A Feynman-Kac approach for Logarithmic Sobolev Inequalities

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Abstract

This note presents a method based on Feynman-Kac semigroups for logarithmic Sobolev inequalities. It follows the recent work of Bonnefont and Joulin on intertwining relations for diffusion operators, formerly used for spectral gap inequalities. In particular, it goes beyond the Bakry-Émery criterion and allows to investigate high-dimensional effects on the optimal logarithmic Sobolev constant. The method is finally illustrated on particular examples, for which explicit dimension-free bounds on the latter constant are provided.

1. Introduction

Since their introduction by Gross in 1975, the Logarithmic Sobolev Inequalities (LSI) became a widely used tool in infinite dimensional analysis. Initially studied in relation to the hypercontractivity property for Markov semigroups, they turned out to be prominent in many various domains, at the interface of analysis, probability theory and geometry.

For μ a probability measure on $\mathbf{R}^d$, this inequality provides a control on the entropy of any smooth function f in term of its gradient:

$$\mathrm{Ent}_\mu(f^2) \leq c \int_{\mathbf{R}^d} |\nabla f|^2 d\mu,$$

for some $c > 0$, where $\mathrm{Ent}_\mu(f^2) = \int_{\mathbf{R}^d} f^2 \log(f^2)\, d\mu - \left(\int_{\mathbf{R}^d} f^2\, d\mu\right) \log\left(\int_{\mathbf{R}^d} f^2\, d\mu\right)$. The optimal constant for the latter inequality to hold, that will be denoted by $c_{LSI}(\mu)$, is of primary importance in the study of the measure μ, since it encodes many of its properties. For instance, among many results in this area, Otto and Villani established in [18] a connection between LSI and some transportation inequalities (see also the related work by Bobkov and Götze in [7]), and Herbst provided a powerful argument that connects LSI to Gaussian concentration inequalities (see the lecture notes by Ledoux [16] for more details and his reference monograph [17] about concentration of measure).

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The case where μ is the invariant measure of some Markov process is also of great interest. For example, apart from Gross' initial results on hypercontractivity in [13], $c_{LSI}(\mu)$ encodes the decay in entropy of the related semigroup, and is linked to the Fisher information through the de Bruijn's identity. Significant advances in this setting were due to Bakry and Émery in [4], who stated their eponymous criterion, also known as "curvature-dimension criterion", that connects the logarithmic Sobolev inequality (and many functional inequalities) to geometric properties of μ. We refer to [5] for a comprehensive overview of this theory.

Although the value of $c_{LSI}(\mu)$ is of primary importance in the study of μ, its explicit value is hardly ever known explicitly. Bakry-Émery theory provides sharp estimates on this constant for some log-concave measures, assumption that might be weakened according to some perturbation arguments, such as the well-known Holley-Stroock method. Stability of LSI by tensorization is also a key property of such inequalities, since it exhibits dimension-free behaviours for product measures, but fails to provide efficient bounds beyond this case. In particular, one may wish to keep track of the geometry of μ (dimension of the space, log-concavity, curvature, etc.) through $c_{LSI}(\mu)$, which can be difficult in many settings (as will be discussed in Section 4). We refer to the remarkably synthetic monograph [1] for further informations on LSI.

In this note, we provide a method based on the study of some Feynman-Kac semigroups (a somewhat similar approach can be found in the recent work of Sturm and his collaborators on metric measure spaces [11]) to derive new estimates on the logarithmic Sobolev constant. This technique encompasses the curvature-dimension criterion, yet is derived in a very general setting, allowed by our probabilistic point of view. To this end, we follow the recent work of Bonnefont and Joulin involving intertwinings and functional inequalities of spectral flavour [8, 9] and extend their approach to the logarithmic Sobolev inequalities. We also briefly discuss a comparison to the Holley-Stroock method.

Main results on intertwinings are recalled in Section 2, along with the framework of this article. In Section 3, we state and prove our main results. First, a probabilistic representation of Feynman-Kac semigroups is provided in Theorem 3.3 (using tangent processes and a Girsanov transformation), from which estimates on $c_{LSI}(\mu)$ are derived in Theorem 3.5, following an idea from Wang [19]. The last section is devoted to the computation of explicit bounds on the logarithmic Sobolev constant in some particular cases.

2. Basic framework

In this first section, we recall the framework of our analysis, basic results and definitions about intertwinings and Feynman-Kac semigroups (as introduced in [8, 2]).

2.1 Setting

The whole analysis shall be performed on the d-dimensional Euclidean space $(\mathbf{R}^d, |\cdot|)$, for $d \in \mathbf{N}^\star$. We let $\mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R})$ and $\mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R}^d)$ be respectively the set of smooth functions and vector fields on $\mathbf{R}^d$, and let $\mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$ and $\mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$ denote respectively the set of compactly supported and positive smooth functions on $\mathbf{R}^d$. We endow those spaces with the supremum norm $\|\cdot\|_\infty$. We consider throughout this article a probability measure μ on $\mathbf{R}^d$ whose density with respect to the Lebesgue measure is proportional to e^{-V}, for some smooth potential V. To this measure, one can associate a Markov diffusion operator defined as

$$\mathbf{L} = \Delta - \nabla V \cdot \nabla,$$

where Δ and ∇ respectively stand for the usual Laplace operator and gradient on $\mathbf{R}^d$. The flow of $\mathbf{L}$ over $\mathbf{R}_+$ defines a Markov semigroup $(\mathbf{P}_t)_{t\geq 0}$, invariant with respect to μ, which is, under standard assumptions on V, ergodic in $L^2(\mu)$. Moreover, this semigroup describes the dynamics of a diffusion process $(X_t^x)_{t\geq 0}$ that solves the following Stochastic Differential Equation (SDE):

$$dX_t^x = \sqrt{2}\, dB_t - \nabla V(X_t^x)dt, \quad X_0^x = x \in \mathbf{R}^d \text{ a.s.}, \tag{E}$$

where $(B_t)_{t\geq 0}$ denotes the standard d-dimensional Brownian motion. Under mild assumptions on V (such as a Lyapunov or a drift condition), this process is non-explosive and converges in distribution towards μ, its invariant distribution (see for example [3]). Moreover, regularity of V ensures that $x \mapsto X_t^x$ is smooth over $\mathbf{R}^d$, for any $t \geq 0$. Such assumptions shall be implicitly made throughout the paper.

