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NAṢĪR AL-DĪN AL-ṬŪSĪ'S TREATISE ON THE QUADRILATERAL: THE ART OF BEING EXHAUSTIVE

ATHANASE PAPADOPOULOS

Abstract. We comment on some combinatorial aspects of Naṣīr al-Dīn al-Ṭūsī's treatise on the quadrilateral, a 13th century work on spherical trigonometry. The final version of this paper will appear in Ganita Bharati, the Bulletin of the Indian Society for History of Mathematics

AMS classification: 01A20, 01A30, 01A35.

Keywords: Naṣīr al-Dīn al-Ṭūsī, spherical trigonometry, spherical geometry, quadrilateral, complete spherical quadrilateral, sector figure, Menelaus Theorem.

1. Introduction

A *spherical quadrilateral* is a convex figure drawn on the sphere which is bounded by four arcs of great circles (the sides of the quadrilateral). A spherical quadrilateral becomes a *complete spherical quadrilateral* if one adds to it two spherical triangles by extending the two pairs of adjacent sides until each two sides in a pair meet. See Figure 1.[1] Thus, each spherical quadrilateral gives rise to four complete spherical quadrilaterals, depending on the choice of two adjacent sides.

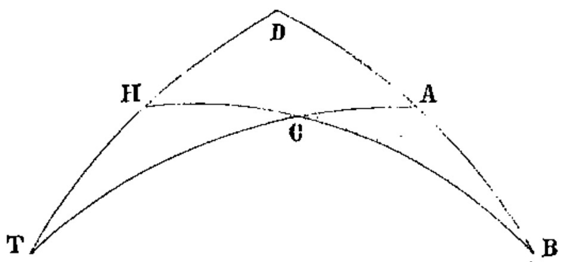

Figure 1. A complete spherical quadrilateral obtained by adjoining to the spherical quadrilateral $ACHD$ the two spherical triangles ABC and CTH

The study of such quadrilaterals was carried out (conceivably for the first time)[2] by Menelaus of Alexandria in his *Spherics*, a treatise on spherical

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[1] In all this article, the figures are extracted from Carathéodory's edition of Naṣīr al-Dīn al-Ṭūsī's treatise [15].

[2] The question is controversial but this does not concern us here. Sir Thomas Heath suggests that the main result about these quadrilaterals, namely, the theorem called the

geometry that dates back to the first or the beginning of the second century of our era. This figure is associated with a proposition of the *Spherics* which has far-reaching applications and which is known in the modern mathematical literature (and stated in a slightly different form) as *Menelaus' theorem* (see [8, Proposition 66]). The proposition was used by Ptolemy in his *Almagest* (Book I, Chapter 11), written around the middle of the second century, as the main lemma for the astronomical calculations that are carried out there, see [2, p. 50–55] and [14, p. 64–69]. We have elaborated on this proposition and its applications in the commentary contained in our edition of the *Spherics*; see [8, p. 292–337].

The complete spherical quadrilateral, together with the proposition to which it belongs, also bears the names "quadrilateral figure", "complete quadrilateral", "quadrilateral", "sector figure" or "secant figure", and there are others. The name "secant figure" is the one that was used by Ptolemy in his *Almagest*, and it is also the name that was adopted by the Arabs[3] in their commentaries on Menelaus and Ptolemy's works and in the treatises they edited on spherical geometry.

Naṣīr al-Dīn al-Ṭūsī's treatise on the quadrilateral[4] is an invaluable thirteenth century text on spherical geometry. It was translated into French and edited by Alexandre Carathéodory in 1891 [15]. The "quadrilateral" that is referred to in the title is a complete spherical quadrilateral.

For us today, the importance of Naṣīr al-Dīn's treatise lies mainly in the exposition of the spherical trigonometric formulae that is carried out in the last part (Book V). It contains the main formulae that are used today in this field, namely, the spherical laws of sines, cosines, tangents, etc. These formulae are established using the notion of polar triangle and they are preceded by introductory material (Books I to IV). As a matter of fact, this treatise is in the tradition of other treatises written by the Arabs whose object was to provide proofs of the trigonometric formulae *without* using the sector figure.[5] We refer the reader to the concise and clear exposition by Rosenfeld in his book on the history of non-Euclidean geometry [12, p. 20ff]. At the same time, Naṣīr al-Dīn's treatise is a remarkable historical document that gives a very good idea of the state of spherical geometry, as developed by the Arabs between the ninth and the thirteenth centuries.

Our aim in this article is to highlight one characteristic of the treatise, namely, the fact that the author, for each topic he discussed, was keen to consider all the cases that occur at each step of the mathematical development and to count them in an exhaustive manner, at the expense of being redundant. We note that this was not the approach of the Greek mathematicians. For instance, Menelaus, in the proofs of several propositions of his *Spherics*, did not bother considering more than one case. His style was extremely concise, and even at several places where several cases deserve to be discussed, he considered only one case. (One should not conclude from

Sector Figure, which we shall comment on below, was already known to Hipparchus of Nicaea (2nd c. BCE) [4, Vol. II p. 270].

[3]For instance, this is the name used by Thābit ibn Qurra (826–921), الشَّكل القَطَّاع.

[4]We are using the name that the translator, Alexandre Carathéodory, used, *Traité du quadrilatère* [15]. There are other possibilities for the translation of the Arabic title.

[5]In this sense, the title of the work may seem inappropriate. But one must accept the fact that if the part of the book which for us is the more important from the mathematical point of view is Book V, this may not have been the case in the thirteenth century.

this that Menelaus' proofs are incomplete; it was part of his style to give the arguments for a single typical case and to leave the discussion of the other cases to the care of the reader.) Several Arab editors of Menelaus' work took it as a duty to expand his proofs by considering the various cases. All this is discussed in detail in our edition of the al-Māhānī/al-Harawī version of Menelaus' *Spherics* [8].

Thus, in particular, our aim in this article is not to discuss the Sector Figure or the other purely mathematical issues in Naṣīr al-Dīn's treatise, which pertain to spherical geometry. This will be discussed in a sequel to the present paper.

Before entering into the subject matter of our paper, let us quickly review the content of Naṣīr al-Dīn's treatise.

