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Dedicated to A. V. Arkhangel'skii and S.A. Naimpally

Abstract. In the work using interval mathematics, we develop knowledge for cardinal numbers from the viewpoint of uncertainty analysis. In the finite non-T_1 topological spaces, the inclusion-exclusion formula provide interval estimations for the closure and interior of given sets. This paper introduces a novel approach that combines combinatorial and point-set topology, which leads to a number of results. Among these is the cardinality estimation for the intersection of two open sets that cover a hyperconnected topological space.

Contents

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1. Introduction

In point-set topology, trends of last five decades are connected with investigations of infinite cardinal functions of topological spaces. However, many combinatorial properties of finite topological spaces are in the shadow. Even so, discrete mathematics and combinatorics use finite sets for naturally formulated inclusion-exclusion identities via closure operators [4] and improved Bonferroni inequalities via abstract tubes [5], [8]. Also, we must emphasize, that characterizations of finite sets yield important results in computational topology (see, *e.g.*, [9], [2]).

It seems that cardinality counting problems in discrete mathematics and combinatorics began using famous inclusion-exclusion formula. That is, for two given finite sets, we have the well-known view, namely,

$$card(A \cup B) = card(A) + card(B) - card(A \cap B).$$

This formula is successfully applicable, if we know exact values of cardinalities (see, *e.g.*, [8], [4], [3]). However, it is interesting that the evaluation of cardinalities of corresponding sets occurs while we have imprecise information about cardinals of the particular sets. Such a situation arises when we have, for example, so-called big data sets and molecular structures.

2. Preliminaries

In [7], R. Moore developed interval mathematics for computational problems, where parameters of investigating models are uncertain and we are only able to describe parameters by closed interval estimations. In this section, we briefly recall basic operations of interval arithmetic.

Let $a_1, a_2, b_1, b_2, x \in \mathbb{R}$. A closed interval of the reals is denoted by $[a_1, a_2] = \{x \in \mathbb{R} : a_1 \leq x \leq a_2\}$. From [7], we have following interval arithmetic:

(1) $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$;
(2) $[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$;
(3) $[a_1, a_2] \times [b_1, b_2] = [minP, maxP]$, where $P = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$;
(4) If $0 \notin [b_1, b_2]$, then $\frac{[a_1,a_2]}{[b_1,b_2]} = [a_1, a_2] \times [\frac{1}{b_2}, \frac{1}{b_1}]$.

It should be especially noticed that any real number k is identified with interval $[k, k]$. Moreover, if a_1 and b_1 are non-negative real numbers then interval multiplication (3) should be change in the following way $[a_1, a_2] \times [b_1, b_2] = [a_1b_1, a_2b_2]$. Also, if $a_1 \geq 0$ and $b_1 > 0$ then the rule (4) can be simplified as following $\frac{[a_1,a_2]}{[b_1,b_2]} = [\frac{a_1}{b_2}, \frac{a_2}{b_1}]$.

There are established several important computational differences between interval arithmetic from real one, but we do not need more information than we present here about interval mathematics.

In the sequel, the sets of natural and rational numbers are denoted by symbols $\mathbb{N}$ and $\mathbb{Q}$, but $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For a rational number $q \in \mathbb{Q}$ we have to use two well-known notations:
$\lfloor q \rfloor = max\{m \in \mathbb{Z} | m \leq q\}$ and $\lceil q \rceil = min\{n \in \mathbb{Z} | n \geq q\}$.

For topological spaces we use notions from [1]. If $O \subset X$ is an open subset of a topological space (X, τ) then we will write $O \in \tau$. The complement of an open set is called to be closed set. The collection of all closed subsets of (X, τ) we denote by $co(\tau)$. Also, in a finite topological space (X, τ) denote by $cl(A)$ closure (resp. $int(A)$ interior of) $A \subset X$, which is minimal closed (resp. maximal open) set containing (resp. contained in) a set A.

Recall that a topological space (X, τ) is T_1 space if and only if $\{x\}$ is closed set, for every $x \in X$. Therefore, in T_1 space (X, τ), we have $\{x\} = cl(\{x\})$ for every $x \in X$.

For the finite, non-T_1 topological spaces cardinal estimations using closure and interior operators is less lightened part of discrete mathematics.

