An innovative procedure to improve integration algorithm for modified Cam-Clay plasticity model

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Abstract

An innovative numerical scheme to improve integration algorithm for critical state elastoplasticity is presented and detailed with special consideration to the modified Cam-Clay model. The scheme is based on a rotation of the principal stresses and represent the elastoplastic model in a rotated Haigh-Westergaard space. Such an approach allows the calculation of modified Cam-Clay to be computationally efficient and easy to implement. The validity of the proposed scheme is verified by comparing the numerical results with analytic solutions.

Keywords: Critical state elastoplasticity; Modified Cam-Clay; Integration algorithm; Return mapping; Rotated Haigh-Westergaard space.

1. Introduction

Cam-Clay models of critical state soil mechanics are widely applied in various geotechnical applications involving numerical modeling and analysis of soil structure [1]. The original Cam-Clay (*OCC*) model was presented by Roscoe et al. [2], Schofield & Wroth [3], and the modified Cam-Clay (*MCC*)

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6 model was developed by Roscoe & Burland [4]. The general features of Cam-
7 Clay models include pressure sensitivity, hardening and softening responses
8 and demand a small number of material parameters which can be directly
9 obtained from conventional laboratory tests [5]. In particular, the popularity
10 of the *MCC* model is due to its simplicity and capability to represent the
11 actual behavior of soils [6].

12 The *MCC* model depends essentially on two invariants (P and Q). The
13 extension of the *MCC* model to three-invariant by introducing the Lode's
14 angle β was proposed by Alawaji et al. [7]. The extended model was rep-
15 resented as the generalized Cam-Clay (*GCC*) model [8]. The *MCC* model
16 requires five material parameters, while the *GCC* model demands six param-
17 eters [8].

18 The implementation of the *MCC* model in finite element demands the
19 use of numerical integration algorithms for presenting the incremental evo-
20 lution of stresses and hardening parameters [6]. Within the last decade,
21 various integration algorithms have been proposed and categorized within
22 two techniques: explicit and implicit. The implicit algorithms have become
23 predominant because of their efficiency and robustness.

24 Throughout the years, Borja and his co-authors developed an implicit
25 integration for *MCC* model by considering either a linear elastic predictor [6]
26 or a nonlinear elastic predictor [9], and later extended the *MCC* model from
27 the infinitesimal to finite strain [10]. In addition, many integration algorithms
28 have been proposed for *MCC* model, such as documented in [11, 12, 13, 14].

29 The numerical integration scheme of elastoplastic constitutive models is
30 described in the reference book [15] where the numerical approach is based on
31 the two main steps: the elastic trial step and a subsequent return-mapping
32 scheme. The convergence rate of the iterative method for solving nonlin-
33 ear elastoplastic equations is strongly dependent on the choice of variables
34 to represent the residual vector. It can be enhanced by representing the
35 elastoplastic equations in terms of the principal stresses [16].

36 In the present paper, we propose an innovative numerical scheme of the
37 plastic corrector step for *MCC* model. The scheme is developed by intro-
38 ducing an additional dependence for *MCC* model on the Lode's angle β and
39 representing the *MCC* model from two-invariant $P - Q$ space to the rotated
40 principal stress (three-invariant) space. Then, we consider a distance function
41 in the rotated Haigh–Westergaard space that should be minimized, in order
42 to compute plastic deformation from closed point projection. The proposed
43 scheme is probably more efficient due to the expression of the yield surface

within a Haigh–Westergaard cylindrical coordinates [17] and the application of the rotated Haigh–Westergaard space [16].

The proposed approach for the *MCC* model is verified by comparing the numerical results of *MCC* model with analytical solutions provided by Perić & Ayari [8], Perić [18]. The implementation of the present study is done in the NeoPZ library [19], which is an object-oriented scientific computational environment, providing a framework for developing finite element algorithms [20].

2. Elastoplastic constitutive model

A nonlinear elastoplastic constitutive model is described by the theory of elastoplasticity, when a material undergoes an irreversible deformations after some specific loading conditions. The total strain tensor $\boldsymbol{\epsilon}$ is decomposed into two parts as follows [15]:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_e + \boldsymbol{\epsilon}_p \quad (1)$$

where $\boldsymbol{\epsilon}_e$ is the elastic component and $\boldsymbol{\epsilon}_p$ is the plastic component. The elastic part is reversible and the plastic part represents a permanent deformation, and it is related to the history of irreversible deformations [15]. Once a displacement increment $\dot{\mathbf{u}}$ is determined, the relationship between strain and displacement under the infinitesimal strain theory is defined as:

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2} \left(\nabla \dot{\mathbf{u}} + \nabla^T \dot{\mathbf{u}} \right) \quad (2)$$

The elastoplastic deformation can be as a result of various microstructural changes and it can be described by the set of internal damage variables [21]:

$$\boldsymbol{\chi}_i; \quad i = 1, 2, \dots \quad (3)$$

where $\boldsymbol{\chi}_i$ may be scalars, vectors, or higher rank tensors. With reference to the internal variables, the Helmholtz free energy is defined as in [22]:

$$\mathbf{F} = \mathbf{F}(\boldsymbol{\epsilon}_e, \boldsymbol{\chi}_i) \quad (4)$$

From the point of view of thermodynamics, the free energy can be decomposed into elastic $\mathbf{F}^e(\boldsymbol{\epsilon}_e)$ and plastic $\mathbf{F}^p(\boldsymbol{\chi}_i)$ part. By differentiating the elastic part of free energy $\mathbf{F}^e$ and substituting the result into the Clausius–Duhem inequality, the expression of stress is written as [21]:

$$\boldsymbol{\sigma} = \bar{\rho} \frac{\partial \mathbf{F}^e}{\partial \boldsymbol{\epsilon}_e} \quad (5)$$

where $\bar{\rho}$ and $\boldsymbol{\sigma}$ are the mass density and the stress tensor, respectively. Moreover, the thermodynamic conjugate force $\mathbf{A}_i$ is represented by:

$$\mathbf{A}_i = \bar{\rho} \frac{\partial \mathbf{F}^p}{\partial \boldsymbol{\chi}_i} \quad (6)$$

The elastoplastic deformation is mathematically described by four fundamental axioms as follows [15]:

Elastic law. The elastic law can be presented by two types of constitutive behaviors: linear and nonlinear elasticity. The linear elasticity is described as [23]:

$$\dot{\boldsymbol{\sigma}} = 2\mu\dot{\boldsymbol{\epsilon}}_e + \lambda \text{tr}(\dot{\boldsymbol{\epsilon}}_e) \mathbf{I} \quad (7)$$

where μ and λ are the Lamé coefficients and $\dot{\boldsymbol{\sigma}}$ is the effective stress increment. In many cases the Cam-Clay elastoplastic model is applied to poroelastic materials. In this case the elastoplastic behavior is determined by the value of the effective stress whose relation with the total stress is given by $\boldsymbol{\sigma}_t = \boldsymbol{\sigma} - \alpha p \mathbf{I}$ and in which p is the pore pressure and α is the Biot coefficient. The nonlinear stress-strain relationship presented in this paper relates only to the effective stress. No mention to the Biot model will be made in this paper.

Yield criterion. Describes the elastic limit and the plastic part through a plasticity yield function $\Phi = \Phi(\boldsymbol{\sigma}, A)$, where $A = \bar{\rho} \partial \mathbf{F}^p / \partial \boldsymbol{\chi}$ is the hardening thermodynamic force and $\boldsymbol{\chi}$ is the internal damage variable. The plasticity function assumes negative values in the elastic part and null values in the plastic part [24].

Flow rule. Assumes the existence of a plastic potential function $\Psi = \Psi(\boldsymbol{\sigma}, A)$, which specifies how the plastic deformation tensor $\boldsymbol{\epsilon}_p$ evolves in the plasticity process $\dot{\boldsymbol{\epsilon}}_p = \dot{\gamma} \mathbf{N}$, in which $\mathbf{N}(\boldsymbol{\sigma}, A) = \partial \Psi / \partial \boldsymbol{\sigma}$ is the flow direction and $\dot{\gamma}$ is the plastic multiplier. The flow rule is called associative if the plastic potential function equals to yield function, namely $\Psi = \Phi$. For the associative case, the direction of strain rate is the outward normal to yield surface, whereas for non-associative flow rule it is

the gradient of plastic potential surface [25]. It is required to mention that the *MCC* model is an associative plasticity model.

Hardening law. Specifies how the internal damage variable $\dot{\chi} = \dot{\gamma}H$ evolves, in which, $H(\boldsymbol{\sigma}, A) = -\partial\Psi/\partial A$ is the hardening modulus.