The treatise is divided into five books, with the following titles:

- **Book I**. On compounded ratios and their rules.
- **Book II**. On the plane quadrilateral figure and the ratios that we find therein.
- **Book III**. On the lemmas for the spherical quadrilateral figure and on what is necessary for using it profitably.
- **Book IV**. On the spherical quadrilateral figure and the ratios that we find therein.
- **Book V**. An exposition of the methods that liberate one from the theory of the quadrilateral figure, for what concerns the knowledge of the arcs of great circles.

More details on the content of the treatise are given in the rest of the present article. Each of the following sections is concerned with one of the books of Naṣīr al-Dīn's treatise. To make the article easy to read for the modern reader, we use modern notation (as Carathéodory does in his translation).

2. Book I

Book I, in one chapter, is concerned with compounded ratios. In modern notation, a compounded ratio is an expression of the form $\frac{A}{B}\frac{C}{D}$. Menelaus' *Spherics*, starting from the proposition on the Sector Figure, makes heavy use of such expressions. Therefore, a skill in their manipulation was necessary in the study of spherical geometry that he developed.

There is a large literature on compounded ratios, written by historians (who are usually not mathematicians). It is not our goal here to repeat what is said about this, but only to explain in plain words, to a potential reader who likes to understand things, what are the problems behind compounded ratios.

To understand the usefulness of Naṣīr al-Dīn's exposition in Book I, we remind the reader that results on compounded ratios that seem obvious to us, like the equality $\frac{A}{B}\frac{C}{D} = \frac{A}{D}\frac{C}{B}$, were not so for the Ancients. The reason has to do with the differences in our ways of understanding the meaning of such expressions and the operations they involve.

Today, from the mathematical point of view, we are trained to think of division and multiplication as operations pertaining to a certain mathematical structure (ring or field) even though we do this unconsciously, and the identities that they satisfy follow from the axioms: commutativity, distributivity, etc. For a mathematician belonging to the period of Greek antiquity or to the Arabic period that followed, this was not part of the way mathematics was conceived, and to use such expressions and equalities between them, one had to provide proofs, starting from what was known about products, ratios and compounded ratio. This was essentially the material contained in Euclid's *Elements*, a treatise where the operations we mentioned had a geometric significance. Thinking geometrically is easy but writing proofs of geometric facts is cumbersome. The theory of ratios is developed in Book V of the *Elements*. The definition of ratio that is given there was a subject of debates and controversies since Euclid's times until the modern period. Most of the commentators pointed out relations between the definition of equality of ratios and the theory of irrational numbers. Heath, who, from our point of view, remains the best modern specialist of Euclid, in his comments on this definition in [3, Vol. 2, p. 120–129], discusses the relation with Dedekind's construction of the real numbers. We do not need to enter into these details; we only note that the fact of defining equality between two objects without defining the objects is not an unusual feature of the *Elements*, where, for instance, the notion of area is not defined and never used, but there is a use of the notion of "two figures having the same area".

Likewise, there is hardly any definition of compounded ratios in Euclid's *Elements*. The first allusion to them is in Definition 5 of Book VI, which is considered to be a later addition; see Heath's comments in [3, Vol. 2, p. 189]. The definition, in Heath's interpretation, says the following: *A ratio is said to be compounded of ratios when the sizes of the ratios multiplied together make some (? ratio or size).*[6]

I would like to illustrate the Greek usage of compounded ratios, by quoting Proposition 23 of Book VI, a proposition from the *Elements* which is usually referred to as a place where compounded ratios are implicitly used. The proposition is purely geometric and it concerns areas of parallelograms. It says the following [3, Vol. 2, p. 247]: *Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.* It is interesting to read Heath's explanation of this proposition. In his *History of Greek Mathematics*, he writes [4, Vo. 1, p. 393] (see Figure 2 here):

> Proposition 23 is important in itself, and also because it introduces us to the practical use of the method of compounding, i.e. multiplying, ratios which is of such extraordinarily wide application in Greek geometry. Euclid has never defined "compound ratio" or the "compounding" of ratios; but the meaning of the terms and the way to compound ratios are made clear in this proposition. The equiangular parallelograms are placed so that two equal angles as BCD, GCE are vertically opposite at C. Complete the parallelogram $DCGH$. Take any straight line K, and find another L, such that
>
> $$BG : CG = K : L,$$

[6]The uncertainty expressed by the question mark is in Heath's edition. The correctness of Heath's writing has been the subject of multiple debates, but again, entering into these controversies is not the part of the goal of this article and of the author's competence.

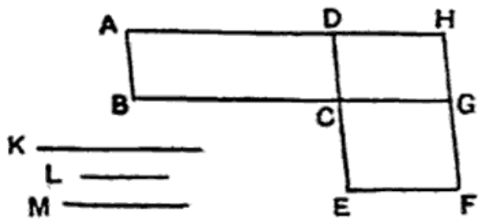

FIGURE 2. Figure for Euclid's Proposition 23 of Book VI (from Heath's *History of Greek Mathematics*)

and again another straight line M, such that

$$DC : CE = L : M.$$

Now the ratio compounded of $K : L$ and $L : M$ is $K : M$; therefore $K : M$ is the "ratio compounded of the ratios of the sides". And

$$(ABCD) : (DCGH) = BC : CG = K : L;$$

$$(DCGH) : (CEFG) = DC : CE = L : M,$$

therefore, *ex aequali*,

$$(ABCD) : (CEFG) : K : M.$$

Naṣīr al-Dīn makes a systematic study of compounded ratios, in 11 propositions of his first book. Let us review two of these propositions; this will give us an idea of the way he addresses these questions.

Proposition 10 says that if we have an equality between a ratio of two quantities and a compounded ratio of four quantities expressed (in modern terms) as $\frac{A}{B} = \frac{C}{E}\frac{D}{F}$, then the ratio of any one of the three quantities A, E, F to any one of the three quantities B, C, D is equal to a compounded ratio of the four remaining quantities, whose numerators are taken among the remaining quantities in A, E, F and the denominators among the remaining quantities in B, C, D.

To be more precise, a special case of this proposition says that if $\frac{A}{B} = \frac{C}{E}\frac{D}{F}$, then $\frac{A}{C} = \frac{B}{E}\frac{D}{F}$.

