discrete time approximation of fractional derivative is used in this paper.

Q6: I do not fully understand how Lemma 4 goes into Theorem 5. The matrix R seems to be related to the matrix A? but in Lemma 4, this matrix R is given and P is determined, and in the formulation of Theorem 5, the matrix P is fixed and then A is defined. Could the authors please clarify their reasoning here?

R6: The reviewer is totally right. There is no link between Lemma 4 and Theorem 5. Theorem 5 has been reformulated accordingly. In the paper, Lemma 4 is used to refer to works on rational systems.

Q7. Quite a few editorial comments...

R7: All the editorial comments pointed out by the reviewer have been corrected.

Author's reply to Reviewer 2

Q: stands for reviewer's Question or Comment.
R: stands for authors' response.

Q: Reviewed paper deals with an interval observer which is synthesized for fractional linear systems with additive noise and disturbances. Some corrections are necessary.

R: The reviewer is gratefully acknowledged for his careful reading, encouragements, and guidance for improving the manuscript.

Q1. Since order ν is real, defined in (1), definition of binomial coefficients in form of factorial (2) cannot be used, it is valid only for integer number, not real one, rather use definition by Gamma function.

R1: Since j is an integer, the presented definition of Newton binomial is correct. The authors do not use factorial of ν. The definition of Newton's binomial exists and is an alternative for the previous version of equation (2). As suggested by the reviewer, the two definitions of Newton's binomial are now presented in equation (2).

Q2. Laplace transform (3) is valid for all kind of definitions, but only for zero initial conditions.

R2: The reviewer is completely right. That's why, the authors stated in the first version of the paper :$x(t)$ relaxed at $t = 0$.

Q3. approximation ((7)) is called Grünwald-Letnikov, not Grunwald.

R3: The authors agree fully with the reviewer and the sentence pointed out has been modified. Check the sentence in red.

Q4. Fig.(1) legend is not readable, check also other figures.

R4: As suggested by the reviewer, all figures legend are checked.

Q5. Example 1: why you use different notation for fractional derivative in (40) than notation in (4)?

R5: As advised by the reviewer, a unique notation of fractional derivative is now used in equations (4) and (40).

Q6. check References, e.g. Ref. [19] and [20] are the same, etc.

R5: Authors have now updated and checked all the references.