2.1. Modified Cam-Clay elastoplasticity model

The modified Cam-Clay *MCC* includes four key ingredients: (i) the elastic law, (ii) the yield function, (iii) the flow rule, and (iv) the hardening (softening) law. These four components are defined using the following stress invariants [13]:

$$P = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \quad (8)$$

$$Q = \left[\frac{1}{2}(\sigma_1 - \sigma_2)^2 + \frac{1}{2}(\sigma_2 - \sigma_3)^2 + \frac{1}{2}(\sigma_3 - \sigma_1)^2 \right]^{\frac{1}{2}} \quad (9)$$

where P is the effective mean stress [MPa], Q is the von Mises equivalent stress [MPa], and σ_1 , σ_2 , and σ_3 are the principal of effective stresses [MPa]. Note that P is positive in compression while the general sign conventions are maintained for the Cauchy stress.

2.1.1. Elastic law of modified Cam-Clay

In soil mechanics, the *MCC* model is based on the nonlinear elasticity law that can be linearized (as expressed in the equation (7)) whenever the change of mean effective stress is sufficiently small [15]. The elastic law of *MCC* establishes a nonlinear relation between the hydrostatic stress (i.e. P) and the volumetric deformation. For the shear stress, two flavours of constitutive relations are commonly used [26]: First, by establishing a constant shear modulus G [MPa] and Second, by establishing the Poisson's ratio ν as a constant (in this case G depends on the bulk modulus).

- **First.** The nonlinear elastic law for the constant shear modulus is:

$$\boldsymbol{\sigma} - \boldsymbol{\sigma}^\circ = 2G(\boldsymbol{\epsilon}_{ed} - \boldsymbol{\epsilon}_{ed}^\circ) - (P_{cc}(\epsilon_{ev}) - P^\circ)\mathbf{I} \quad (10)$$

- **Second.** The nonlinear elastic law for the constant Poisson's ratio is:

$$\boldsymbol{\sigma} - \boldsymbol{\sigma}^\circ = 2G(P_{cc}(\epsilon_{ev}))(\boldsymbol{\epsilon}_{ed} - \boldsymbol{\epsilon}_{ed}^\circ) - (P_{cc}(\epsilon_{ev}) - P^\circ)\mathbf{I} \quad (11)$$

121 where the shear modulus for the constant Poisson's ratio is:

$$G(P_{cc}(\epsilon_{ev})) = \frac{3K(P_{cc}(\epsilon_{ev}))(1-2\nu)}{2(1+\nu)} \quad (12)$$

122 where $\boldsymbol{\sigma}^\circ$ and $\boldsymbol{\sigma}$ are the initial and current effective stress tensor, re-
 123 spectively [MPa]. P° is the initial effective mean stress [MPa]. $\boldsymbol{\epsilon}_{ed}^\circ$ and
 124 $\boldsymbol{\epsilon}_{ed}$ are the initial and current elastic deviatoric strain, respectively.
 125 The elastic volumetric strain ϵ_{ev} is:

$$\epsilon_{ev} = tr(\boldsymbol{\epsilon}_e) \quad (13)$$

126 The elastic deviatoric strain tensor $\boldsymbol{\epsilon}_{ed}$ is given by:

$$\boldsymbol{\epsilon}_{ed} = \boldsymbol{\epsilon}_e - \frac{1}{3}tr(\boldsymbol{\epsilon}_e)\mathbf{I} \quad (14)$$

127 The constitutive law for the Cam-Clay effective mean stress $P_{cc}(\epsilon_{ev})$ is
 128 [27]:

$$P_{cc}(\epsilon_{ev}) = -p_t + (P^\circ + p_t) \exp \left[-\frac{1+e^\circ}{C^e} (\epsilon_{ev} - \epsilon_{ev}^\circ) \right] \quad (15)$$

129 where e° is the initial void ratio, C^e is the recompression index, ϵ_{ev}° is
 130 the initial elastic volumetric strain, and p_t is the tensile strength [MPa].
 131 The initial mean effective stress P° is represented based on the initial
 132 conditions. For a zero tensile strength $P^\circ > 0$ is required because the
 133 *MCC* constitutive model is generally applied to problems with non-zero
 134 initial stresses [26].

135 The Bulk modulus $K(P_{cc}(\epsilon_{ev}))$ [MPa] in equation (60) is computed as:

$$K(P_{cc}(\epsilon_{ev})) = -\frac{dP_{cc}(\epsilon_{ev})}{d\epsilon_{ev}} \quad (16)$$

136 leading to

$$K(\epsilon_{ev}) = K(P_{cc}(\epsilon_{ev})) = \frac{1+e^\circ}{C^e} (P_{cc}(\epsilon_{ev}) + p_t) \quad (17)$$

137 In order to implement the *MCC* model, it is required to present the
 138 elastic tangent operator. The elastic tangent operator $\mathbf{C}$ is derived from the
 139 derivative of the effective stress tensor in equation (7) with respect to elastic
 140 strain tensor. Here, an elastic tangent operator is presented separately for
 141 constant shear modulus and constant Poisson's ratio, as:

- 142 • The elastic tangent operator for constant shear modulus is:

$$\mathbf{C} = \nabla_{\epsilon_e} \boldsymbol{\sigma} = 2G \mathbf{I}^4 + \left(K(\epsilon_{ev}) - \frac{2}{3}G \right) \mathbf{I} \otimes \mathbf{I} \quad (18)$$

- 143 • The elastic tangent operator for constant Poisson's ratio is:

$$\mathbf{C} = \nabla_{\epsilon_e} \boldsymbol{\sigma} = 2G (P_{cc}(\epsilon_{ev})) \mathbf{I}^4 + \left(\frac{3K(\epsilon_{ev})\nu}{1+\nu} \right) \mathbf{I} \otimes \mathbf{I} \quad (19)$$

144 where $\otimes$ is the tensor product operator, $\mathbf{I}$ is the second-order identity
 145 tensor, and $\mathbf{I}^4$ is the 4th-order identity tensor.

- 146 • Comparing the elastic tangent operator with the tangent for linear
 147 elasticity:

$$\mathbf{C} = \nabla_{\epsilon_e} \boldsymbol{\sigma} = 2G \mathbf{I}^4 + \lambda \mathbf{I} \otimes \mathbf{I} \quad (20)$$

148 one can easily identify the equivalent elasticity parameters for a given
 149 state of stress or strain.

150 2.1.2. Yield function of modified Cam-Clay

151 The yield function of the *MCC* model (Fig. 1) is defined as [15]:

$$\Phi(P, Q, a) = \left[\frac{P - p_t + a}{b a} \right]^2 + \left[\frac{Q}{M a} \right]^2 - 1 \quad (21)$$

152 where M is the ratio between the two radii of the *MCC* ellipse, a is the radius
 153 of the ellipse along the effective mean stress axis [MPa]. The parameter b is:

$$b = \begin{cases} 1 & \text{if } P \geq p_t - a \\ \zeta & \text{if } P < p_t - a \end{cases} \quad (22)$$

154 where ζ is the material constant which modifies the radius of the second half
 155 of the ellipse on the compressive side of hydrostatic axis. If $\zeta = 1$, the yield

156 locus becomes an ellipse with radii a and Ma , respectively, along P and Q .
 157 The dashed line in Fig. 1 is named the critical state line. The parameter M is
 158 the slope of the critical state line and the area to its right is the supercritical
 159 region (Dry, softening) and the area to its left is the subcritical region (Wet,
 160 hardening). The critical state line is given by:

$$Q = MP \quad (23)$$

Fig. 1. Modified Cam-Clay *MCC* model, yield surface [15].

161 2.1.3. Flow rule of modified Cam-Clay

162 The plastic flow rule is described by postulating associativity, [13] as:

$$\dot{\epsilon}_p = \dot{\gamma} \frac{\partial \Phi}{\partial \sigma} \quad (24)$$

163 2.1.4. Hardening law of modified Cam-Clay

164 The hardening law of *MCC* model is classified as hardening when the
 165 yield surface expands ($Q < MP$), and softening when the surface shrinks
 166 ($Q > MP$) and it undergoes no further changes when $Q = MP$ [10].

167 The hardening law is expressed by writing the yield surface parameter
 168 (a) as a function of hardening variable (χ). For many plasticity compressible
 169 materials, the hardening variable χ is corresponds to the plastic volumetric
 170 strain ϵ_{pv} , namely $\chi \equiv -\epsilon_{pv}$, in which $\epsilon_{pv} = tr(\epsilon_p)$. The hardening law can

171 be defined by a piecewise linear form or an exponential form [26]. The linear
 172 form of hardening law is given in terms of preconsolidation pressure p_c [MPa],
 173 as follows [15]:

$$p_c(\chi) \equiv (1 + \zeta) a(\chi) - p_t \quad (25)$$

174 The exponential form of the hardening law is written by replacing $a(\chi)$
 175 in equation (25), as follows [27]:

$$a(\chi) \equiv a(-\epsilon_{pv}) = a^\circ \exp \left[-\frac{(1 + e^\circ)}{C^p - C^e} (\epsilon_{pv} - \epsilon_{pv}^\circ) \right] \quad (26)$$

176 where a° is the initial value of the hardening parameter, C^p is the compres-
 177 sion index. ϵ_{pv}° and ϵ_{pv} are the initial and current plastic volumetric strain,
 178 respectively.