The proof that Naṣīr al-Dīn gives of Proposition 10 is based on Propositions 33 and 34[7] of Book XI of the *Elements* which say respectively [3, Vol. 3, p. 342 and 345]:

Similar parallepipedal solids are to one another in the triplicate ratio of their corresponding sides,

and

In equal parallelepipedal solids the bases are reciprocally proportional to the heights; and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal.

After the proof of Proposition 10 of his treatise, Naṣīr al-Dīn makes a count of the various possible cases, and he find 36 cases, reproduced here in Figure 3. In this figure, case 1 on the left hand side says that $\frac{A}{B} = \frac{C}{E}\frac{D}{F}$, case 2 on the same side says that $\frac{A}{B} = \frac{C}{F}\frac{D}{E}$, etc. Naṣīr al-Dīn will refer to these 36 cases in the rest of the treatise.

[7]We are using the numbering in Heath's edition. Naṣīr al-Dīn's numbering is different.

At the end of Chapter 4 of Book IV, Naṣīr al-Dīn writes that there is a striking resemblance between the propositions he proved and the forms of syllogisms in Logic, where the first form plays the role of a principle from which all the others follow.

1	A	B	C	E	D	F	1	B	A	E	C	F	D
2	A	B	C	F	D	E	2	B	A	E	D	F	C
3	A	C	B	E	D	F	3	B	E	A	C	F	D
4	A	C	B	F	D	E	4	B	E	A	D	F	C
5	A	D	B	E	C	F	5	B	F	A	C	E	D
6	A	D	B	F	C	E	6	B	F	A	D	E	C
7	E	B	C	A	D	F	7	C	A	E	B	F	D
8	E	B	C	F	D	A	8	C	A	E	D	F	B
9	E	C	B	A	D	F	9	C	E	A	B	F	D
10	E	C	B	F	D	A	10	C	E	A	D	F	B
11	E	D	B	A	C	E	11	C	F	A	B	E	D
12	E	D	B	F	C	A	12	C	F	A	D	E	B
13	F	B	C	A	D	E	13	D	A	E	B	F	C
14	F	B	C	E	D	A	14	D	A	E	C	F	B
15	F	C	B	A	D	E	15	D	E	A	B	F	C
16	F	C	B	E	D	A	16	D	E	A	C	F	B
17	F	D	C	E	B	A	17	D	F	C	C	A	B
18	F	D	C	A	B	E	18	D	F	E	B	A	C

FIGURE 3. Carathéodory's table of the 36 cases of Proposition 10 of Book I

Let us consider now a second example, Proposition 11. It says that if we have the two equalities $\frac{A}{B} = \frac{C}{E}\frac{F}{D}$ and $A = C$, then $\frac{B}{D} = \frac{E}{F}$.

Like in the case of Proposition 10, there are variations on this equality. For instance, the conclusion, under the same hypothesis, may be $\frac{B}{E} = \frac{D}{F}$, etc. One may also take as a hypothesis the equality $\frac{A}{B} = \frac{C}{E}\frac{D}{F}$ together with the equality $B = D$ instead of $A = C$, and obtain other conclusions. Al-Ṭūsī gives a classification into 9 possibilities, and he gives them in a table which we reproduce here in Figure 4. In this table, the second and the third columns indicate the quantities that are set to be equal, under the hypothesis $\frac{A}{B} = \frac{C}{E}\frac{D}{F}$, and the the four columns starting from the fourth indicate the four quantities that appear pairwise in the equality among ratios obtained in the conclusion.

For instance, the first row says that if $A = B$, then $\frac{C}{E} = \frac{F}{D}$; the second raw says that if $A = C$, then $\frac{B}{E} = \frac{F}{D}$, and so on.

For the proof, al-Ṭūsī refers to the same propositions in Euclid's Book XI that he used in the proof of Proposition 10, concerning solids that have the same height.

1	A	B	C	E	F	D
2	A	C	B	E	F	D
3	A	D	B	E	F	C
4	E	B	A	C	D	F
5	E	C	A	D	B	F
6	E	D	A	B	C	F
7	F	B	A	C	D	E
8	F	C	A	B	D	E
9	F	D	A	B	C	E

FIGURE 4. The 9 cases of Proposition 11 of Book III

3. BOOK II

Book II of Naṣīr al-Dīn's treatise, in eleven chapters, concerns the Euclidean complete quadrilateral. This is a complete quadrilateral in the sense of the definition we gave in the introduction, except that it is Euclidean instead of being spherical (see Figure 5).

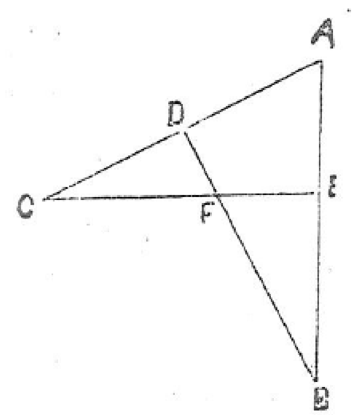

FIGURE 5. A Euclidean complete quadrilateral

In the first chapter, Naṣīr al-Dīn al-Ṭūsī introduces the plane quadrilateral as a figure made by four lines in the plane that intersect pairwise and such that no three of these lines intersect at the same point. He declares that there are 12 cases, and his aim in the first chapter of Book II is to make a list of them. Before doing this, he considers a simpler case, that of *plane triangles*, that is, figures formed by the intersection of three pairwise intersecting lines such that the three intersection points they form are distinct. The various cases correspond in fact to the different *triangles with labels*, that is, the names of the three lines, and the combinatorics of the letters being marked, is part of the structure. (The same occurs in his consideration of quadrilaterals, which he does next).

There are three sorts of triangles in this sense, which al-Ṭūsī describes as follows:

We start with two lines, called AB, AC (thus, we assume that they intersect at the point A). The third line cuts the line AB at a point different from A, which we can assume to be the point B, and it cuts the line AC in a point different from C, which we call E. There are three cases for the point E: it may be outside the segment AC and on the same side as A, or between A and C, or outside the segment AC and from the same side as C. The three cases are represented in Figure 6.