As a basis of the well-known $\mathbf{\Gamma}_2$-calculus, introduced by Bakry and Émery in [4], we let $\mathbf{\Gamma}$ be the *carré du champ* operator defined on $\mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R}) \times \mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R})$ as

$$\mathbf{\Gamma}(f, g) = \frac{1}{2}\left[\mathbf{L}(fg) - f\mathbf{L}g - g\mathbf{L}f\right] = \nabla f \cdot \nabla g, \quad f, g \in \mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R}).$$

Moreover, $\mathbf{L}$ is symmetric with respect to μ, and the integration by parts formula rewrites as follows: for any $f, g \in \mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$,

$$\int_{\mathbf{R}^d} \mathbf{\Gamma}(f, g)\, d\mu = -\int_{\mathbf{R}^d} f\mathbf{L}g\, d\mu = -\int_{\mathbf{R}^d} g\mathbf{L}f\, d\mu = \int_{\mathbf{R}^d} \nabla f \cdot \nabla g\, d\mu.$$

In particular, $\mathbf{L}$ is non-positive on $\mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$. Hence by completeness, this operator admits a unique self-adjoint extension (which shall still be denoted by $\mathbf{L}$) on some domain $\mathcal{D}(\mathbf{L}) \subset L^2(\mu)$ for which $\mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$ is a core, *i.e.* is dense for the norm induced by $\mathbf{L}$.

Finally, let us recall the definition of the logarithmic Sobolev inequality we will refer to.

Definition 2.1. The measure μ is said to satisfy a *Logarithmic Sobolev Inequality* with constant $c > 0$ (in short $LSI(c)$) if for any $f \in \mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$ one has

$$\mathrm{Ent}_\mu(f^2) \leq c \int_{\mathbf{R}^d} |\nabla f|^2 d\mu.$$

We let $c_{LSI}(\mu)$ denote the optimal constant in the latter inequality.

2.2 Intertwinings

We now focus on intertwinings (for a comprehensive introduction, see [8, 2]). Basically, we are interested in commutation relations between gradients and Markov generators, which gives rise to the so-called Feynman-Kac semigroups.

Definition 2.2 (Diagonal generator)**.** Let $F = (F_1, \dots, F_d) \in \mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R}^d)$ be a smooth vector field. We define the *diagonal Markov generator* $\mathcal{L}$ as the diagonal action of $\mathbf{L}$ on $(F_1, \dots, F_d)$, that is

$$\mathcal{L}F := (\mathbf{L}F_1, \dots, \mathbf{L}F_d).$$

If F is moreover compactly supported, one can define $(\mathcal{P}_t F)_{t\geq 0}$ as the Markov semigroup associated to $\mathcal{L}$. In terms of stochastic processes, this writes as

$$\mathcal{P}_t F = \mathbf{E}[F(X_t)], \quad t \geq 0,$$

with $(X_t)_{t\geq 0}$ the diffusion process defined by Equation (E), where the initial condition is omitted (which will be the case from now on, except when necessary).

One can write then the intended intertwining and associated Feynman-Kac semigroup. This idea takes roots in various works in differential geometry and operators analysis, and relates (in some more general setting) to the Weitzenböck formula. See also the works around Witten Laplacians arising in statistical mecanics, for which we refer to Helffer's monograph [14].

A straightforward computation leads to the following result.

Proposition 2.3. *Let $f \in \mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R})$. One has:*

$$\nabla \mathbf{L} f = (\mathcal{L} - \nabla^2 V)(\nabla f),$$

where $\nabla^2 V$ acts above as a product (zero-order) operator. The semigroup associated to the Schrödinger-like generator $\mathcal{L}^{\nabla^2 V} := \mathcal{L} - \nabla^2 V$, denoted $(\mathcal{P}_t^{\nabla^2 V})_{t\geq 0}$, is a Feynman-Kac semigroup *and satisfies the following intertwining relation:*

$$\nabla \mathbf{P}_t f = \mathcal{P}_t^{\nabla^2 V}(\nabla f), \quad t \geq 0,$$

provided that f has compact support.

Remark. We can still define the Feynman-Kac semigroup associated to $\mathcal{L}$ and a general smooth map $M : \mathbf{R}^d \to \mathcal{M}_d(\mathbf{R})$ as the flow of the following PDE system:

$$\begin{cases} \partial_t u &= (\mathcal{L} - M)u \\ u(0, \cdot) &= u_0 \end{cases},$$

denoted by $(\mathcal{P}_t^M)_{t\geq 0}$, provided that solutions to this system do not explode in finite time. Such an extension will be implicitly used later.

3. Main results

In this section, we state and prove our main results in two steps: we first provide a representation theorem, related to Feynman-Kac semigroups, then apply it to estimates on the logarithmic Sobolev constant.

3.1 Representation of Feynman–Kac semigroups

This first part is devoted to the main representation theorem we shall make use of. It is presented for Feynman-Kac semigroups acting on gradients, but still holds for more general vector fields (in which case the proof relies on a classical martingale argument).

The perturbation technique that will be set up in the next section strongly relies on a Girsanov representation of the semigroup $(\mathbf{P}_t)_{t\geq 0}$. To this end, we introduce a smooth perturbation function in V and study the relation between $(X_t)_{t\geq 0}$ and the process obtained from this new potential.

Definition 3.1. Let $a \in \mathcal{C}^\infty_+(\mathbf{R}^d, \mathbf{R})$. We let $(X_{t,a})_{t\geq 0}$ denote the solution of the SDE

$$dX_{t,a} = \sqrt{2}dB_t - \nabla V_a(X_{t,a})\,dt,$$

where $V_a = V + \log(a^2)$.

Straightforward computations show that the generator of this process writes down

$$\mathbf{L}_a = \mathbf{L} + 2a\nabla(a^{-1})\cdot\nabla,$$

and we let $(\mathbf{P}_{t,a})_{t\geq 0}$ denote the associated Markov semigroup (in particular, for any $f \in \mathcal{C}^\infty_0(\mathbf{R}^d, \mathbf{R})$, $\mathbf{P}_{t,a}f = \mathbf{E}[f(X_{t,a})]$). Moreover, if μ_a stands for the Boltzmann measure associated to V_a ($d\mu_a(x) \propto e^{-V_a}dx$, in particular $d\mu_a = d\mu/a^2$), then $(\mathbf{P}_{t,a})_{t\geq 0}$ is μ_a-invariant and $\mathbf{L}_a$ is (essentially) self-adjoint in $L^2(\mu_a)$.

Provided that everything is well-defined, the intertwining relation of Proposition 2.3 still holds for $\mathbf{P}_{t,a}f$, and writes as follows:

$$\nabla\mathbf{P}_{t,a}f =: \mathcal{P}^{\nabla^2 V_a}_{t,a}(\nabla f).$$

Before we state the main result of this section, let us define a condition on the perturbation function that naturally arises in the computations involving Girsanov's theorem.

Definition 3.2. A function $a \in \mathcal{C}^\infty_+(\mathbf{R}^d, \mathbf{R})$ is said to satisfy the *(G) condition* whenever $|\nabla a|/a$ is bounded.