179 2.2. Yield function of generalized modified Cam-Clay

180 The *MCC* model is generally implemented by a numerical integration
 181 of two invariant [18]. Inspired by the articles [7, 28], to control the plastic
 182 surface format, we consider a generalized modified Cam-Clay (*GMCC*) model
 183 by introducing an additional dependence on the Lode angle β . To convert
 184 the *MCC* model in equation (21) from $P - Q$ space to Haigh-Westergaard
 185 space, the invariant stress's are expressed as:

$$P = -\frac{I_1}{3} \quad Q = \sqrt{3 J_2} \quad (27)$$

186 The *GMCC* model is defined by considering the invariant stress's in equa-
 187 tion (27) and applying the Lode's angle β as follows:

$$\Phi \left(I_1, \sqrt{J_2}, a, \beta \right) = \left[\frac{-\frac{I_1}{3} - p_t + a}{b a} \right]^2 + \left[\frac{\sqrt{3} \Gamma(\beta) \sqrt{J_2}}{M a} \right]^2 - 1 \quad (28)$$

188 Here the factor $\Gamma(\beta)$ is given by:

$$\Gamma(\beta) = \frac{1}{2} [(1 + \sin(3\beta)) + \frac{1}{\psi} (1 - \sin(3\beta))] \quad (29)$$

189 where, ψ has the range from 7/9 to 9/7. The *MCC* model refers to $\psi = 1$,
 190 such that, $\Gamma(\beta) = 1$.

191 *2.2.1. Hardening law associated with generalized modified Cam-Clay*

192 The *MCC* model is an associative plasticity model where the hardening
193 modulus and flow relation can be related as:

$$H = -tr(\mathbf{N}) \quad (30)$$

194 by considering the above property and the definition of incremental plastic
195 deformation $\dot{\epsilon}_p = \dot{\gamma}\mathbf{N}$, the relationship for the hardening variable at the
196 plastic correction step can be expressed as:

$$\dot{\chi} = \dot{\gamma} H = -\dot{\gamma} tr(\mathbf{N}) = -tr(\dot{\epsilon}_p) = -\dot{\epsilon}_{pv} \quad (31)$$

197 Then, the incremental plastic volumetric strain while applying the im-
198 plicit Euler method can be represented by:

$$\Delta\epsilon_{pv} = \frac{\Delta I_1}{3K(\epsilon_{ev})} \quad (32)$$

199 where the evaluation of plastic volumetric strain is defined through equations
200 (26), as follow:

$$\epsilon_{pv}(a) = \epsilon_{pv}^{\circ} + \frac{C^e - C^p}{1 + e^{\circ}} \ln\left(\frac{a}{a^{\circ}}\right) \quad (33)$$

201 This derivation will permit the elimination of $\Delta\gamma$ as a variable in return-
202 mapping step.

203 **3. Numerical integration algorithm for the elastoplastic model**

204 The numerical integration is divided into two main steps: the elastic trial
205 step and the plastic corrector step (or return-mapping algorithm). If the
206 elastic trial state lies within the elastic domain or on the yield surface, the
207 solution is accepted. Otherwise, if the trial stress in the first step fails to
208 verify the plastic admissible condition, it is projected onto the yield surface
209 by the return-mapping algorithm [15].

210 *3.1. Incremental elastoplastic constitutive problem (general setting)*

211 The incremental constitutive model is presented by applying an implicit
212 Euler method. It is formed by giving the elastic deformation ϵ_e^{n-1} , the plastic
213 deformation ϵ_p^{n-1} , and the hardening variable χ^{n-1} at a (pseudo) time step
214 t^{n-1} , and also given a prescribed incremental strain tensor $\Delta\epsilon$ for the time

interval $[t^{n-1}, t^n]$ in order to find the following system of algebraic equations
at a time-step t^n [29, 15]:

$$\begin{aligned}\boldsymbol{\epsilon}_e^n &= \boldsymbol{\epsilon}_e^{n-1} + \Delta\boldsymbol{\epsilon} - \Delta\gamma \mathbf{N}(\boldsymbol{\sigma}^n, A^n) \\ \chi^n &= \chi^{n-1} + \Delta\gamma H(\boldsymbol{\sigma}^n, A^n)\end{aligned}\quad (34)$$

for the unknowns $\boldsymbol{\epsilon}_e^n$, χ^n and incremental of plastic multiplier $\Delta\gamma$, it is
restricted to the Kuhn-Tucker conditions [30], as:

$$\Delta\gamma \geq 0, \Phi(\boldsymbol{\sigma}^n, A^n) \leq 0, \Delta\gamma \Phi(\boldsymbol{\sigma}^n, A^n) = 0 \quad (35)$$

where

$$\boldsymbol{\sigma}^n = \bar{\rho} \frac{\partial \mathbf{F}^e}{\partial \boldsymbol{\epsilon}_e} |_n \quad A^n = A^n(\chi^n) = \bar{\rho} \frac{\partial \mathbf{F}^p}{\partial \chi} |_n \quad (36)$$

and

$$\mathbf{N}(\boldsymbol{\sigma}^n, A^n) = \frac{\partial \Psi}{\partial \boldsymbol{\sigma}} |_n \quad H(\boldsymbol{\sigma}^n, A^n) = -\frac{\partial \Psi}{\partial A} |_n \quad (37)$$

Solving the elastoplastic problem occurs in two steps. First a an elastic
response is computed (i.e. elastic predictor), where $\Delta\gamma = 0$ leading to
the elastic trial stress. $\boldsymbol{\epsilon}_{e,trial}^n = \boldsymbol{\epsilon}_e^{n-1} + \Delta\boldsymbol{\epsilon}$, and internal variables $\chi_{trial}^n =$
 χ^{n-1} . $\boldsymbol{\sigma}_{trial}^n$ and $\Phi(\boldsymbol{\sigma}_{trial}^n, A_{trial}^n)$ are computed as a function of $\boldsymbol{\epsilon}_{e,trial}^n$. If
 $\Phi(\boldsymbol{\sigma}_{trial}^n, A_{trial}^n) \leq 0$, the elastic response is a valid solution and the elasto
plastic variables are updated from the trial values $(\cdot)^n := (\cdot)_{trial}^n$. Otherwise,
the return-mapping algorithm is applied and a set of nonlinear equations
needs to be solved [15]:

$$\begin{aligned}\boldsymbol{\epsilon}_e^n &= \boldsymbol{\epsilon}_{e,trial}^n - \Delta\gamma \mathbf{N}(\boldsymbol{\sigma}^n, A^n) \\ \chi^n &= \chi_{trial}^n + \Delta\gamma H(\boldsymbol{\sigma}^n, A^n) \\ \Delta\gamma &> 0, \Phi(\boldsymbol{\sigma}^n, A^n) = 0\end{aligned}\quad (38)$$

Once the solution $\boldsymbol{\epsilon}_e^n$ has been calculated, the plastic strain at a load-step
 t^n can be computed as:

$$\boldsymbol{\epsilon}_p^n = \boldsymbol{\epsilon}_p^{n-1} + \Delta\boldsymbol{\epsilon} - \Delta\boldsymbol{\epsilon}_e \quad (39)$$

For the particular case of modified Cam-clay model :

$$\begin{aligned}\chi^n &= -\epsilon_{pv}^n \\ \boldsymbol{\sigma}^n &= \boldsymbol{\sigma}(\boldsymbol{\epsilon}_e^n) \\ \Psi(\boldsymbol{\sigma}^n) &= \Phi(\boldsymbol{\sigma}^n, \chi^n) \\ H(\boldsymbol{\sigma}^n, A^n) &= tr(\mathbf{N}(\boldsymbol{\sigma}^n, \chi^n))\end{aligned}\quad (40)$$

232 3.2. Plastic return-mapping in rotated principal stress

233 In order to present a simplified procedure to improve integration scheme
 234 for *MCC* model, a numerical integration algorithm provided by [16] for
 235 DiMaggio & Sandler [31] elastoplastic model is used. The proposed scheme
 236 is implemented by using the closest point projection in the rotated principal
 237 stresses. In this algorithm, instead of using the six stress components for the
 238 representation of the stress state at a point, an alternative representation of
 239 the stress is used composed of the principal stresses $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3]^T$. The
 240 principal stresses are then rotated to the Haigh–Westergaard stress space.
 241 $\boldsymbol{\sigma}^* = [\sigma_1^*, \sigma_2^*, \sigma_3^*]^T$ is used for the stress representation in the Haigh–Westergaard
 242 stress space. The definition of the constitutive model in the rotated principal
 243 variables is simpler and computationally more efficient to implement. The
 244 coordinate system of rotated principal variables is similar to the decomposi-
 245 tion used in [30, 32].