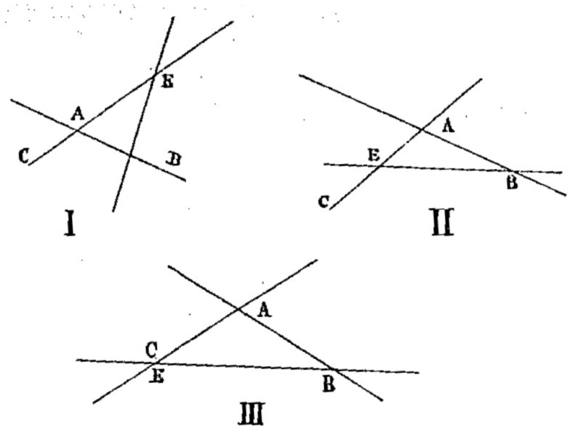

FIGURE 6. The three cases of the intersection of a line EB with the two lines AB and AC. In case III, two of the intersection points coincide.

Now we pass to the case of four lines (the complete quadrilateral).

We consider a line CD that cuts the preceding three lines. Let it cut the line AC in C and the line AB in D. There are three cases for this point D: either it lies outside the segment AB and on the same side as A, or between A and B, or outside the segment AB and on the same side as B. Thus, each of the three preceding three figures gives rise to three new figures. This makes nine cases and they are represented in Figure 7.

It remains now to study the intersection of the lines BE, CD. From the cases I.1, II.3 and III.2 of Figure 7, one can see that depending on whether F is on the side of B or on the side of E, the figure is not the same. In the other cases, F necessarily falls on one side. Thus, in cases I.2, II.2 and III.1, F falls between B and E, and in cases I.3, II.1 and III.3, the line CD cuts the line BE before it cuts the line AB. In conclusion, we have 12 cases, represented in Figure 8.

We learn from Naṣīr al-Dīn al-Ṭūsī that he was not the first to do such a classification of quadrilaterals. He mentions indeed the mathematician Ḥussām al-Dīn 'Ali Ibn Faḍlullāh Assālār, who, according to Naṣīr al-Dīn, made a similar study and who (again, according to Naṣīr al-Dīn), despite being one of the best experts in this matter, thought that there are only 9 cases (namely, the three cases of Figure 7) instead of 12 (those of Figure 8). Naṣīr al-Dīn quotes Ḥussām al-Dīn saying: "Some people pretend indeed that there are 12 case, but, as for me, I do not see them."

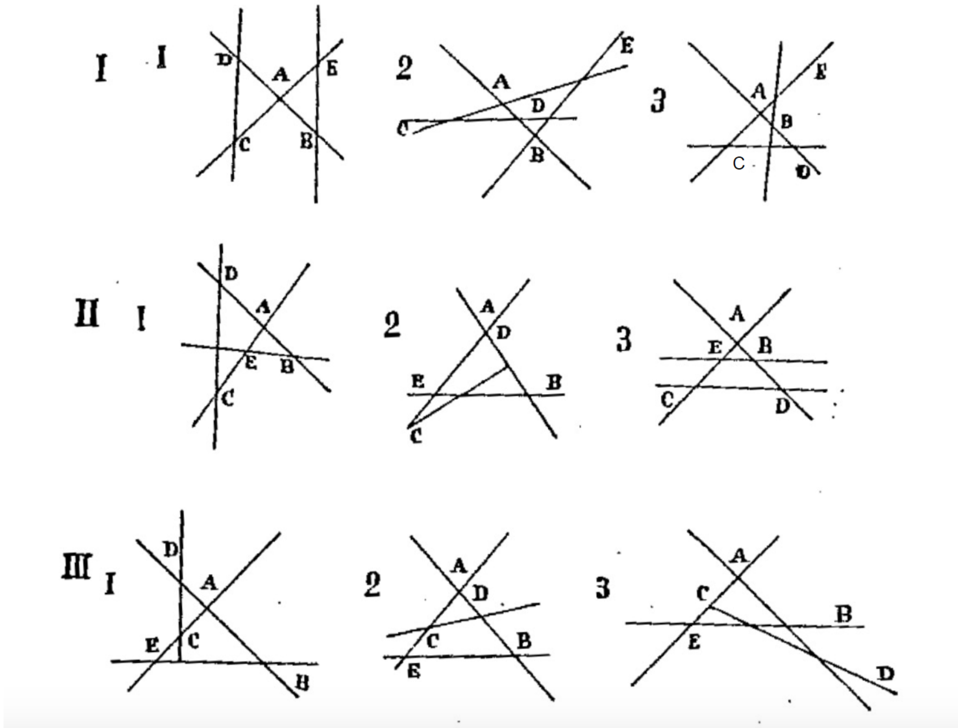

FIGURE 7. The nine cases of the intersection pattern of the line CD with the three lines AB, AC and BE

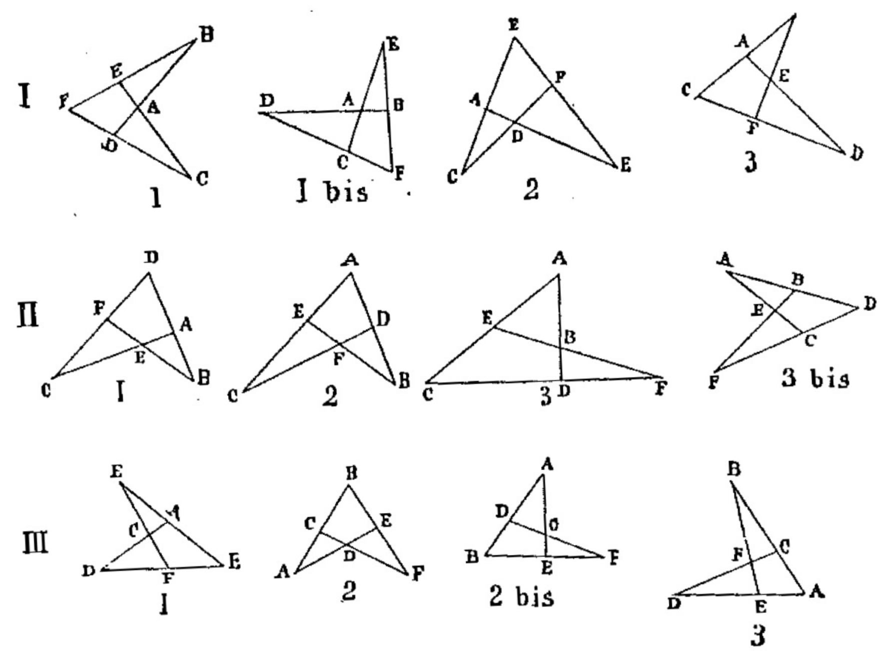

FIGURE 8. The twelve cases of the complete quadrilateral formed by the intersections of the four lines AB, AC, BE, CD in Figure 7