Naturally, if we know about a set A that both of estimations $card(A) \in [a_1, a_2]$ and $card(A) \in [b_1, b_2]$ are valid, where $[a_1, a_2] \cap [b_1, b_2] \neq \{\emptyset\}$ then we should declare $card(A) \in [\max\{a_1, b_1\}, min\{a_2, b_2\}]$.

Theorem 2.1. *Let $A \subset B$ be subsets of a set X where $card(X) = n$ and $card(A) \in [a_1, a_2]$. Then $card(B) \in [a_1 + 1, n - 1]$.*

Proof. It is obvious that $A \subset B$ implies that $card(A) < card(B)$. Since the minimal value of cardinality of a set A can be equal to a_1, then $a_1 + 1 \leq card(B)$. On the other hand we have, $B \subset X$ and $card(B) < card(X) = n$. Hence it can be write $card(B) \leq n - 1$.

□

Theorem 2.2. *Let A, B and C be finite subsets of a set X such that $C = A \times B$, $card(C) \in [c_1, c_2]$ and $card(A) \in [a_1, a_2]$. Then $card(B) \in [\lceil \frac{c_1}{a_2} \rceil, \lfloor \frac{c_2}{a_1} \rfloor]$.*

Proof. Since for Cartesian product $C = A \times B$, we can write following cardinal equality: $card(B) = \frac{card(C)}{card(A)}$, then applying above mentioned operation of the interval division we get $card(B) \in [\frac{c_1}{a_2}, \frac{c_2}{a_1}] \cap \mathbb{N}_0 = [\lceil \frac{c_1}{a_2} \rceil, \lfloor \frac{c_2}{a_1} \rfloor]$.
□

Theorem 2.3. *Let $X = A \cup B$ be a finite set with $card(X) \in [m, n]$, but $card(A) \in [a_1, a_2]$ and $card(B) \in [b_1, b_2]$. Then $card(A \cap B) \in [a_1 + b_1 - n, a_2 + b_2 - m] \cap \mathbb{N}_0$.*

Proof. Applying the famous inclusion-exclusion formula, we can write $card(A \cap B) = card(A) + card(B) - card(A \cup B) = card(A) + card(B) - [m, n]$. By substitution of given cardinal estimations we obtain $card(A \cap B) \in [a_1 + b_1 - n, a_2 + b_2 - m] \cap \mathbb{N}_0$. □

3. Main Results

In section, we work with topological spaces which are not even T_1 topologies. Examples of such topological spaces are known in the point-set topology as T_0 and R_0 spaces.

In the Theorem 3.1, Theorem 3.2 and Theorem 3.3, we assume that (X, τ) is a non-T_1 space with $card(X) = n$ and $min\{card(F)|F \in co(\tau) \setminus \{\emptyset\}\} = 2$.

Theorem 3.1. *Let $A = \{a_1, a_2, ..., a_m\}$ be a subset of a topological space (X, τ) with $m \in [1, \lfloor \frac{n}{2} \rfloor]$. If $cl(\{a_i\}) \cap cl(\{a_j\}) = \emptyset$, for $i, j \in \{1, 2, ..., m\}$ and $i \neq j$ then $card(cl(A)) \in [2m, n]$.*

Proof. It is known that in T_1 topological space (X, τ) with $card(X) = n$ we have $card(cl(\{x\})) = 1$, for every $x \in X$. Therefore, in view our conditions we conclude that $n \geq card(cl(\{x\})) \geq 2$, for every $x \in X$. Note that for the set $A = \{a_1, a_2, ..., a_m\}$ we can write its closure as following: $cl(A) = cl(\{a_1\}) \cup cl(\{a_2\}) \cup ... \cup cl(\{a_m\})$. Therefore, the inequalities hold $n \geq card(cl(A)) = card(cl(\{a_1\})) + card(cl(\{a_2\})) + ... + card(cl(\{a_m\})) \geq 2m$. □

Theorem 3.2. *Let $A = \{x_1, x_2, ..., x_p\}$ be a subset of a topological space (X, τ) with $p \in [\lceil \frac{n}{2} \rceil, n]$. If $card(cl(x)) \in [2, k_x]$, for any point $x \in (X \setminus A)$ and $cl(\{x_i\}) \cap cl(\{x_j\}) = \emptyset$, for $i, j \in \{p+1, p+2, ..., n\}, i \neq j$, then*

$card(int(A)) \in [0, 2p - n]$, if $n \leq \sum_{x \in (X \setminus A)} k_x$

and

$card(int(A)) \in [n - \sum_{x \in (X \setminus A)} k_x, 2p - n]$, if $\sum_{x \in (X \setminus A)} k_x < n$.