246 3.2.1. Haigh–Westergaard stress space

247 The stress tensor $\boldsymbol{\sigma}$ [MPa] sorted in descending order $\sigma_1 > \sigma_2 > \sigma_3$ can be
 248 written as a function of the hydrostatic stress, deviatoric stress, and Lode’s
 249 angle (ξ, ρ, β) [33], as:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}\rho \cos(\beta) \\ \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}\rho \cos\left(\beta - \frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}\rho \cos\left(\beta + \frac{2\pi}{3}\right) \end{bmatrix} \quad (41)$$

$$\xi = \frac{I_1}{\sqrt{3}} \quad \rho = \sqrt{2J_2} \quad \beta = \frac{1}{3} \cos^{-1} \left(\frac{3\sqrt{2}}{2} \frac{J_3}{J_2^{3/2}} \right) \quad (42)$$

250 The above expression is valid for the $\beta \in [0, \frac{\pi}{3}]$. I_1 , J_2 , and J_3 are the first
 251 invariant of the stress tensor, the second and third deviatoric stress tensor,
 252 respectively. I_1 , J_2 , and β are defined as:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$J_2 = \frac{1}{3} [\sigma_1^2 + \sigma_2^2 - \sigma_2\sigma_3 + \sigma_3^2 - \sigma_1(\sigma_2 + \sigma_3)] \quad (43)$$

$$\beta = \frac{1}{3} \arccos \left(\frac{(2\sigma_1 - \sigma_2 - \sigma_3)(\sigma_1 - 2\sigma_2 + \sigma_3)(\sigma_1 + \sigma_2 - 2\sigma_3)}{2[\sigma_1^2 + \sigma_2^2 - \sigma_2\sigma_3 + \sigma_3^2 - \sigma_1(\sigma_2 + \sigma_3)]^{3/2}} \right)$$

253 The relationship between principal stresses and principal strains is ex-
 254 pressed as:

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = (\mathbf{D}_{HW})^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \quad (44)$$

255 where

$$\mathbf{D}_{HW} = \begin{bmatrix} \left(K + \frac{4G}{3}\right) & \left(K - \frac{2G}{3}\right) & \left(K - \frac{2G}{3}\right) \\ \left(K - \frac{2G}{3}\right) & \left(K + \frac{4G}{3}\right) & \left(K - \frac{2G}{3}\right) \\ \left(K - \frac{2G}{3}\right) & \left(K - \frac{2G}{3}\right) & \left(K + \frac{4G}{3}\right) \end{bmatrix} \quad (45)$$

256 where K [MPa] and G [MPa] are the Bulk and the shear modulus, respec-
257 tively.

258 3.2.2. Rotated Haigh-Westergaard space

259 The tensor of stress $\boldsymbol{\sigma}$ [MPa] can be represented in terms of a rotated
260 Haigh-Westergaard (RHW). The RHW of stress $\boldsymbol{\sigma}^* = [\sigma_1^*, \sigma_2^*, \sigma_3^*]^T$ is defined
261 as:

$$\begin{bmatrix} \sigma_1^* \\ \sigma_2^* \\ \sigma_3^* \end{bmatrix} = \mathbf{R} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \quad (46)$$

262 where

$$\mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad (47)$$

263 The invariant stresses I_1 and J_2 and Lode's angle β in the rotated space
264 can be computed as:

$$I_1 = \sqrt{3}\sigma_1^* \quad J_2 = \frac{\sigma_2^{*2} + \sigma_3^{*2}}{2} \quad \beta = \arctan(\sigma_3^*/\sigma_2^*) \quad (48)$$

Fig. 2. Transformation of the principal stresses to the rotated Haigh-Westergaard stresses.

265 The relationship between stress and strain in the rotated Haigh–Westergaard
 266 space is:

$$\begin{bmatrix} \delta\epsilon_1^* \\ \delta\epsilon_2^* \\ \delta\epsilon_3^* \end{bmatrix} = (\mathbf{D}_{RHW})^{-1} \begin{bmatrix} \delta\sigma_1^* \\ \delta\sigma_2^* \\ \delta\sigma_3^* \end{bmatrix} \quad (49)$$

267 where

$$\mathbf{D}_{RHW} = \begin{bmatrix} 3K & 0 & 0 \\ 0 & 2G & 0 \\ 0 & 0 & 2G \end{bmatrix} \quad (50)$$

268 where K and/or G for nonlinear elasticity are a function of ϵ_{ev} .

269 3.3. Incremental elastoplastic constitutive problem in RHW space

270 The incremental associative elastoplastic problem can be interpreted us-
 271 ing a distance minimization of the trial stress to the admissibility surface
 272 in the RHW space [34, 35]. The admissibility surface for a given variable

273 χ is $S^*(\chi) = \{\boldsymbol{\sigma}^* : \Phi^*(\boldsymbol{\sigma}^*, A(\chi)) = 0\}$, where $\Phi^*(\boldsymbol{\sigma}^*, A)$ is the yield func-
 274 tion. The expression of the return map in the RHM space corresponding to
 275 equation (38) is:

$$\begin{aligned} \boldsymbol{\epsilon}_{e_{trial}}^{*,n} - \boldsymbol{\epsilon}_e^{*,n} &= \Delta\gamma \mathbf{N}^{*,n} \\ \chi_{trial}^n - \chi^n &= -\Delta\gamma H^{*,n} \end{aligned} \quad (51)$$

276 where the flow direction $\mathbf{N}^*$ is:

$$\mathbf{N}^{*,n} = \frac{\partial \Phi^*(\boldsymbol{\sigma}^{*,n}, A(\chi^n))}{\partial \boldsymbol{\sigma}^*} \quad (52)$$

277 and the hardening modulus H^* is:

$$H^{*,n} = -\frac{\partial \Phi^*(\boldsymbol{\sigma}^{*,n}, A(\chi^n))}{\partial A} \quad (53)$$

278 Considering that the direction $\mathbf{N}^{*,n}$ is orthogonal to the surface $S^*(\chi)$,
 279 the inner product of a variation of $\delta\boldsymbol{\sigma}^*$ along the tangent plane at $\boldsymbol{\sigma}^{*,n} \in$
 280 $S^*(\chi^n)$ should satisfy the following orthogonality relationship:

$$\begin{aligned} 0 &= (\delta\boldsymbol{\sigma}^*, \boldsymbol{\epsilon}_{trial}^{*,n} - \boldsymbol{\epsilon}^{*,n}) = \left(\delta\boldsymbol{\sigma}^*, \mathbf{D}_{RHW}^{-1} (\boldsymbol{\sigma}_{trial}^{*,n} - \boldsymbol{\sigma}^{*,n}) \right) = \\ &\quad \left\langle \delta\boldsymbol{\sigma}^*, \boldsymbol{\sigma}_{trial}^{*,n} - \boldsymbol{\sigma}^{*,n} \right\rangle_{RHW} \end{aligned} \quad (54)$$

281 where $\boldsymbol{\sigma}^{*,n}$ is the projected point in $S^*(\chi^n)$. Define the internal product for
 282 vectors $\mathbf{a}^*$ and $\mathbf{b}^*$ in the RHW space as:

$$\left\langle \mathbf{a}^*, \mathbf{b}^* \right\rangle_{RHW} = \left(\mathbf{a}^*, \mathbf{D}_{RHW}^{-1} \mathbf{b}^* \right) \quad (55)$$

283 The equation (54) is a distance function which shows minimizing the
 284 distance induced by the RHW internal product of the trial stress to the
 285 admissibility surface. The distance function for nonlinear elasticity can be
 286 written as:

$$d(\mathbf{a}^*, \mathbf{b}^*) := \left\{ \frac{(\mathbf{a}_1^* - \mathbf{b}_1^*)^2}{3K(\epsilon_{ev})} + \frac{(\mathbf{a}_2^* - \mathbf{b}_2^*)^2}{2G(\epsilon_{ev})} + \frac{(\mathbf{a}_3^* - \mathbf{b}_3^*)^2}{K(\epsilon_{ev})} \right\}^{1/2} \quad (56)$$

287 where the function $K(\epsilon_{ev})$ depends on the elastic volumetric strain and the
 288 function $G(\epsilon_{ev})$ depends on the flavour of the elastic constitutive law (i.e. G
 289 constant or ν constant). It means:

$$G(\epsilon_{ev}) = \begin{cases} G \text{ constant :} & G \\ \nu \text{ constant :} & G(\epsilon_{ev}) = \frac{3K(P_{cc}(\epsilon_{ev}))(1-2\nu)}{2(1+\nu)} \end{cases} \quad (57)$$