Naṣīr al-Dīn then proves an equality which holds in all these cases, namely,

$$\frac{AB}{BD} = \frac{AE}{EC}\frac{CF}{FD}.$$

This is the famous *Sector Figure* or *Menelaus Theorem* in the planar case. The proof follows easily from the construction of a parallel AH to CD, with H on BE, and using similar triangles. The various possibilities are represented in Figure 9 and the same proof holds in all these cases.

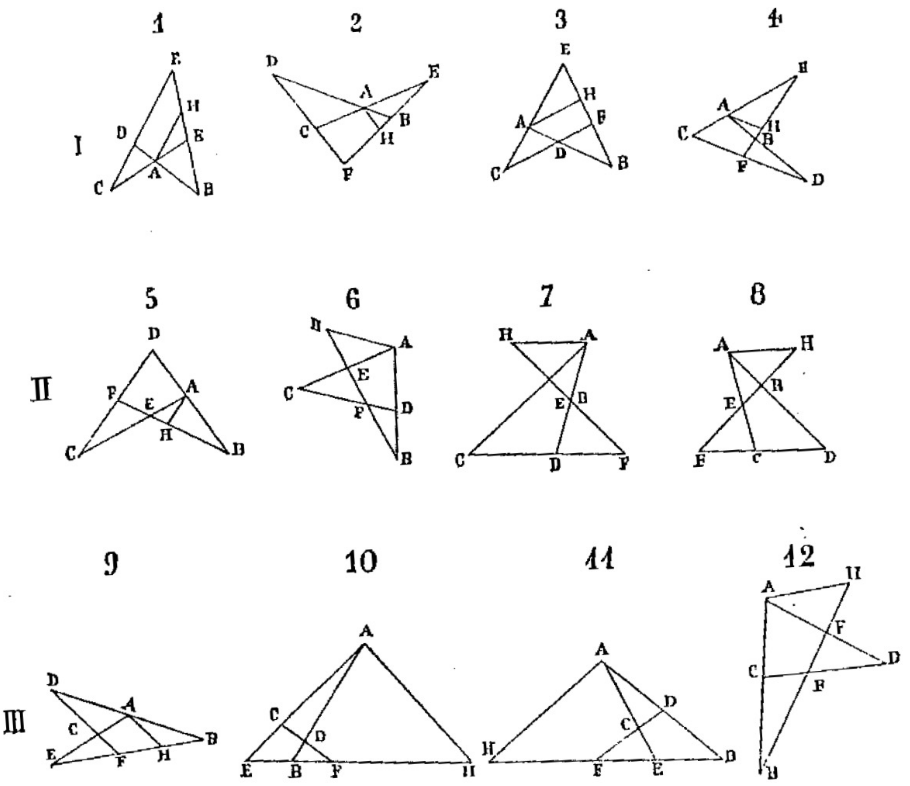

FIGURE 9. Drawing the parallel AH to CD, in the proof of the Sector Figure.

After presenting the 12 cases and proving the corresponding Euclidean Sector Figure, Naṣīr al-Dīn notes that "the geometers", taking into account left and right directions, multiplied by two the number of cases, and considered that there are 24 cases (all with a common statement and a common proof), instead of 12. He goes on that, in fact, if one takes into account left and right, then one has also to take into account the up and down directions, and in this case, we have 48 cases and not only 24. He then declares that elaborating on this point is not useful, but that he mentions it only in order to comply with the standing practice, despite the fact that it will lengthen the exposition.

The rest of the book (Chapters 2 to 11) is dedicated to a discussion of these 48 cases.

In Chapter 2, Naṣīr al-Dīn introduces two perpendicular axes that he uses for the distinction between the left/right and up/down directions. He then introduces some terminology for his classification, involving notions of *column* and *line* of a figure, of *inactive* triangle, of *associated* and *dissociated* lines, with three kinds of associations and three kinds of propositions corresponding to them. This leads him to the classification of the propositions he proves into three types, each type subdivided into several subtypes, depending on whether the ratios it involves are equal or not. This classification is conducted in the next chapters.

Chapter 3 of the book is titled "On the various cases of the first proposition". The author introduces there the notions of *implicit* and *explicit* ratio, of *inactive column* and *inactive triangle* of a ratio, of *antecedent* and *consequent*, of *angle of the antecedent*, *angle of the consequent*, *common angle* and many other similar notions. The overall aim is to recover the species of a quadrilateral from the properties of the statement of the proposition that concerns it.

In Chapter 4 and 5, the same study is made for the second and third propositions respectively.

The rest of the book is dedicated to the proofs of the propositions. This involves the drawing of parallel lines (which is something that cannot be performed in the spherical case), with a tedious consideration of various cases, depending on the angle from which the parallel used in the proof is drawn.

Naṣīr al-Dīn starts Chapter 10 by declaring that some authors constructed tables where, to each kind of question to which they dedicated a proof, they made explicit 18 different ratios, and he notes that this consideration is useless. He makes a count of 288 expressions, with 6 proofs for each, which makes 1728 proofs, and since there are 12 figures, he gets a total of 3456 cases and 20736 proofs. He says that on the other hand, if one also considers the various sides, there are 288×48 cases, and $6 \times 13824 = 82944$ proofs. Taking into account the fact that each compounded ratio gives rise to 35 others, we are led to $13824 \times 36 = 497664$ expressions, each one comprising three ratios. His aim in this chapter is to put some limits to this growing number of cases. He concludes by saying: "See the number of ratio to which such a small figure leads! This is the effect of the omniscient will of the Supreme Being!"