Proof. Assume that $X = \{x_1, x_2, ..., x_n\}$ and $A = \{x_1, x_2, ..., x_p\}$. It is known that $int(A) = X \setminus cl(X \setminus A)$. Since $card(X \setminus A) = n - p$ then using Theorem 3.1. we can write $card(cl(X \setminus A)) \in [2(n-p), n]$. But, taking into account condition $card(cl(x_i)) \in [2, k_i], i = \overline{p+1, n}$ we obtain better estimation than previous, namely: $card(cl(X \setminus A)) = card(cl(x_{p+1})) +$

$card(cl(x_{p+2})) + ... + card(cl(x_n)) \in [2, k_{p+1}] + [2, k_{p+2}] + ... + [2, k_n] = [2(n-p), min\{n, \sum_{i=p+1}^{n} k_i\}]$.

It is clear that if $n \leq \sum_{i=p+1}^{n} k_i$ then $card(cl(X \setminus A)) \in [2(n-p), n]$. Hence we get $card(int(A)) \in [0, 2p-n]$. If $\sum_{i=p+1}^{n} k_i < n$ then $card(cl(X \setminus A)) \in [2(n-p), \sum_{i=p+1}^{n} k_i]$ and we obtain $card(int(A)) \in [n - \sum_{i=p+1}^{n} k_i, 2p-n]$. □

Recall that a set A of a topological space (X, τ) is called semi-open if there exists $O \in \tau \backslash \{\emptyset\}$ such that $O \subset A \subset cl(O)$ [6]. The complement of an semi-open set is called semi-closed. The class of all semi-open (resp. semi-closed) subsets of a space (X, τ) we denote usually as $SO(X)$ (resp. $SC(X)$). It can be easily to verify that $A \in SO(X)$ if and only if $A \subset cl(intA)$, but $B \in SC(X)$ if and only if $int(clB) \subset B$.

Theorem 3.3. *Let $A \in SO(X)$ be a nonempty subset of (X, τ). Then there exists $k \in \mathbb{N}$ such that $card(A) \in [k+1, 2k-1]$, where $k \in [1, \lfloor \frac{n}{2} \rfloor]$.*

Proof. For a set $A \in SO(X)$ we can choose $O \in \tau \backslash \{\emptyset\}$ such that $O \subset A \subset cl(O)$. Hence $card(O) < card(A) < card(cl(O)) \leq n$. Denote by $k = card(O)$, then it is obvious that $k \in [1, n-1]$. Hence $card(A) \in [k+1, n-1]$, but by Theorem 3.1. we can write $card(clO) \in [2k, n]$. Note that the inequality $2k < n$ implies $k \in [1, \lfloor \frac{n}{2} \rfloor]$. Collecting our estimations we get $k+1 \leq card(A) < [2k, n]$, i.e. $card(A) \in [k+1, 2k-1]$. □

A topological space (X, τ) is called hyperconnected if $cl(O) = X$, for every $O \in \tau \setminus \{\emptyset\}$. It is obvious that (X, τ) is hyperconnected if and only if $O_1 \cap O_2 \neq \{\emptyset\}$, for any pair of $O_1, O_2 \in \tau \setminus \{\emptyset\}$.

Now, in contrast of above theorems we remove certain conditions from a topological space.

Theorem 3.4. *Let (X, τ) be a hyperconnected topological space with $card(X) \in [m, n]$. If $X = O_1 \cup O_2$, where $O_1, O_2 \in \tau \backslash \{\emptyset\}$ are sets with $card(O_1) \in [a_1, a_2]$, $card(O_2) \in [b_1, b_2]$ and $max\{a_2, b_2\} < m$. Then $card(O_1 \cap O_2) \in [a_1 + b_1 - n, a_2 + b_2 - m] \cap \mathbb{N}_0$.*

Proof. Since in the hyperconnected space (X, τ) we have $O_1 \cap O_2 \neq \emptyset$, for any pair of $O_1, O_2 \in \tau \setminus \{\emptyset\}$ then it takes place following equality: $card(O_1 \cap O_2) = card(O_1) + card(O_2) - card(O_1 \cup O_2) = [a_1, a_2] + [b_1, b_2] - [m, n] = [a_1 + b_1 - n, a_2 + b_2 - m] \cap \mathbb{N}_0$.

□

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