We can now state the representation result.

Theorem 3.3. *Let $f \in \mathcal{C}^\infty_0(\mathbf{R}^d, \mathbf{R})$ and $a \in \mathcal{C}^\infty_+(\mathbf{R}^d, \mathbf{R})$ satisfying (G). Then for any $t \geq 0$,*

$$\mathcal{P}^{\nabla^2 V}_t(\nabla f) = \mathbf{E}[R_{t,a}J^{X_a}_t\nabla f(X_{t,a})],$$

where $(R_{t,a})_{t\geq 0}$ is a martingale defined as

$$R_{t,a} = \frac{a(X_{t,a})}{a} \exp\left(-\int_0^t \frac{\mathbf{L}_a a}{a}(X_{s,a})\, ds\right), \quad t \geq 0,$$

and $(J_t^{X_a})_{t\geq 0}$ is a matrix-valued process that solves

$$\begin{cases} dJ_t^{X_a} & = -J_t^{X_a} \nabla^2 V(X_{t,a}) dt, \quad t > 0 \\ J_0^{X_a} & = I_d \end{cases}.$$

As mentioned before, this result is based on Girsanov's theorem. Hence, before we turn to its proof, we need the following lemma, that establishes a relation between the Markov semigroups $(\mathbf{P}_t)_{t\geq 0}$ and $(\mathbf{P}_{t,a})_{t\geq 0}$.

Lemma 3.4. *Let $a \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$ satisfying the (G) condition. Then for any function $f \in \mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$, any $t \geq 0$:*

$$\mathbf{P}_t f = \mathbf{E}[f(X_t)] = \mathbf{E}\left[R_{t,a} f(X_{t,a})\right],$$

where $(R_{t,a})_{t\geq 0}$ is the martingale defined above.

Proof. We first set up a suitable exponential martingale before we identify the involved probability distributions with Girsanov's theorem. For the sake of legibility, the initial condition shall be omitted in the following.

We first apply Itō's formula to $\log(a(X_{t,a}))$:

$$\log a(X_{t,a}) = \log a + \sqrt{2} \int_0^t \nabla(\log a(X_{s,a})) \cdot dB_s + \int_0^t \mathbf{L}_a(\log a)(X_{s,a})\, ds.$$

Expanding the right-hand side and taking exponential lead to the following formula for $R_{t,a}$:

$$R_{t,a} = \exp\left(\sqrt{2}\int_0^t \frac{\nabla a}{a}(X_{s,a}) \cdot dB_s - \int_0^t \left|\frac{\nabla a}{a}\right|^2 (X_{s,a})\, ds\right).$$

The (G) condition ensures through standard arguments that the right-hand side is a true martingale, then so is the left-hand one. From now on, we set $Y_{t,a} = \sqrt{2}\frac{\nabla a(X_{t,a})}{a}$.

We let $\mathbf{Q}_a$ be the probability measure defined as

$$\left.\frac{d\mathbf{Q}_a}{d\mathbf{P}}\right|_{\mathcal{F}_t} = R_{t,a},$$

with $\mathbf{P}$ the reference measure and $(\mathcal{F}_t)_{t\geq 0}$ the natural (completed) filtration associated to $(B_t)_{t\geq 0}$. According to Girsanov's theorem, the process $(\tilde{B}_t)_{t\geq 0}$ defined as

$$\tilde{B}_t = B_t - \int_0^t Y_{s,a}\, ds,$$

is a $\mathbf{Q}_a$-Brownian motion. Furthermore, the process $(X_{t,a})_{t\geq 0}$ solves the SDE

$$dX_{t,a} = \sqrt{2}d\tilde{B}_t - \nabla V(X_{t,a})\, dt,$$

hence the law of $X_{t,a}$ under $\mathbf{Q}_a$ coincides with the one of X_t under $\mathbf{P}$. In particular, for any $f \in \mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$,

$$\mathbf{P}_t f = \mathbf{E}[f(X_t)] = \mathbf{E}[R_{t,a} f(X_{t,a})],$$

and the proof is complete. ■

We can now prove Theorem 3.3.

Proof. Recall that under the aforementioned non-explosion assumptions, the diffusion process defined by Equation (E) is differentiable with respect to its initial condition, so that for any $t \geq 0$:

$$\begin{aligned} \mathcal{P}_t^{\nabla^2 V}(\nabla f) &= \nabla \mathbf{P}_t f \\ &= \mathbf{E}[\nabla(f(X_t))] \\ &= \mathbf{E}[J_t^X \nabla f(X_t)], \end{aligned}$$

where $(J_t^X)_{t\geq 0}$ denotes the (matrix-valued) tangent process to $(X_t)_{t\geq 0}$ (that is, the Jacobian matrix of X_t with respect to the initial condition). Differentiating with respect to the initial condition in the SDE (E) provides the following formula for J_t^X:

$$J_t^X = Id - \int_0^t J_s^X \nabla^2 V(X_s)\, ds.$$

Moreover, since Girsanov's theorem relates essentially to trajectories, one can replace X_s by $X_{s,a}$ in the previous expression, to define

$$J_t^{X_a} = Id - \int_0^t J_s^{X_a} \nabla^2 V(X_{s,a})\, ds.$$

Note that the potential V is unchanged in the equation. Lemma 3.4 implies then, since $R_{t,a}$ is scalar-valued,

$$\mathbf{E}[J_t^X \nabla f(X_t)] = \mathbf{E}[R_{t,a} J_t^{X_a} \nabla f(X_{t,a})],$$

and the proof is complete. ■

Remark. In dimension 1, since gradients and functions are both 1-dimensional objects, the Theorem 3.3 rewrites in a more standard way:

$$(\mathbf{P}_t f)' = \mathbf{P}_t^{V''}(f') = \mathbf{E}\left[R_{t,a} f'(X_{t,a}) \exp\left(-\int_0^t V''(X_{s,a})\, ds\right)\right].$$

This writing shall be useful when dealing with monotonic functions in dimension 1, as is briefly discussed at the end of the next section.

3.2 Logarithmic Sobolev inequalities

In this section, we provide a Feynman-Kac-based proof of the logarithmic Sobolev inequality, stated for a scalar perturbation. The method can easily be refined to improve the bound on $c_{LSI}(\mu)$, for example when finer spectral estimates on the generator are available or for a restricted set of test functions. For instance, we adapt the proof to derive estimates in restriction to monotonic (positive) functions.

3.2.1 General case

Notation. The proof of the following theorem requires some matrix analysis. Henceforward, if A is a symmetric matrix, we let $\rho_-(A)$ denote its smallest eigenvalue.