Chapter 11 concerns cases of equality of two terms in the complete quadrilateral. This reminds us of Proposition 11 of Book I which we mentioned. Naṣīr al-Dīn presents the conclusion of his discussion in a table that we reproduce in Figure 10. It contains 28 cases of a certain equality implying another. The first two columns represent the two equal lines, and the rest of the columns the proportional lines. For instance, the first row means the following: if $AB = AD$, then $\frac{BE}{EF} = \frac{DC}{FC}$, and so on. The author declares at the end of this chapter that it is possible to write mamy more equalities, by exchanging, inverting, etc. several of the ratios.

After giving an exhaustive classification of all the cases of the Sector Figure theorem associated with the various figures obtained as the intersection

1	AB	AD	BE	EF	DC	FC
2	AB	BD	AE	EC	FD	FC
3	AB	AC	AE	FD	EC	BD
4			EB	DC	EF	AD
5	AE	EB	AC	CD	FB	DF
6	AE	AC	EB	BF	DC	DF
7	AE	EC	AB	BD	FC	DF
8	AE	DF	AB	FC	DB	EC
9			EB	DC	BF	AC
10	EB	EF	AB	AD	FC	DC
11	EB	BF	AE	AC	DF	CB
12	EB	DC	AB	FC	AD	BF
13			AE	DF	AC	BF
14	AC	EC	AD	DB	EF	BF
15	AC	BF	AD	EF	BD	EC
16			DC	EB	FD	AE
17	AD	DC	AB	BE	FC	FE
18	AD	BD	AC	CE	BF	EF
19	AD	EF	AC	BF	EC	BD
20			DC	EB	FC	AB
21	DC	DF	CA	EA	BF	EB
22	DC	FC	DA	AB	EF	BE
23	BD	EC	BF	AC	EF	AD
24			DF	EA	FC	AB
25	BF	FD	BE	EA	CD	BA
26	BF	EF	BD	DA	CE	CA
27	FD	AC	FB	AB	EC	EA
28	EF	FC	EB	AB	CD	AC

FIGURE 10. The various cases of the Sector Figure in the case of equality between two lines: the first two columns represent the two lines that are equal and the other four columns represent the proportional lines

of four lines in the plane, Naṣīr al-Dīn notes that Ptolemy studied only two cases, although he made use of several others.

Although Ptolemy's treatment of the Sector Figure theorem is very concise, it is much more detailed than the version of that theorem that in the al-Māhānī/al-Harawīversion of the *Spherics* [8]. This is at least what appears from the texts and the later editions of these texts that survive.

4. Book III

Book III of Naṣīr al-Dīn's treatise, in three chapters, despite its title ("On the lemmas for the spherical quadrilateral figure and what is necessary for using it profitably"), is in reality concerned with Euclidean geometry. It contains Euclidean results that are used in the proof of the spherical Sector Figure Theorem.

In the first chapter, Naṣīr al-Dīn proves the following theorem:

Let AB and AC be unequal arcs on a circle having the same extremity A and such that $B \neq C$. Let D be the intersection of the chord BC and the diameter AC. Then,

$$\frac{BD}{DC} = \frac{\sin AB}{\sin AC}.$$

In the figure that is given in the proof (Figure 11 here), two cases are considered: either AB and AC intersect only at the point A, or one of these arcs contains the other.

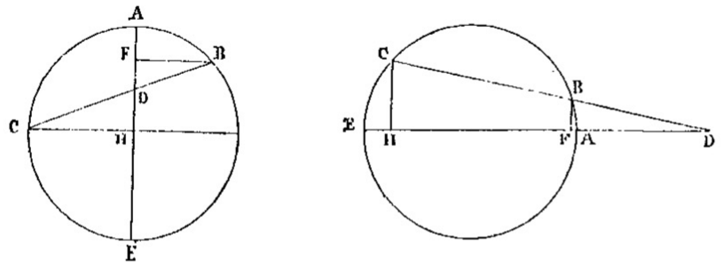

FIGURE 11. In the left hand side figure, the arcs AB and AC intersect only at the point A. In the right hand side figure, the arc AB is contained in the arc AC.

Naṣīr al-Dīn, after the proof, discusses six cases:

1) The two arcs AB and AC are smaller than a semi-circle. (This is the case considered in the first proof he gives.)

2) The two arcs AB and AC are equal to a semi-circle.

3) The two arcs AB and AC are greater than a semi-circle.

4) One of the two arcs is smaller than a semi-circle and the other one is equal to a semi-circle.

5) One of the two arcs is smaller than a semi-circle and the other one is greater than a semi-circle

6) One of the two arcs is greater than a semi-circle and the other one is equal to a semi-circle

In Chapter 2, Naṣīr al-Dīn discusses the various ways of obtaining all the sides and angles of a triangle if we are given some of them. In particular, he treats the following four cases:

1) Two angles are given. (In this case, he says, we know the form of the triangle, but not the magnitude of its sides.)

2) Two angles and a side are given.

3) Two sides and an angle are given.

4) The three sides are given.

Naṣīr al-Dīn then gives a proof of the sine formula for triangles. In the construction he uses for the proof, he considers two cases: the three angles are acute, or one angle is obtuse and the two others acute (Figure 12).

5. Book IV

Book IV, in five chapters, contains a detailed proof of all the cases of the Sector Figure that are associated with a spherical sector figure, with a consideration of the various ratios that are involved.

As the definition of a spherical quadrilateral involves four great circles, in the first chapter, Naṣīr al-Dīn starts by studying the various figures that are

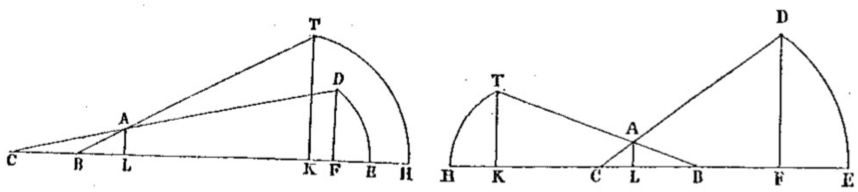

FIGURE 12. The construction used for the proof of the sine formula: either the three angles of the triangle ABC are acute (right hand side), or one of them is obtuse and the two others acute (left hand side)

formed by a configuration of four great circles such that no three of them intersect at the same point. He counts the number of complete quadrilaterals arising from such a configuration.