The projected point, which minimizes the aforementioned distance function is computed as:

$$\boldsymbol{\sigma}^{*,n} = \min_{\boldsymbol{\sigma}^{*,n} \in S^*(\chi^n)} d(\boldsymbol{\sigma}_{trial}^{*,n}, \boldsymbol{\sigma}^{*,n})$$

290 3.4. Solution of the incremental return-mapping step

In the RHW space, a stress point on the yield surface depends on three parameters as:

$$S^*(\chi^n) = \{\boldsymbol{\sigma}^{*,n}(r_1, r_2, \chi^n) : (r_1, r_2, \chi^n) \in \mathcal{I}\} \quad (58)$$

291 where the internal damage parameter χ^n is included to define the position
 292 of the surface and the other two parameters r_1 and r_2 which determine the
 293 position of a point on the surface. The square distance of a given $\boldsymbol{\sigma}_{trial}^{*,n}$, to a
 294 point on the surface $\boldsymbol{\sigma}^{*,n}(r_1, r_2, \chi^n) \in S^*(\chi^n)$ is defined as:

$$\delta(\boldsymbol{\sigma}_{trial}^{*,n}; r_1, r_2, \chi^n) = (d(\boldsymbol{\sigma}_{trial}^{*,n}, \boldsymbol{\sigma}^{*,n}(r_1, r_2, \chi^n)))^2 \quad (59)$$

295 where $d(\cdot)$ is defined in equation (56).

296 Given the trial elastic strain $\boldsymbol{\epsilon}_{trial}^{*,n}$ and internal variable $\chi_{trial}^n = \chi^{n-1}$, the
 297 quantity of $\boldsymbol{\sigma}_{trial}^{*,n}$ is computed as:

$$\boldsymbol{\sigma}_{trial}^n = \boldsymbol{\sigma}_{trial}^{n-1} + 2G(\epsilon_{ev}^n) \boldsymbol{\epsilon}_{ed_{trial}}^n - 2G(\epsilon_{ev}^{n-1}) \boldsymbol{\epsilon}_{ed_{trial}}^{n-1} - (P_{cc}(\epsilon_{ev}^n) - P_{cc}(\epsilon_{ev}^{n-1})) \mathbf{I} \quad (60)$$

298 The relation between a stress increment and a strain increment for the
 299 Cam-clay nonlinear elasticity in the rotated Haigh–Westergaard space as
 300 documented in equation (50) is:

$$\mathbf{D}_{RHW}(\epsilon_{ev}) = \begin{bmatrix} 3K(\epsilon_{ev}) & 0 & 0 \\ 0 & 2G(\epsilon_{ev}) & 0 \\ 0 & 0 & 2G(\epsilon_{ev}) \end{bmatrix} \quad (61)$$

301 where the expression for $G(\epsilon_{ev})$ depends on the flavour of the nonlinear elastic
 302 law. It is required to mention that the Bulk modulus and shear modulus in
 303 equation (61) are the functions of elastic volumetric strain that are belong

304 to trial stress tensor. Then, in the rest of paper, the following notations are
 305 applied.

$$\begin{cases} K_{trial} = K(\epsilon_{ev_{trial}}) \\ G_{trial} = G(\epsilon_{ev_{trial}}) \end{cases} \quad (62)$$

306 After computing the quantity of $\sigma_{trial}^{*,n}$, the stress $\sigma^{*,n}(r_1, r_2, \chi^n) \in S^*(\chi^n)$
 307 at a time-step t^n is computed by minimizing the distance of $\sigma_{trial}^{*,n}$ to the ad-
 308 missibility surface $S^*(\chi^n)$, as follows:

$$\begin{aligned} \frac{\partial \delta(\sigma_{trial}^{*,n}; r_1, r_2, \chi^n)}{\partial r_1} &= 0 \\ \frac{\partial \delta(\sigma_{trial}^{*,n}; r_1, r_2, \chi^n)}{\partial r_2} &= 0 \\ \chi^n - \chi^{n-1} - \Delta\gamma H^*(r_1, r_2, \chi^n) &= 0 \\ \epsilon_{e_{trial}}^{*,n} - \epsilon_e^{*,n} - \Delta\gamma \mathbf{N}^*(r_1, r_2, \chi^n) &= 0 \end{aligned} \quad (63)$$

309 For the particular case of modified Cam-clay model, the system of equa-
 310 tions (63) when considering the equation (31), can be simplified as:

$$\begin{aligned} \frac{\partial \delta(\sigma_{trial}^{*,n}; r_1, r_2, \chi^n)}{\partial r_1} &= 0 \\ \frac{\partial \delta(\sigma_{trial}^{*,n}; r_1, r_2, \chi^n)}{\partial r_2} &= 0 \\ \chi^n - \chi^{n-1} + \epsilon_{pv}^n - \epsilon_{pv}^{n-1} &= 0 \end{aligned} \quad (64)$$

311 which is a system of equation of three equations in three unknowns.

312 3.5. The generalized modified Cam-Clay in the RHW space

313 The *GMCC* surface in the RHW space as shown in Fig. 3 is formed by
 314 two parts. There is one softening part S_s^* (Green color) and one hardening
 315 part S_h^* (Red color). The implementation of the return map to any of these
 316 surfaces is similar and the only difference is related to the quantity of b , where
 317 $b = 1.0$ for softening part and $b = \zeta$ for the hardening part. Therefore, only
 318 the return mapping to the hardening part of *GMCC* in the RHW space is
 319 discussed.

Fig. 3. Yield surface represented in the RHW stress space: softening part S_s^* with green color and hardening part S_h^* with red color.

320 The RHW representation of any point on the hardening surface S_h^* of the
 321 *GMCC* model can be represented in term of three parameters $r_1 = \theta$, $r_2 = \beta$,
 322 and $r_3 = a$. That is:

$$\begin{bmatrix} \sigma_1^* \\ \sigma_2^* \\ \sigma_3^* \end{bmatrix} = \begin{bmatrix} \frac{3(a+ab \cos(\theta))}{\sqrt{3}} \\ \cos(\beta) \frac{\sqrt{2}F(a) \sin(\theta)}{\Gamma(\beta)} \\ \sin(\beta) \frac{\sqrt{2}F(a) \sin(\theta)}{\Gamma(\beta)} \end{bmatrix} \quad (65)$$

323 where $F(a)$ is defined by the following expression:

$$F(a) = \frac{M\sqrt{-p_t(2ab + p_t)}}{\sqrt{3}b\Gamma(\beta)} \quad (66)$$

324 3.6. Numerical integration algorithm for generalized modified Cam-Clay

325 The minimization of RHW square distance from $\sigma_{trial}^{*,n}$ to a point $\sigma^{*,n} \in$
 326 $S_h^*(a)$ for *GMCC* model is presented for linear and nonlinear elasticity. In
 327 the rest of paper, for the simplicity of notation, we will use the notation
 328 $\sigma_{trial}^* = \sigma_{trial}^{*,n}$ and $\sigma^{*,n} = \sigma^*$. Using the definition (65), the RHW square
 329 distance for nonlinear elasticity is:

$$\begin{aligned}
d^2(\boldsymbol{\sigma}_{trial}^*, \boldsymbol{\sigma}^*) &= \frac{\left(\sigma_{trial,1}^* - \frac{3(a+ab \cos(\theta))}{\sqrt{3}}\right)^2}{3K_{trial}} + \frac{\left(\sigma_{trial,2}^* - \cos(\beta) \frac{\sqrt{2}F(a) \sin(\theta)}{\Gamma(\beta)}\right)^2}{2G_{trial}} + \\
&\frac{\left(\sigma_{trial,3}^* - \sin(\beta) \frac{\sqrt{2}F(a) \sin(\theta)}{\Gamma(\beta)}\right)^2}{2G_{trial}} := \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)
\end{aligned} \tag{67}$$

where the expression for G_{trial} depends on the flavor of the nonlinear elastic law.