Theorem 3.5. *Let $a \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$. Define*

$$\kappa_a = \inf_{x \in \mathbf{R}^d} \left\{ 2\rho_-(\nabla^2 V(x)) - a\mathbf{L}(a^{-1})(x) \right\}.$$

If a, a^{-1} and $|\nabla a|$ are bounded and $\kappa_a > 0$, then for any $f \in \mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$,

$$\mathrm{Ent}_\mu(f^2) \leq \frac{4\|a\|_\infty \|a^{-1}\|_\infty}{\kappa_a} \int_{\mathbf{R}^d} |\nabla f|^2 \, d\mu.$$

Proof. Let $f \in \mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$ be a non-negative function. Ergodicity and μ-invariance of $(\mathbf{P}_t)_{t \geq 0}$ give:

$$\mathrm{Ent}_\mu(f) = -\int_{\mathbf{R}^d} \int_0^{+\infty} \partial_t \left(\mathbf{P}_t f \log \mathbf{P}_t f \right) dt \, d\mu = -\int_{\mathbf{R}^d} \int_0^{+\infty} \mathbf{L}[\mathbf{P}_t f] \log \mathbf{P}_t f \, dt \, d\mu.$$

The integration by parts formula and the intertwining relation yield then:

$$\mathrm{Ent}_\mu(f) = \int_{\mathbf{R}^d} \int_0^{+\infty} \frac{|\nabla \mathbf{P}_t f|^2}{\mathbf{P}_t f} \, d\mu \, dt = \int_{\mathbf{R}^d} \int_0^{+\infty} \frac{\left| \mathcal{P}_t^{\nabla^2 V}(\nabla f) \right|^2}{\mathbf{P}_t f} \, dt \, d\mu.$$

We focus on the numerator of the right-hand side. More precisely, we aim to cancel out $\mathbf{P}_t f$ at the denominator, which is made possible by Girsanov's theorem. Indeed, the assumptions on a ensure that it satisfies the (G) condition, and Theorem 3.3 leads to

$$\mathcal{P}_t^{\nabla^2 V}(\nabla f) = \mathbf{E}[R_{t,a} J_t^{X_a} \nabla f(X_{t,a})],$$

which rewrites

$$\mathcal{P}_t^{\nabla^2 V}(\nabla f) = 2\mathbf{E}\left[R_{t,a}^{1/2} J_t^{X_a} \nabla \sqrt{f}(X_{t,a}) R_{t,a}^{1/2} \sqrt{f}(X_{t,a}) \right].$$

Cauchy-Schwarz' inequality with Lemma 3.4 finally entail

$$\begin{aligned} \left| \mathcal{P}_t^{\nabla^2 V}(\nabla f) \right|^2 &\leq 4\mathbf{E}\left[\left| R_{t,a}^{1/2} J_t^{X_a} \nabla \sqrt{f}(X_{t,a}) \right|^2 \right] \mathbf{E}\left[R_{t,a} f(X_{t,a}) \right] \\ &= 4\mathbf{E}\left[\nabla \sqrt{f}(X_{t,a})^T J_t^{X_a} R_{t,a} (J_t^{X_a})^T \nabla \sqrt{f}(X_{t,a}) \right] \mathbf{P}_t f. \end{aligned}$$

This implies then for the entropy:

$$\mathrm{Ent}_\mu(f) \leq 4 \int_{\mathbf{R}^d} \int_0^{+\infty} \mathbf{E}\left[\nabla\sqrt{f}(X_{t,a})^T J_t^{X_a} R_{t,a} (J_t^{X_a})^T \nabla\sqrt{f}(X_{t,a})\right] \, dt \, d\mu.$$

In order to recover the energy term in the LSI, one should provide some spectral estimates for $J_t^{X_a} R_{t,a} (J_t^{X_a})^T$. Define then

$$J_t^a = J_t^{X_a} \exp\left(-\frac{1}{2}\int_0^t \frac{\mathbf{L}_a a}{a}(X_{s,a})\, ds\right),$$

which solves the following equation:

$$dJ_t^a = -J_t^a \left(\nabla^2 V(X_{t,a}) - \frac{1}{2} a \mathbf{L}(a^{-1})(X_{t,a}) I_d\right) dt.$$

Indeed, we have on the one hand:

$$dJ_t^{X_a} = -J_t^{X_a} \nabla^2 V(X_{t,a})\, dt,$$

and on the other hand:

$$d\left[\exp\left(-\frac{1}{2}\int_0^t \frac{\mathbf{L}_a a}{a}(X_{s,a})\, ds\right)\right] = -\frac{1}{2}\frac{\mathbf{L}_a a}{a}(X_{t,a}) \exp\left(-\frac{1}{2}\int_0^t \frac{\mathbf{L}_a a}{a}(X_{s,a})\, ds\right) dt.$$

Moreover, $\mathbf{L}_a(a)/a = -a\mathbf{L}(a^{-1})$, so that both previous points and a chain rule give the expected formula. Since $J_t^{X_a} R_{t,a} (J_t^{X_a})^T = \frac{a(X_{t,a})}{a(x)} J_t^a (J_t^a)^T$, one should focus on spectral estimates for the latter term.

Therefore, if we let $\varphi(t) = y^T J_t^a (J_t^a)^T y$, for some $y \in \mathbf{R}^d$, symmetry of $\nabla^2 V$ entails

$$\begin{aligned}
d\varphi(t) &= y^T dJ_t^a (J_t^a)^T y + y^T J_t^a (dJ_t^a)^T y \\
&= -y^T J_t^a \left(\nabla^2 V(X_{t,a}) - \frac{1}{2} a \mathbf{L}(a^{-1})(X_{t,a}) I_d\right) (J_t^a)^T y \, dt \\
&\quad - y^T J_t^a \left(\nabla^2 V(X_{t,a}) - \frac{1}{2} a \mathbf{L}(a^{-1})(X_{t,a}) I_d\right)^T (J_t^a)^T y \, dt \\
&= -y^T J_t^a \left(2\nabla^2 V(X_{t,a}) - a\mathbf{L}(a^{-1})(X_{t,a}) I_d\right) (J_t^a)^T y \, dt \\
&\leq -\kappa_a y^T J_t^a (J_t^a)^T y \, dt = -\kappa_a \varphi(t)\, dt,
\end{aligned}$$

by definition of κ_a. Hence, for any $t \geq 0$, $\varphi(t) \leq e^{-\kappa_a t}\varphi(0)$, which yields

$$y^T J_t^a (J_t^a)^T y \leq e^{-\kappa_a t} |y|^2.$$

We can apply the previous inequality to $y = \sqrt{\frac{a(X_{t,a})}{a(x)}} \nabla\sqrt{f}(X_{t,a})$ to get

$$\mathrm{Ent}_\mu(f) \leq 4 \int_{\mathbf{R}^d} \int_0^{+\infty} e^{-\kappa_a t} \mathbf{E}\left[\frac{a(X_{t,a})}{a(x)} |\nabla\sqrt{f}(X_{t,a})|^2\right] \, dt \, d\mu,$$

which rewrites

$$\mathrm{Ent}_\mu(f) \le 4 \int_0^{+\infty} e^{-\kappa_a t} \int_{\mathbf{R}^d} \frac{1}{a} \mathbf{P}_{t,a}\left(a|\nabla\sqrt{f}|^2\right)\, d\mu\, dt.$$