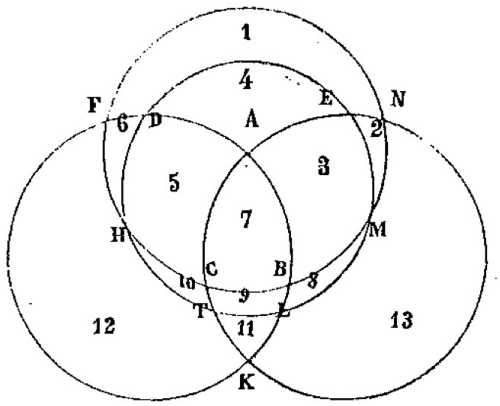

FIGURE 13. The various regions formed by the intersection of four great circles $ABKF$, $BCFN$, $ACKN$, $EDTL$ on the sphere ; the numbers on the digram were introduced by Carathéodory.

The combinatorics of the intersection is represented in Figure 13, where the four great circles are $ABKF$, $BCFN$, $ACKN$, $EDTL$. Since two arbitrary distinct great circles on the sphere intersect in two points, the four great circles will have 12 points in common, namely, A, B, C, D, E, F, H, K, L, M, N, T. Each of the circles $ABKF$, $BCFN$, $ACKN$, $EDTL$ is cut by these points into 6 arcs, and each of these arcs is a side of a unique complementary component of the four great circles which is a quadrilateral. There are 24 such arcs. The complement of the four great circles consist of 14 components, six of them being quadrilaterals (numbered 1, 2, 5, 9, 12, 13 in the figure) and eight of them being triangles (numbered 2, 4, 6, 7, 8, 10, 1, 14).

Several complete spherical quadrilaterals appear in Figure 13; for instance, the quadrilateral $DACH$ together with the two adjacent triangles ABC and HTC. This is the complete quadrilateral represented in Figure 1.

A count shows that any quadruple of great circles such that no three of them intersect at the same point gives rise to 24 spherical complete quadrilaterals.

Naṣīr al-Dīn notes that because of the antipodal map of the sphere, the 24 quadrilaterals are in fact 12 pairs of doubled quadrilaterals (two doubled quadrilaterals being considered as equal because they are images of each other by the antipodal map of the sphere).

In the rest of the book, Naṣīr al-Dīn gives the proofs of the various forms of the Sector Figure. Each such proof, following the one given by Ptolemy in the *Almagest*, is based on a use of the planar Sector Figure. The proof is not straightforward because the is no natural projection of a spherical complete quadrilateral on a Euclidean plane that gives rise to a planar complete quadrilateral. We do not enter here into the details of the proof of the Sector Figure; we only highlight a fact related to the subject of the present paper, namely, that as in the rest of the treatise, at each step of the proof, Naṣīr al-Dīn makes a precise count of the various cases involved.

At the end of Book IV, Naṣīr al-Dīn says that the Ancients have used the quadrilateral figure with confidence, without treating the large number of cases, and he mentions as examples Menelaus' *Spherics* and Ptolemy's *Almagest*. But the moderns, he says, in order to avoid engaging themselves in the examination of the multitude of ratios and the lengthy considerations that the use of compounded ratios requires, imagined and studied other figures, which replace the quadrilateral, which are as much useful and which do not lead to a multitude of cases. The presentation of these methods is the subject of Book V of his treatise.

It should be emphasized that Naṣīr al-Dīn al-Ṭūsī was not the first to study in detail the various cases of the Sector Figure. The first known such study was carried out by Thābit ibn Qurra in the 10th-11th century. See the article by Bellosta [1]

6. Book V

Book V of Naṣīr al-Dīn's treatise contains seven chapters concerning spherical trigonometry, including the spherical Sine Rule together with several other trigonometric formulae, that also involve the cosine and the tangent functions. This book contains in particular the following important topics:

- (Chapter 3) A complete classification of spherical sector figures regarding their side lengths (more precisely, whether they are less, equal to greater than a quadrant, that is, a quarter of a circle) and their angles (acute, right or obtuse) and the consequences of this information on the geometry of these triangles.
- (Chapter 4) How to know the elements (sides and angles) of a triangle from some other given elements of this triangle
- (Chapter 5) The Sine Rule with a large number of different proofs, due to several mathematicians.
- (Chapter 6) Other trigonometric formulae, in particular formulae involving the cosine and the tangent functions
- (Chapter 7) Given the sizes of three elements of a triangle (sides or angles), how to obtain the others using trigonometry and the polar triangle.

A spherical triangle is determined by three distinct points on the sphere together with three arcs of great circles that join them pairwise. Therefore, Naṣīr al-Dīn starts by studying the intersection of three great circles (Chapter 2). He notes that they divide the sphere into 8 regions, each one being a spherical triangle (Figure 14). He also notes that this triple of circles has six intersection points, twelve arcs of circles, and 24 angles. The eight triangles are pairwise equal (they are pairwise images of each other by the antipodal map of the sphere). For instance, in Figure 14, the triangles ACF and DEB are equal. Thus, there are at most four unequal triangles. Among these four triangles any two have at least one side and one angle in common, and the rest of their elements are supplementary of each other (their sum is equal to two right angles; we recall that in spherical geometry, the length of a spherical segment is the angle made by two rays that join the origin to the two extremities). For instance, comparing the triangles ABC and CDF, we have: the angle C is common, $AB = FD$, BC and FC are supplementary, AC and CD are supplementary, the angle ABC is equal to the angle AFC and is supplementary to the angle CFD, and the angle BAC is equal to the angle BDC and is supplementary to the angle CFD.

Consequently, if one of the eight triangles is known then the other seven triangles are also known.