The third relation of equation (64) by considering the equation (32) can be written as:

$$I_1(\boldsymbol{\sigma}_{trial}^*) - I_1(\boldsymbol{\sigma}^*) = 3K_{trial} [\epsilon_{pv}(a_{trial}) - \epsilon_{pv}(a)] \tag{68}$$

The equation (68) is applied to update simultaneously the position of the yield surface which is defined by the variable a . In the RHW space, the invariant $I_1(\boldsymbol{\sigma}^*)$ is expressed by:

$$I_1(\boldsymbol{\sigma}^*) = I_1(\theta, a) = 3(a + ab \cos(\theta)) \tag{69}$$

Define a residual associated with the position of the cap:

$$res(\theta, a) = [I_1(\boldsymbol{\sigma}_{trial}^*) - I_1(\theta, a)] - 3K_{trial} [\epsilon_{pv}(a_{trial}) - \epsilon_{pv}(a)] = 0 \tag{70}$$

The projected point on the Cam-clay surface as a function of a_{trial} and $\boldsymbol{\sigma}_{trial}^*$ can be obtained by computing a and $\boldsymbol{\sigma}_{proj}^* \in S_h^*(a)$ which minimizes the RHW distance to $\boldsymbol{\sigma}_{trial}^*$. The point is found by applying Newton's method to find the values of θ , β , and a such that:

$$\mathbf{R}_h = \begin{bmatrix} \frac{\partial \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)}{\partial \theta} \\ \frac{\partial \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)}{\partial \beta} \\ res(\theta, a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{71}$$

The procedure to solve the system of equations, derivative of the projected stresses, and the derivation of elastoplastic tangent operator for this *GMCC* model is documented in the Appendices A, B, and C.

345 4. Numerical results

346 To verify the implementation of *GMCC* model, we compare the numer-
 347 ical results of our implementation with the analytical solution provided by
 348 Perić [18] for infinitesimal strain assumption. The analytical solutions are
 349 presented separately for volumetric and deviatoric behavior of the material.

350 4.1. The volumetric behavior

351 The rate of volumetric strain is obtained from its nonlinear elasticity and
 352 hardening rule. Then, by integrating the rate over a finite time increment,
 353 the analytical expressions for volumetric elastic strain and plastic strain are
 354 derived as:

$$\epsilon_{ev} = \epsilon_{ev}^{\circ} + \frac{1}{1 + e^{\circ}} \ln \left[\left(\frac{P}{P^{\circ}} \right)^{-C^e} \right] \quad (72)$$

$$\epsilon_{pv} = \epsilon_{pv}^{\circ} + \frac{1}{1 + e^{\circ}} \ln \left[\left(\left(\frac{P}{P^{\circ}} \right) \left(\frac{M^2 + \bar{\eta}^2}{M^2 + \bar{\eta}_o^2} \right) \right)^{C^e - C^p} \right] \quad (73)$$

355 where $\bar{\eta}$ is the modified stress ratio and defines as:

$$\bar{\eta} = \eta g \quad \eta = \frac{Q}{P} \quad (74)$$

356 In the case of a conventional triaxial compression, the function g is equal
 357 to 1.0. The superscript ($^{\circ}$) refers to the initial condition herein.

358 A total volumetric strain is:

$$\epsilon_v = \epsilon_v^{\circ} + \frac{1}{1 + e^{\circ}} \ln \left[\left(\frac{P}{P^{\circ}} \right)^{-C^p} \left(\frac{M^2 + \bar{\eta}^2}{M^2 + \bar{\eta}_o^2} \right)^{C^e - C^p} \right] \quad (75)$$

359 By considering a stress path in $P - Q$ space, with a slope $\bar{k} = k g$, in
 360 which $k = Q/P$, the total volumetric strain is expressed as:

$$\epsilon_v = \epsilon_v^{\circ} + \frac{1}{1 + e^{\circ}} \ln \left[\left(\frac{\bar{k} - \bar{\eta}_o}{\bar{k} - \bar{\eta}} \right)^{-C^p} \left(\frac{M^2 + \bar{\eta}^2}{M^2 + \bar{\eta}_o^2} \right)^{C^e - C^p} \right] \quad (76)$$

361 Note that in the case of conventional triaxial test $\bar{k} = 3.0$.

362 4.2. The deviatoric behavior

363 By integrating the rate of deviatoric strain over a finite time increment,
 364 the analytical expressions for generalized shear elastic strain and plastic
 365 strain are obtained as:

$$\epsilon_{eq} = \epsilon_{eq}^{\circ} + \frac{1}{1+e^{\circ}} \ln \left[\left(\frac{\bar{k} - \bar{\eta}}{\bar{k} - \bar{\eta}_o} \right)^{-\frac{C^e \bar{k}}{3\omega g}} \right] \quad (77)$$

$$\begin{aligned} \epsilon_{pq} = & \epsilon_{pq}^{\circ} - \frac{2C_i}{M(1+e^{\circ})} \left[\arctan \left(\frac{\bar{\eta}}{M} \right) - \arctan \left(\frac{\bar{\eta}_o}{M} \right) \right] + \\ & \frac{1}{1+e^{\circ}} \ln \left[\left(\frac{M-\bar{\eta}}{M-\bar{\eta}_o} \right)^{\frac{C_i \bar{k}}{M(M-\bar{k})}} \left(\frac{M+\bar{\eta}}{M+\bar{\eta}_o} \right)^{\frac{C_i \bar{k}}{M(M+\bar{k})}} \left(\frac{\bar{k}-\bar{\eta}}{\bar{k}-\bar{\eta}_o} \right)^{\frac{2C_i \bar{k}}{\bar{k}^2-M^2}} \right] \end{aligned} \quad (78)$$

366 where

$$C_i = (C^p - C^e) g \quad \omega = \frac{3(1-2\nu)}{2(1+\nu)} \quad (79)$$

367 A total generalized shear strain is:

$$\begin{aligned} \epsilon_q = & \epsilon_q^{\circ} - \frac{2C_i}{M(1+e^{\circ})} \left[\arctan \left(\frac{\bar{\eta}}{M} \right) - \arctan \left(\frac{\bar{\eta}_o}{M} \right) \right] + \\ & \frac{1}{1+e^{\circ}} \ln \left[\left(\frac{M-\bar{\eta}}{M-\bar{\eta}_o} \right)^{\frac{C_i \bar{k}}{M(M-\bar{k})}} \left(\frac{M+\bar{\eta}}{M+\bar{\eta}_o} \right)^{\frac{C_i \bar{k}}{M(M+\bar{k})}} \left(\frac{\bar{k}-\bar{\eta}}{\bar{k}-\bar{\eta}_o} \right)^{\frac{2C_i \bar{k}}{\bar{k}^2-M^2} - \frac{C^e \bar{k}}{3\omega g}} \right] \end{aligned} \quad (80)$$

368 The axial and radial strains in a triaxial test can be calculated from
 369 volumetric and generalized shear strains as follows:

$$\epsilon_a = \frac{1}{3}\epsilon_v + \epsilon_q \quad \epsilon_r = \frac{1}{3}\epsilon_v - \frac{1}{2}\epsilon_q \quad (81)$$

370 4.3. Verification of the implementation

371 The consolidated-drained triaxial test documented in [36] is reproduced.
 372 In this test, the sample is first consolidated under hydrostatic pressure and
 373 thus sheared by applying additional axial load. The material parameters
 374 that were used in this test are listed in Table 1. The test is on a normally
 375 consolidated clay sample which involves elastoplastic loading, a behavior that
 376 considers hardening. The stress paths, initial and final yield surfaces for
 377 softening and hardening part of this test is shown in Fig. 4. The performance
 378 of this test is done in two examples. First by giving the constant shear
 379 modulus and second by considering the constant Poisson's ratio.

Table 1. Material parameters for clay [36].

Parameter	Value
G (Shear modulus)	20 MPa
ν (Poisson's ratio)	0.3
M	1.2
C^e	0.0077
C^p	0.066
e°	0.438
ψ	1
P°	0.2 MPa
p_c°	0.2 MPa
Q°	0.0 MPa

Fig. 4. The normally consolidated-drained triaxial test: stress paths, initial and final yield surfaces for softening and hardening part in the $P - Q$ space.

380 4.3.1. Verification for constant shear modulus

381 For the following triaxial test, all the subfigures in the Fig. 5 presents
382 a comparison for the *GMCC* model implementation with the results calcu-
383 lated from an analytical solution (see in the Appendix D), displaying the
384 verification of the implementation for constant shear modulus.

Fig. 5. Comparison between the numerical results of the *GMCC* model for constant shear modulus with the analytical solution provided by Perić [18]. Relation between axial strain and von Mises stress (top-left), relation between volumetric strain and effective mean stress (top-right), relation between volumetric plastic strain and preconsolidation pressure (middle-left), relation between effective mean stress and Bulk modulus (middle-right), relation between effective mean stress and von Mises stress (bottom-left), and relation between axial strain and volumetric strain (bottom-right).

385 Fig. 6 (left) shows the evolution of the residual normal when solving
 386 the nonlinear systems of equations to project σ_{trial}^* on $S_h^*(a)$ for the *GMCC*
 387 model with constant shear modulus, illustrating that the number of iterations
 388 required for convergence is less than 5. A typical 3D profile of the *GMCC*
 389 model for constant shear modulus is shown in Fig. 6 (right).