Recall that $d\mu_a = d\mu/a^2$. Then, since a is bounded,

$$\mathrm{Ent}_\mu(f) \le 4\|a\|_\infty \int_0^{+\infty} e^{-\kappa_a t} \int_{\mathbf{R}^d} \mathbf{P}_{t,a}\left(a|\nabla\sqrt{f}|^2\right)\, d\mu_a\, dt.$$

One can use invariance of $\mathbf{P}_{t,a}$ with respect to μ_a, then assumption on κ_a to get

$$\mathrm{Ent}_\mu(f) \le \frac{4\|a\|_\infty}{\kappa_a} \int_{\mathbf{R}^d} a|\nabla\sqrt{f}|^2\, d\mu_a.$$

Finally, boundedness of a^{-1} entails

$$\mathrm{Ent}_\mu(f) \le \frac{4\|a\|_\infty \|a^{-1}\|_\infty}{\kappa_a} \int_{\mathbf{R}^d} |\nabla\sqrt{f}|^2\, d\mu,$$

and the proof is complete replacing f by f^2. ■

Remark. In terms of perturbation matrices (as presented in [2] through weighted intertwinings) one has here $A = aI_d$. To take into account the geometry of $\nabla^2 V$, a natural extension to this result would be to consider non-homothetic perturbations, that is of the form $A = \mathrm{diag}(a_1, \ldots, a_d)$, where $a_1, \ldots, a_d \in \mathcal{C}^\infty_+(\mathbf{R}^d, \mathbf{R})$ are distinct functions. In spite of many attempts, the above proof does not seem to transpose to this case, and more general spectral estimates are besides much harder to derive. Generalisation of the representation result and Grönwall-like estimates for non-homothetic perturbation would then allow an interesting extension to this result.

Remark. One may wish to compare this technique to the well-known Holley-Stroock method (introduced in [15] for the Ising model). As a reminder, if ν is a probability measure that satisfies a LSI and if there exists $\Phi : \mathbf{R}^d \to \mathbf{R}$ a bounded continuous function such that $d\mu \propto e^\Phi d\nu$, then μ satisfies a LSI and

$$c_{LSI}(\mu) \le e^{2osc(\Phi)} c_{LSI}(\nu),$$

where $osc(\Phi) = \sup(\Phi) - \inf(\Phi)$. Note that $osc(\Phi)$ can poorly depend on the dimension, for example if $\Phi(x) = \sum_{i=1}^d \varphi(x_i)$, in which case $osc(\Phi) = d \cdot osc(\varphi)$. To stick to our framework, one might choose $\Phi = \log(a^2)$ for some bounded perturbation function $a \in \mathcal{C}^\infty_+(\mathbf{R}^d, \mathbf{R})$. The above inequality becomes

$$c_{LSI}(\mu) \le \|a\|^4_\infty \|a^{-1}\|^4_\infty c_{LSI}(\mu_a),$$

so that Holley-Stroock method leads to show that μ_a satisfies a LSI. This is conveniently ensured as soon as μ_a satisfies the Bakry-Émery criterion, namely

$$\inf_{x\in\mathbf{R}^d} \{\rho_-(\nabla^2 V_a(x))\} > 0.$$

In terms of a and V, the above condition rewrites explicitly:

$$\inf_{\mathbf{R}^d} \left\{ \rho_- \left(\nabla^2 V + \frac{2}{a}\nabla^2 a - \frac{2}{a^2}\nabla a (\nabla a)^T \right) \right\} > 0,$$

which shall be compared to the spectral estimates involved in κ_a, that can be expressed as:

$$\inf_{\mathbf{R}^d} \left\{ \rho_-(\nabla^2 V) + \frac{\Delta a}{a} - \nabla V \cdot \nabla a - \frac{2}{a^2}|\nabla a|^2 \right\} > 0.$$

Both expressions do not compare to each other, yet the second one seems to be far more tractable, as it could be illustrated on various examples.

As mentioned before, the above proof can be adapted in some particular cases to improve the estimate on $c_{LSI}(\mu)$. In the next section, we thus study the restriction of the latter to monotonic (positive) functions.

3.2.2 Monotonic functions

Definition 3.6. A measurable function $f : \mathbf{R}^d \to \mathbf{R}$ is said to be *monotonic in each direction* if for any $i = 1, \dots, d$, for any fixed $(x_1, \dots, x_{i-1}, x_{i-1}, \dots, x_d) \in \mathbf{R}^{d-1}$, $f_i : x_i \mapsto f(x_1, \dots, x_d)$ is monotonic.

In particular, if f is differentiable, then f is monotonic if and only if $\partial_i f$ has a constant sign on $\mathbf{R}^d$ for any i.

Remark. In the following, we shall focus on smooth functions f such that all f_i are non-decreasing (resp. non-increasing). In such cases, f will be said to be itself non-decreasing (resp. non-increasing).

Definition 3.7 (*(BM)* condition)**.** Given the potential V, a function $a \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$ is said to satisfy the *Bakry-Michel condition* (in short *(BM)*) if:

1. for any $i, j \in [\![1, d]\!]$, $i \neq j$, $\partial^2_{ij} V_a \leq 0$;
2. for any $i \in [\![1, d]\!]$, $\sum_{j=1}^d \partial^2_{ij} V_a$ is upper bounded,

The following proposition is one of the main arguments that allows to improve the estimate on $c_{LSI}(\mu)$ for monotonic functions.

Proposition 3.8. *Let $f \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$ and $a \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$ satisfying* (BM)*. Assume furthermore that f and a are both non-decreasing. Then*

$$\mathbf{P}_{t,a} f \leq \mathbf{P}_t f, \quad t \geq 0.$$

This proposition is based on a lemma provided by Bakry and Michel in [6], used initially to infer some FKG inequalities in $\mathbf{R}^d$.

Lemma 3.9. *Let $M : \mathbf{R}^d \to \mathcal{M}_d(\mathbf{R})$ be a measurable map such that $M_{ij} \leq 0$ for any $i \neq j$ and $\sum_{j=1}^d M_{ij}$ is upper bounded for any i, and let F be a smooth vector field on $\mathbf{R}^d$. Then all components of $\mathcal{P}_t^M F$ are non-negative whenever all components of F are so.*

We refer the reader to [6] for the proof. We can now provide a proof of Proposition 3.8.

Proof. The proof relies on very classical techniques. Let $t \geq 0$ be fixed and take $f \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$ a non-decreasing function. Define, for any $s \in [0, t]$,

$$\Psi(s) = \mathbf{P}_s(\mathbf{P}_{t-s,a} f).$$

Since $\Psi(0) = \mathbf{P}_{t,a} f$ and $\Psi(t) = \mathbf{P}_t f$, we aim to prove that Ψ is non-decreasing. One has, for any $s \in [0, t]$,

$$\Psi'(s) = \mathbf{P}_s[(\mathbf{L} - \mathbf{L}_a)\mathbf{P}_{t-s,a} f],$$

which rewrites accordingly

$$\Psi'(s) = \mathbf{P}_s \left[\frac{\nabla a}{a} \cdot \nabla \mathbf{P}_{t-s,a} f \right] = \mathbf{P}_s \left[\frac{\nabla a}{a} \cdot \mathcal{P}_{t-s,a}^{\nabla^2 V_a} (\nabla f) \right].$$

Since f is non-decreasing, all entries of ∇f are non-negative, and since a satisfies *(BM)*, Lemma 3.9 implies that all entries of $\mathcal{P}_{t-s,a}^{\nabla^2 V_a}(\nabla f)$ are non-negative. Moreover, a is positive and non decreasing, so that

$$\frac{\nabla a}{a} \cdot \mathcal{P}_{t-s,a}^{\nabla^2 V_a}(\nabla f) \geq 0.$$

Hence, since $\mathbf{P}_s$ preserves the positivity, $\Psi'(s) \geq 0$ and the proof is over. ■

Remark. In dimension 1, due to the particular form of the Feynman-Kac semigroup $(\mathcal{P}_{t,a}^{\nabla^2 V_a})_{t \geq 0}$, Proposition 3.8 still holds if one only assumes that a is positive and a and f are both non-decreasing.

Proposition 3.8 enables us to adapt the proof of Theorem 3.5 and improve the estimate on $c_{LSI}(\mu)$. Moreover, the proof allows to handle unbounded perturbation functions (as long as the (G) condition is satisfied).

Theorem 3.10. *Let $a \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$ be non-decreasing. Define*

$$\tilde{\kappa}_a = \inf_{x \in \mathbf{R}^d} \left\{ \rho_-(\nabla^2 V(x)) - a \mathbf{L}(a^{-1})(x) \right\}.$$

If a satisfies (BM), (G) and $\tilde{\kappa}_a > 0$, then for any non-decreasing $f \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$,

$$\mathrm{Ent}_\mu(f^2) \leq \frac{2}{\tilde{\kappa}_a} \int_{\mathbf{R}^d} |\nabla f|^2 \, d\mu.$$

Proof. Let $f \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$ be non-decreasing. The beginning of the proof is very similar to the one of Theorem 3.5. Indeed, the entropy rewrites

$$\mathrm{Ent}_\mu(f) = \int_{\mathbf{R}^d} \int_0^{+\infty} \frac{\left|\mathcal{P}_t^{\nabla^2 V}(\nabla f)\right|^2}{\mathbf{P}_t f} \, dt \, d\mu,$$

with the representation

$$\mathcal{P}_t^{\nabla^2 V}(\nabla f) = 2\mathbf{E}\left[R_{t,a} J_t^{X_a} \nabla \sqrt{f}(X_{t,a}) \sqrt{f}(X_{t,a})\right],$$

since a satisfies (G). Theorem 3.3 and Cauchy-Schwartz' inequality imply here

$$\begin{aligned}\left|\mathcal{P}_t^{\nabla^2 V}(\nabla f)\right|^2 &\leq 4\mathbf{E}\left[R_{t,a}^2 |J_t^{X_a} \nabla \sqrt{f}(X_{t,a})|^2\right] \overbrace{\mathbf{E}[f(X_{t,a})]}^{\mathbf{P}_{t,a} f} \\ &\leq 4\mathbf{E}\left[R_{t,a}^2 |J_t^{X_a} \nabla \sqrt{f}(X_{t,a})|^2\right] \mathbf{P}_t f,\end{aligned}$$

using Proposition 3.8. Plugged into the entropy, this yields

$$\mathrm{Ent}_\mu(f) \leq 4 \int_{\mathbf{R}^d} \int_0^{+\infty} \mathbf{E}\left[\nabla \sqrt{f}(X_{t,a})^T J_t^{X_a} R_{t,a}^2 (J_t^{X_a})^T \nabla \sqrt{f}(X_{t,a})\right] \, dt \, d\mu.$$

Here we let

$$J_t^a = J_t^{X_a} \exp\left(-\int_0^t \frac{\mathbf{L}_a a}{a}(X_{s,a}) \, ds\right),$$

and the same reasoning as in the proof of Theorem 3.5 gives then

$$\mathrm{Ent}_\mu(f) \leq 4 \int_{\mathbf{R}^d} \int_0^{+\infty} e^{-2\tilde{\kappa}_a t} \mathbf{E}\left[\frac{a(X_{t,a})^2}{a(x)^2} |\nabla \sqrt{f}(X_{t,a})|^2\right] \, dt \, d\mu.$$

Hence, using μ_a-invariance of $(\mathbf{P}_{t,a})_{t \geq 0}$,

$$\begin{aligned}\mathrm{Ent}_\mu(f) &\leq 4 \int_0^{+\infty} e^{-2\tilde{\kappa}_a t} \int_{\mathbf{R}^d} \mathbf{P}_{t,a}\left(a^2 |\nabla \sqrt{f}|^2\right) \, d\mu_a \, dt \\ &= 4 \int_0^{+\infty} e^{-2\tilde{\kappa}_a t} \int_{\mathbf{R}^d} |\nabla \sqrt{f}|^2 \, d\mu \, dt = \frac{2}{\tilde{\kappa}_a} \int_{\mathbf{R}^d} |\nabla \sqrt{f}|^2 \, d\mu,\end{aligned}$$

and the proof is achieved replacing f by f^2. ∎

4. Examples

In this section, we illustrate the Feynman-Kac approach on some examples. Since the perturbation function we introduce is scalar-valued, the method will be particularly suitable for potentials whose Hessian matrix admits many symmetries, for instance radial potentials. The examples we focus on shall then pertain to this class of potentials, namely here Subbotin and double-well potentials. Let us mention that, using other techniques, similar results for compactly supported radial measures were recently derived by Cattiaux, Guillin and Wu in [12].