Fig. 6. (Left) Residual norm per iteration for each load step for the *GMCC* model with constant shear modulus; (Right) The generalized modified Cam-Clay plasticity yield function for constant shear modulus in which the arrow with a blue color shows the evolution of elastoplastic model.

390 4.3.2. Verification for constant Poisson's ratio

391 All the subfigures in the Fig. 7 illustrates a comparison for the *GMCC*
 392 model implementation with the results calculated from an analytical solution
 393 (see in the Appendix D), demonstrating the verification of the implementa-
 394 tion for constant Poisson's ratio.

Fig. 7. Comparison between the numerical results of the *GMCC* model for constant Poisson's ratio with the analytical solution provided by Perić [18]. Relation between axial strain and von Mises stress (top-left), relation between volumetric strain and effective mean stress (top-right), relation between volumetric plastic strain and preconsolidation pressure (middle-left), relation between effective mean stress and Bulk modulus (middle-right), relation between effective mean stress and von Mises stress (bottom-left), and relation between axial strain and volumetric strain (bottom-right).

Fig. 8 (left) displays that the evolution of the residual normal when solving the nonlinear systems of equations to project σ_{trial}^* on $S_h^*(a)$ for the *GMCC* model with constant shear modulus, showing that the number of iterations required for convergence is less than 5. A typical 3D profile of the *GMCC* model for constant Poisson's ratio is shown in Fig. 8 (right).

Fig. 8. (Left) Residual norm per iteration for each load step for the *GMCC* model with constant Poisson's ratio; (Right) The generalized modified Cam-Clay plasticity yield function for constant Poisson's ratio in which the arrow with a blue color shows the evolution of elastoplastic model.

5. Conclusions

An innovative description for the modified Cam-Clay plasticity model is developed. The description allows to contemplate the dependence on the Lode's angle. The modified Cam-Clay usually described in the $P - Q$ space is reformulated using the rotated Haigh-Westergaard space. By doing so the plastic return-mapping amounts to closest point projection to the yield surface as a function of a only three independent parameters. The advantage of the new scheme for the modified Cam-Clay is its compactness and numerical efficiency. The Newton method applied to the minimization of a convex function which leads to fast convergence and improved numerical stability. The average number of iterations required for convergence is less than 5.

411 Appendix

412 *Appendix A: Procedure for solving the elasto plastic return map*

413 It is done by representing the RHW square distance separately for linear
414 and nonlinear elasticity:

415 The RHW square distance for nonlinear elasticity is:

$$\begin{aligned} \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a) = & \frac{\left(\sigma_{trial,1}^* - \frac{3(a+a \cos(\theta))}{\sqrt{3}}\right)^2}{3K_{trial}} + \frac{\left(\sigma_{trial,2}^* - \cos(\beta) \frac{\sqrt{2}F(a) \sin(\theta)}{\Gamma(\beta)}\right)^2}{2G_{trial}} + \\ & \frac{\left(\sigma_{trial,3}^* - \sin(\beta) \frac{\sqrt{2}F(a) \sin(\theta)}{\Gamma(\beta)}\right)^2}{2G_{trial}} \end{aligned} \quad (82)$$

416 where the expression for G_{trial} depends on the flavour of the nonlinear
417 elastic law. And,

$$res(\theta, a) = [I_1(\boldsymbol{\sigma}_{trial}^*) - I_1(\theta, a)] - 3K_{trial} [\epsilon_{pv}(a_{trial}) - \epsilon_{pv}(a)] = 0 \quad (83)$$

$$\mathbf{R}_h = \begin{bmatrix} \frac{\partial \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)}{\partial \theta} \\ \frac{\partial \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)}{\partial \beta} \\ res(\theta, a) \end{bmatrix} \quad (84)$$

418 and the corresponding tangent matrix is:

$$\mathbf{J}_h = \begin{bmatrix} \frac{\partial^2 \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)}{\partial \theta^2} & \frac{\partial^2 \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)}{\partial \theta \partial \beta} & \frac{\partial^2 \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)}{\partial \theta \partial a} \\ \frac{\partial^2 \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)}{\partial \beta \partial \theta} & \frac{\partial^2 \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)}{\partial \beta^2} & \frac{\partial^2 \delta_h(\boldsymbol{\sigma}_{trial}^*; \theta, \beta, a)}{\partial \beta \partial a} \\ \frac{\partial res(\theta, a)}{\partial \theta} & \frac{\partial res(\theta, a)}{\partial \beta} & \frac{\partial res(\theta, a)}{\partial a} \end{bmatrix} \quad (85)$$

419 *Appendix B: Derivative of project stresses with respect to trial* 420 *stresses*

421 The chain rule for the hardening surface S_h^* is:

$$\frac{\partial \boldsymbol{\sigma}_{proj}}{\partial \boldsymbol{\sigma}_{trial}} = \frac{\partial \boldsymbol{\sigma}_{proj}}{\partial (\theta, \beta, a)} \frac{\partial (\theta, \beta, a)}{\partial \boldsymbol{\sigma}_{trial}} \quad (86)$$

422 By taking the derivative of the RHW square distance as:

$$\mathbf{R}_h = 0 \quad (87)$$

423 And applying the Taylor expansion on it, which is:

$$\frac{\partial \mathbf{R}_h}{\partial \boldsymbol{\sigma}_{trial}} \delta \boldsymbol{\sigma}_{trial} + \frac{\partial \mathbf{R}_h}{\partial (\theta, \beta, a)} \delta (\theta, \beta, a) + O(\boldsymbol{\sigma}_{trial}, \theta, \beta, a) = 0 \quad (88)$$

424 By rearranging it:

$$\frac{\partial (\theta, \beta, a)}{\partial \boldsymbol{\sigma}_{trial}} = - \left(\frac{\partial \mathbf{R}_h}{\partial (\theta, \beta, a)} \right)^{-1} \frac{\partial \mathbf{R}_h}{\partial \boldsymbol{\sigma}_{trial}} \quad (89)$$

$$\frac{\partial (\theta, \beta, a)}{\partial \boldsymbol{\sigma}_{trial}} = - (\mathbf{J}_h)^{-1} \frac{\partial \mathbf{R}_h}{\partial \boldsymbol{\sigma}_{trial}} \quad (90)$$

$$\frac{\partial \mathbf{R}_h}{\partial \boldsymbol{\sigma}_{trial}} = \begin{bmatrix} \frac{\partial^2 \delta_h}{\partial \theta \partial \sigma_1} & \frac{\partial^2 \delta_h}{\partial \theta \partial \sigma_2} & \frac{\partial^2 \delta_h}{\partial \theta \partial \sigma_3} \\ \frac{\partial^2 \delta_h}{\partial \beta \partial \sigma_1} & \frac{\partial^2 \delta_h}{\partial \beta \partial \sigma_2} & \frac{\partial^2 \delta_h}{\partial \beta \partial \sigma_3} \\ \frac{\partial^2 \delta_h}{\partial a \partial \sigma_1} & \frac{\partial^2 \delta_h}{\partial a \partial \sigma_2} & \frac{\partial^2 \delta_h}{\partial a \partial \sigma_3} \end{bmatrix} \quad (91)$$

$$\frac{\partial \boldsymbol{\sigma}_{proj}}{\partial (\theta, \beta, a)} = \begin{bmatrix} \frac{\partial \sigma_1}{\partial \theta} & \frac{\partial \sigma_1}{\partial \beta} & \frac{\partial \sigma_1}{\partial a} \\ \frac{\partial \sigma_2}{\partial \theta} & \frac{\partial \sigma_2}{\partial \beta} & \frac{\partial \sigma_2}{\partial a} \\ \frac{\partial \sigma_3}{\partial \theta} & \frac{\partial \sigma_3}{\partial \beta} & \frac{\partial \sigma_3}{\partial a} \end{bmatrix} \quad (92)$$

425 The variation of the eigenvectors between $\boldsymbol{\sigma}_{proj}$ and $\boldsymbol{\sigma}_{trial}$ is represented
 426 by a rotation and its inclusion in the tangent matrix can be demonstrated.
 427 The detail of the rotation can be found in [16].