For the sake of concision, we restrain ourselves to the illustration of Theorem 3.5. We eventually briefly resume the comparison to Holley-Stroock method.

4.1 Subbotin potentials

The first example we focus on is the general Subbotin distribution [2]. We take then $V(x) = |x|^\alpha/\alpha$ for $\alpha > 2$, to ensure that μ satisfies a *LSI* (see [5]).

Lemma 4.1. *Let $a \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$. Then for any $x \in \mathbf{R}^d$,*

$$\rho_-(2\nabla^2 V(x)) - a\mathbf{L}(a^{-1})(x) = 2|x|^{\alpha-2} - a\mathbf{L}(a^{-1})(x).$$

Proof. First, notice that for any fixed $x \in \mathbf{R}^d$,

$$\nabla^2 V(x) = (\alpha - 2)|x|^{\alpha-4}xx^T + |x|^{\alpha-2}I_d.$$

Hence, $T_x := 2\nabla^2 V(x) - a\mathbf{L}(a^{-1})(x)I_d$ (seen as an element of $\mathcal{L}(\mathbf{R}^d)$), can be written as the sum of a rank 1 operator (projection on $\mathbf{R}x$) and a full-rank operator (multiple of the identity). One can then write $\mathbf{R}^d = \mathbf{R}x \oplus (\mathbf{R}x)^\perp$. Let λ be a non-zero eigenvalue of T_x and y be an associated eigenvector. Then

- either $y \in \mathbf{R}x$, that is, $y = \beta x$ for some $\beta \in \mathbf{R}^*$, and one can write

  $$\lambda y = T_x y = 2\beta(\alpha - 2)|x|^{\alpha-2}x + 2\beta|x|^{\alpha-2}x - \beta a\mathbf{L}(a^{-1})(x)x,$$

  which leads to

  $$\lambda = 2(\alpha - 1)|x|^{\alpha-2} - a\mathbf{L}(a^{-1})(x);$$

- either $y \in (\mathbf{R}x)^\perp$, in which case

  $$\lambda y = T_x y = 2|x|^{\alpha-2}y - a\mathbf{L}(a^{-1})(x)y,$$

  which entails

  $$\lambda = 2|x|^{\alpha-2} - a\mathbf{L}(a^{-1})(x).$$

Hence for any $x \in \mathbf{R}^d$, since $\alpha > 2$,

$$\rho_-(2\nabla^2 V(x)) - a\mathbf{L}(a^{-1})(x) = \rho_-(T_x) = 2|x|^{\alpha-2} - a\mathbf{L}(a^{-1})(x).$$

■

In the following, we may focus on the $\alpha = 4$ (quadric) case. Indeed, computations turn out to be particularly difficult in full generality, as well as keeping track of dependency in both parameters d and α. Bakry-Émery criterion clearly does not apply to this particular potential, since $\rho_-(\nabla^2 V(x))$ vanishes at point $x = 0$.

Theorem 4.2. *There exists $c > 0$ a universal explicit constant such that for any $f \in \mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$, one has*

$$\mathrm{Ent}_\mu(f^2) \le c \int_{\mathbf{R}^d} |\nabla f|^2 d\mu.$$

In particular, c does not depend on the dimension.

[2] after Mikhail Fedorovich Subbotin, 1893–1966, Soviet mathematician

Proof. The first concern about making use of Theorem 3.5 stands in the choice of the perturbation function a. In practice, a should correct a lack of convexity of V where it occurs (namely where $\nabla^2 V(x) \leq 0$, here at $x = 0$). One of the first choices turns out to be the function

$$a(x) = \exp\left(\frac{\varepsilon}{2}\arctan(|x|^2)\right), \quad x \in \mathbf{R}^d.$$

Indeed, the arctangent function behaves like the identity near zero (where lies the lack of convexity of V) and like a constant at infinity (ensuring that a is bounded above and below). Furthermore, the square function is uniformly convex on $\mathbf{R}^d$, so that the Hessian matrix of the above is positive definite near the origin. Finally, taking exponential, a is indeed positive and computations are easier. Note that this choice is motivated by some results on the spectral gap, in which case the choice of a perturbation function that is close to non-integrability can provide relevant estimates on the Poincaré constant (see for example [8, 2]).

The next step in the method consists in the explicit computation of κ_a. With this definition of a, one has

$$-a\mathbf{L}(a^{-1})(x) = \varepsilon\frac{d + |x|^4(d-4)}{(1+|x|^4)^2} - \varepsilon\frac{|x|^4}{1+|x|^4} - \varepsilon^2\frac{|x|^2}{(1+|x|^4)^2}, \quad x \in \mathbf{R}^d,$$

and shall then minimize in $x \in \mathbf{R}^d$:

$$2|x|^2 + \varepsilon\frac{d + |x|^4(d-4)}{(1+|x|^4)^2} - \varepsilon\frac{|x|^4}{1+|x|^4} - \varepsilon^2\frac{|x|^2}{(1+|x|^4)^2},$$

which rewrites, setting $t = |x|^2$,

$$\kappa_a = \inf_{t\geq 0}\left(2t + \varepsilon\frac{d + t^2(d-4)}{(1+t^2)^2} - \varepsilon\frac{t^2}{1+t^2} - \varepsilon^2\frac{t}{(1+t^2)^2}\right).$$

Optimization of polynomials is hardly explicit in most cases, especially when one must keep track of all parameters (namely ε and d). We shall then focus here on the case where the infimum is reached for $t = 0$, that is, for any $t \geq 0$,

$$2t^4 - \varepsilon(d+1)t^3 + 4t^2 - \varepsilon(d+5)t + 2 - \varepsilon^2 \geq 0.$$

Let us denote by g the above polynomial function. Clearly, $\varepsilon \leq \sqrt{2}$ is a necessary, yet not sufficient condition for g to be non negative. In order to make computations more tractable, let us assume that g'' is positive. This is true as soon as

$$\varepsilon < \frac{8}{\sqrt{3}(d+1)}.$$

Consider then $\varepsilon \leq \dfrac{8}{3\sqrt{3}(d+1)}$. With this choice of ε, given that $d \geq 1$, one has for any $t \geq 0$

$$g(t) \geq 2t^4 - \frac{8t^3}{3\sqrt{3}} + 4t^2 - \frac{8t}{\sqrt{3}} + 2 - \frac{16}{27}.$$

It is easy to see that the above right-hand side is non-negative, so that g is non-negative either over $\mathbf{R}_+$. We can then take $\kappa_a = \varepsilon d$, and Theorem 3.5 entails the following estimate:

$$c_{LSI}(\mu) \leq \frac{4e^{\varepsilon\pi/4}}{\varepsilon d},$$

with $\varepsilon \leq \dfrac{8}{3\sqrt{3}(d+1)}$ (which implies that $\varepsilon \leq \sqrt{2}$). We finally minimize this bound with respect to $\varepsilon \in \left(0, \frac{8}{3\sqrt{3}(d+1)}\right]$ to get

$$c_{LSI}(\mu) \leq \frac{3\sqrt{3}(d+1)}{2d} e^{2\pi/3\sqrt{3}(d+1)}.$$

The above is uniformly bounded with respect to $d \in \mathbf{N}^\star$, and one can take $c = 3\sqrt{3}/2$ as the universal constant mentioned in the theorem. ■

Remark. This proof points out the main concerns about Theorem 3.5. Indeed, the choice of the function (or family of functions) a is a key point. Nevertheless, the most important, yet technical, part of the proof is the explicit computation of κ_a, given that track should be kept of all parameters.