428 **Appendix C: Derivation of elastoplastic tangent operator**

429 The elastoplastic tangent operator is derived from the derivative of the
 430 projected stress tensor with respect to the elastic strain tensor and it is
 431 defined by the chain rule as follows:

$$\mathbb{D}^{ep} = \frac{\partial \boldsymbol{\sigma}_{proj}}{\partial \boldsymbol{\epsilon}_e} = \frac{\partial \boldsymbol{\sigma}_{proj}}{\partial \boldsymbol{\sigma}_{trial}} \frac{\partial \boldsymbol{\sigma}_{trial}}{\partial \boldsymbol{\epsilon}_e} \quad (93)$$

432 By straightforward derivation, the elastoplastic tangent operator can be
 433 written as:

$$\frac{\partial \boldsymbol{\sigma}_{proj}}{\partial \boldsymbol{\epsilon}_e} = \left(\sum \frac{\partial \boldsymbol{\sigma}_{proj,i}}{\partial \boldsymbol{\sigma}_{trial,j}} (x_i \otimes x_i) \otimes (x_j \otimes x_j) \right) \mathbf{C} \quad (94)$$

434 The matrix form of elastoplastic tangent operator is:

$$\frac{\partial \sigma_{proj}}{\partial \epsilon_e} \delta \epsilon_e = \left(\sum \frac{\partial \sigma_{proj,i}}{\partial \sigma_{trial,j}} (x_i \otimes x_i) \otimes (x_j \otimes x_j) \right) \mathbf{C} \delta \epsilon_e \quad (95)$$

435 where $\mathbf{C}$ is the elastic tangent operator. To compute elastic tangent op-
 436 erator for constant Poisson's ratio in equation (18), it is required to consider
 437 the following remark:

438 **Remark 1:** The Bulk modulus in equation (17) is a function of the elastic
 439 volumetric deformation. Inspired by the article [6], to prevent a non-
 440 conservative model for constant Poisson's ratio in which energy may
 441 not be conserved, we propose to maintain the Bulk modulus constant
 442 during the time interval $[t^{n-1}, t^n]$ and between timesteps is updated.

443 *Appendix D: Triaxial test on a normally consolidated clay sample*

444 The triaxial test results are presented for effective mean stress, von Mises
 445 stress, elastic volumetric strain, plastic volumetric strain, volumetric strain,
 446 generalized shear elastic strain, generalized shear plastic strain, generalized
 447 shear strain, preconsolidation pressure, and Bulk modulus, in order to verify
 448 the modified Cam-Clay model. The results are expressed separately for case
 449 of constant shear modulus and case of constant Poisson's ratio, as follows:

450 *Triaxial test: results for case of constant shear modulus*

Table 2. Triaxial results for case of constant shear modulus.

No.	P	Q	ϵ_{ev}	ϵ_{pv}	ϵ_v	ϵ_{eq}	ϵ_{pq}	ϵ_q	ϵ_a	ϵ_r	p_c	K
Unit	[MPa]	[MPa]									[MPa]	[MPa]
1	0.2	0.	-0.00382	0.	-0.00382	0.	0.	0.	-0.00127	-0.00127	0.2	52.3
2	0.207	0.0204	-0.00395	-0.00169	-0.00564	0.00034	0.000123	0.000462	-0.00234	-0.00165	0.208	54.08
3	0.214	0.0407	-0.00408	-0.00381	-0.00789	0.000679	0.000563	0.00124	-0.00387	-0.00201	0.219	55.85
4	0.22	0.0611	-0.00419	-0.00627	-0.0105	0.00102	0.0014	0.00242	-0.00591	-0.00228	0.232	57.63
5	0.227	0.0815	-0.00431	-0.00896	-0.0133	0.00136	0.00268	0.00404	-0.00846	-0.0024	0.247	59.41
6	0.234	0.102	-0.00442	-0.0118	-0.0162	0.0017	0.00445	0.00615	-0.0116	-0.00233	0.265	61.18
7	0.241	0.122	-0.00453	-0.0147	-0.0193	0.00204	0.00672	0.00876	-0.0152	-0.00204	0.284	62.96
8	0.248	0.143	-0.00464	-0.0177	-0.0223	0.00238	0.00953	0.0119	-0.0194	-0.00149	0.305	64.73
9	0.254	0.163	-0.00474	-0.0207	-0.0254	0.00272	0.0129	0.0156	-0.0241	-0.000651	0.327	66.51
10	0.261	0.183	-0.00484	-0.0236	-0.0284	0.00306	0.0169	0.02	-0.0295	0.000506	0.351	68.29
11	0.268	0.204	-0.00494	-0.0265	-0.0314	0.0034	0.0216	0.025	-0.0355	0.00202	0.376	70.06
12	0.275	0.224	-0.00504	-0.0293	-0.0344	0.00374	0.0271	0.0308	-0.0423	0.00395	0.402	71.84
13	0.281	0.244	-0.00513	-0.0321	-0.0372	0.00407	0.0335	0.0375	-0.0499	0.00636	0.429	73.61
14	0.288	0.265	-0.00522	-0.0348	-0.04	0.00441	0.041	0.0454	-0.0587	0.00936	0.457	75.39
15	0.295	0.285	-0.00531	-0.0374	-0.0427	0.00475	0.0499	0.0547	-0.0689	0.0131	0.487	77.17
16	0.302	0.306	-0.0054	-0.0399	-0.0453	0.00509	0.0609	0.0659	-0.0811	0.0179	0.517	78.94
17	0.309	0.326	-0.00548	-0.0424	-0.0479	0.00543	0.0746	0.0801	-0.096	0.0241	0.548	80.72
18	0.315	0.346	-0.00557	-0.0448	-0.0503	0.00577	0.0931	0.0989	-0.116	0.0327	0.58	82.49
19	0.322	0.367	-0.00565	-0.047	-0.0527	0.00611	0.121	0.127	-0.144	0.0458	0.612	84.27
20	0.329	0.387	-0.00573	-0.0493	-0.055	0.00645	0.176	0.182	-0.201	0.0728	0.645	86.05

Table 3. Triaxial results for case of constant Poisson's ratio.

No.	P	Q	ϵ_{ev}	ϵ_{pv}	ϵ_v	ϵ_{eq}	ϵ_{pq}	ϵ_q	ϵ_a	ϵ_r	p_c	K
Unit	[MPa]	[MPa]									[MPa]	[MPa]
1	0.2	0.	-0.00382	0.	-0.00382	0.	0.	0.	-0.00127	-0.00127	0.2	52.3
2	0.207	0.0204	-0.00395	-0.00169	-0.00564	0.000277	0.000123	0.000399	-0.00228	-0.00168	0.208	54.08
3	0.214	0.0407	-0.00408	-0.00381	-0.00789	0.000544	0.000563	0.00111	-0.00374	-0.00208	0.219	55.85
4	0.22	0.0611	-0.00419	-0.00627	-0.0105	0.000804	0.0014	0.0022	-0.00569	-0.00239	0.232	57.63
5	0.227	0.0815	-0.00431	-0.00896	-0.0133	0.00106	0.00268	0.00374	-0.00816	-0.00255	0.247	59.41
6	0.234	0.102	-0.00442	-0.0118	-0.0162	0.0013	0.00445	0.00575	-0.0112	-0.00253	0.265	61.18
7	0.241	0.122	-0.00453	-0.0147	-0.0193	0.00154	0.00672	0.00826	-0.0147	-0.00229	0.284	62.96
8	0.248	0.143	-0.00464	-0.0177	-0.0223	0.00177	0.00953	0.0113	-0.0187	-0.00179	0.305	64.73
9	0.254	0.163	-0.00474	-0.0207	-0.0254	0.00199	0.0129	0.0149	-0.0234	-0.00101	0.327	66.51
10	0.261	0.183	-0.00484	-0.0236	-0.0284	0.00221	0.0169	0.0191	-0.0286	0.0000824	0.351	68.29
11	0.268	0.204	-0.00494	-0.0265	-0.0314	0.00242	0.0216	0.024	-0.0345	0.00153	0.376	70.06
12	0.275	0.224	-0.00504	-0.0293	-0.0344	0.00263	0.0271	0.0297	-0.0412	0.0034	0.402	71.84
13	0.281	0.244	-0.00513	-0.0321	-0.0372	0.00283	0.0335	0.0363	-0.0487	0.00574	0.429	73.61
14	0.288	0.265	-0.00522	-0.0348	-0.04	0.00303	0.041	0.044	-0.0573	0.00867	0.457	75.39
15	0.295	0.285	-0.00531	-0.0374	-0.0427	0.00322	0.0499	0.0532	-0.0674	0.0123	0.487	77.17
16	0.302	0.306	-0.0054	-0.0399	-0.0453	0.00341	0.0609	0.0643	-0.0794	0.017	0.517	78.94
17	0.309	0.326	-0.00548	-0.0424	-0.0479	0.0036	0.0746	0.0782	-0.0942	0.0232	0.548	80.72
18	0.315	0.346	-0.00557	-0.0448	-0.0503	0.00378	0.0931	0.0969	-0.114	0.0317	0.58	82.49
19	0.322	0.367	-0.00565	-0.047	-0.0527	0.00395	0.121	0.125	-0.142	0.0448	0.612	84.27
20	0.329	0.387	-0.00573	-0.0493	-0.055	0.00412	0.176	0.18	-0.198	0.0716	0.645	86.05

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