Nonetheless, up to some numerical constant, the bound on ε in the previous proof is optimal (with this optimization method). Recall that the problem reduces to the prove that the function g defined on $\mathbf{R}_+$ as

$$g(t) = 2t^4 - \varepsilon(d+1)t^3 + 4t^2 - \varepsilon(d+5)t + 2 - \varepsilon^2, \quad t \geq 0,$$

is non-negative. If we assume that ε is of order $(d+1)^{-r}$ for some $r \in (0,1)$, then when d is large, for any fixed positive t,

$$g(t) \sim 2t^4 - d^{1-r}t^3 + 4t^2 - d^{1-r}t + 2 - d^{-2r},$$

and taking $t = 3/d^{1-r}$ leads to

$$g(3/(d+1)^r) \sim \frac{162}{d^{4(1-r)}} + \frac{9}{d^{2(1-r)}} - \frac{1}{2^{2r}} - 1 < 0$$

when d increases, which prevents the infimum of $t \mapsto \varepsilon d + tg(t)$ to be reached at $t = 0$.

We do not know if the constant we inferred is optimal (in terms of the dimension). Yet, one can note that, for example from [10], since the spectral gap for the quadric Subbotin distribution is of order $\sqrt{d}$, it is reasonnable to expect $c_{LSI}(\mu)$ to be of order $1/\sqrt{d}$ (since μ satisfies a Poincaré inequality with constant c (which is the inverse of the spectral gap) as soon as is satisfies $LSI(2c)$, see [5]). It is then unclear that we can reach optimality with this very optimization procedure. More reliable optimization techniques would be then a good improvement regarding explicit estimates using this result.

Remark. The Holley-Stroock method announced previously leads, in the present case and after tedious computations, to a conclusion somewhat comparable to ours. Nevertheless, the involved constants are not fully explicit and leave less room for improvement than our above approach.

4.2 Double-well potentials

The following example is a perturbation of the previous one known as the double-well potential. Consider $V(x) = |x|^4/4 - \beta|x|^2/2$, where $\beta > 0$ controls the size of the concave region. Although V is convex at infinity, its Hessian matrix is negative definite near the origin, and Bakry-Émery criterion does not apply. Still, one can expect to recover the behaviour inferred in Theorem 4.2 when β is small.

Similarly to the Subbotin case, one can explicitly compute the Hessian matrix of V to get the following lemma.

Lemma 4.3. *Let $a \in \mathcal{C}_+^\infty(\mathbf{R}^d, \mathbf{R})$. Then for any $x \in \mathbf{R}^d$,*

$$\rho_-(2\nabla^2 V(x) - a\mathbf{L}(a^{-1})(x)I_d) = 2|x|^2 - 2\beta - a\mathbf{L}(a^{-1})(x).$$

Proof. The proof is identical to the one of Lemma 4.1. ∎

Theorem 4.4. *For any $\beta \in (0, 1/2)$, there exists $c_\beta > 0$ a universal constant such that, for any function $f \in \mathcal{C}_0^\infty(\mathbf{R}^d, \mathbf{R})$, one has*

$$\mathrm{Ent}_\mu(f^2) \leq c_\beta \int_{\mathbf{R}^d} |\nabla f|^2 d\mu.$$

Again, c_β does not depend on the dimension.

Proof. This proof is very similar to the previous one. In particular, we set for any $x \in \mathbf{R}^d$

$$a(x) = \exp\left(\frac{\varepsilon}{2}\arctan(|x|^2)\right),$$

so that, for $t = |x|^2$,

$$\kappa_a = \inf_{t\geq 0}\left(2t - 2\beta + \varepsilon\frac{d + t^2(d-4)}{(1+t^2)^2} - \varepsilon(t-\beta)\frac{t}{1+t^2} - \varepsilon^2\frac{t}{(1+t^2)^2}\right).$$

Again, we aim to show that this infimum is equal to $\varepsilon d - 2\beta$, reached for $t = 0$, which amounts to prove that, for any $t \geq 0$,

$$g(t) := 2t^4 - \varepsilon(d+1)t^3 + (4+\beta)t^2 - \varepsilon(d+5)t + 2 - \varepsilon^2 + \beta \geq 0,$$

along with, to ensure positivity of κ_a, $\varepsilon > 2\beta/d$.

The first necessary condition that arises is $\varepsilon \leq \sqrt{\beta+2}$. Moreover, in light of both previous proof and remark, ε should be of order $\frac{1}{d+1}$. To make computations easier, we take $\varepsilon = \frac{2}{d+1}$. Plugging this into both conditions $\varepsilon > 2\beta/d$ and $\varepsilon \leq \sqrt{\beta+2}$ imply that β should not exceed $d/d+1$ for any d, which equates to $\beta < 1/2$. To summarize, we have

$$\varepsilon = \frac{2}{d+1} \quad \text{and} \quad 0 \leq \beta < \frac{1}{2}.$$

Under those assumptions, g can be bounded from below as follows

$$g(t) \geq 2t^4 - 2t^3 + 4t^2 - 2t + 1 + \beta, \quad t \geq 0.$$

The right-hand term is positive on $\mathbf{R}_+$, so that with this choice of ε, one has

$$\kappa_a = \frac{2d}{d+1} - 2\beta.$$

This amounts, using Theorem 3.5,

$$c_{LSI}(\mu) \leq \frac{4(d+1)}{2d(1-\beta)-2\beta} e^{\frac{\pi}{2(d+1)}}.$$

The above is uniformly bounded with respect to $d \in \mathbf{N}^\star$, and one can take $c_\beta = \dfrac{4}{1-2\beta}$ as the aforementioned universal constant. ■

Remark. Note that the restriction on β is a computation artefact, and one has more $c_\beta \xrightarrow[\beta \to \frac{1}{2}^-]{} +\infty$. Nevertheless, the behaviour in term of the dimension is similar to what was derived for the Subbotin distribution in Theorem 4.2.

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