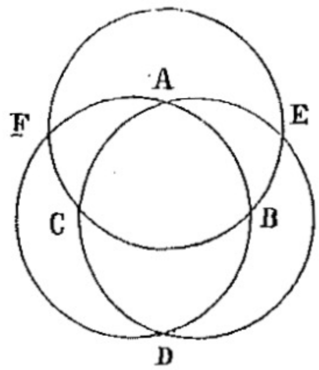

FIGURE 14. The eight triangles made by three great circles on the sphere (note that there is a triangle with sides EF, FD, DE)

Naṣīr al-Dīn gives the following classification of spherical triangles into 10 types, depending on whether their sides are equal, less, or greater than a quadrant:

(1) each of the three sides is a quadrant (q);
(2) two sides = q; one side < q;
(3) two sides = q; one side > q;
(4) one side = q; two sides < q;
(5) one side = q; two sides > q;
(6) one side = q; one side < q; one side > q;
(7) two side < q;
(8) two sides > q; one side < q;
(9) two sides < q; one side > q;
(10) three sides > q.

Using a short argument, Naṣīr al-Dīn shows that in fact, there are five species of intersections of three great circles. The numbers at the end of each item refer to the above classification of triangles into 10 types depending on their side lengths:

(a) the triangles are of type (1);
(b) the triangles are of type (2) and (3);
(c) the triangles are of type (4), (5) and (6);
(d) the triangles are of type (7) and (8);
(e) the triangles are of type (9) and (10).

Then, Naṣīr al-Dīn gives a classification of spherical triangles in terms of their angles; again, there are 10 types:

(1) Each of the three angles is a right angle (R);
(2) two angles = R and one angle < R;
(3) two angles = R and one angle > R;
(4) one angle = R and two angles < R;
(5) one angle = R and two angles > R;
(6) one angle = R, one angle < R and one angle > R;
(7) three angle < R;
(8) two angles > R and one angle < R;
(9) 2 angles < R and one angle > R;
(10) three angles > R.

The two classifications of triangles, in terms of their side lengths and angle measures are used later in the book.

Likewise, there are five species of intersections of three great circles in terms of their angle measures. The numbers at the end of each item refer to the above classification of triangles into 10 types depending on their angles:

(a) The triangles are of type (1).
(b) The triangles are of type (2) and (3).
(c) The triangles are of type (4), (5) and (6).
(d) The triangles are of type (7) and (8).
(e) The triangles are of type (9) and (10).

In Chapter 3, Naṣīr al-Dīn gives 10 propositions in which, for each of the 10 types in the above classification according to sides, he gives the possibilities for the angles. For instance, Proposition 8 says the following:

If a triangle satisfies the following: two sides > q and one side < q, then there are only the following 5 possibilities for its angles:

(1) one angle = R and two angles < R;
(2) one angle = R, one angle < R and one angle > R;
(3) one angle < R and two angles < R;
(4) one angle > R and two angles < R;
(5) three angles > R.

After this, he gives 10 propositions in which he discusses, for each of the 10 types in the above classification according to angles, the various possibilities for the sides. For instance, the tenth proposition says the following:

If a triangle has an angle >R and two angles < R, then there are only the five following possibilities for its sides:

(1) the three sides are < q;

(2) two sides are $<$ q and one side is $= q$;
(3) two sides are $<$ q and one side is $>$ q;
(4) two sides are $>$ q and one side is $<$ q;
(5) one side is $=$ q, one side is $<$ q and one side is $>$ q.

Chapters 4 to 6 are dedicated to the proofs of the spherical trigonometric formulae. These proofs are based on the notion of polar triangle. As a matter of fact, Naṣīr al-Dīn makes use of two kinds of polar triangles: one which he calls the "supplementary figure" and another one he calls the "shadowed figure". This is a substantial mathematical part of his treatise, but it is not the subject of the present paper. We shall comment on it in a sequel to this paper. Sticking to our subject, we now review the last chapter of Book III.

In Chapter 7, Naṣīr al-Dīn summarizes, in two series of propositions, the fundamental trigonometric formulae for spherical triangles. The first series consists of the formulae derived from the supplementary figure, and the second series those derived from the shadowed figure. Let us summarize them:

The propositions concern a right triangle with right angle at C, opposite to a side c, and with angles A and B opposite to the sides a and b respectively.

We do not need to go in detail into these formulae; let us state the content of the propositions.

The first set of propositions give the following information:

1. We know c, the side opposite to the right angle, and another side, b. Then, he gives the formula for the cosine of the unknown side and for the sine of the angle B. (There is then a similar formula for the angle C.)

2. We know the two sides a and b that contain the right angle. Then, he gives a formula for the cosine of the side opposite to the right angle. From the first proposition, we can then use the known sides to find the two angles.

3. We know the angle A and the side a opposite to it. Then, he gives a formula for the sine of the side opposite to the right angle. Using the first proposition, we can then determine the third side and the third angle.

4. We know the angle A and the side c opposite to the right angle. He gives then a formula for the side opposite to the known angle. Using the first proposition, we can then determine the remaining angle and side.

5. We know the angle A and the side b contained by the right angle and this angle. Then, he gives a formula for the angle opposite to the known side. using Proposition 3, we can then determine the two remaining sides.

6. We know the two angles A and B. He then a formula for the side a. As in Proposition 3, we can then determine the remaining sides.

The second set of propositions give the following information:

1. We know the side c opposite to the right angle and another side, b. Then, he gives formulae for the angles A and B and the remaining side a.

2. We know the two sides a and b that contain the right angle. Then, he gives a formula for the angles A and B and the side c opposite to the right angle.

3. We know an angle A and the side a opposite to it. Then, he gives a formula for the side b contained by A and the right angleThe other unknowns are found as in Proposition 2.

4. We know an angle A and the side c opposite to the right angle. Then, he gives a formula for the side b situated between A and the right angle.

5. We know an angle A and the side b contained by this angle and the right angle. Then, he gives a formula for the tangent opposite to the known angle. The remaining elements of the triangle are found as in Propositions 2 and 3.

6. We know the two sides a and b. Then, he gives a formula for the sine of the side opposite to the right angle. The other unknown elements of the triangle are found as in proposition 4.

After these propositions, Naṣīr al-Dīn considers the case of an arbitrary (not necessarily right) triangle. Here, instead of the trigonometric formulae, he gives constructions for the six elements of a triangle, if we are given three of them, and then he establishes the relation between these constructions and the theory of the quadrilateral.

As a conclusion, Naṣīr al-Dīn's treatise is most of all an important piece of work in spherical trigonometry, but the author at the same time, in his systematic count of the various cases, acts as a combinatorician.

Appendix A. Naṣīr al-Dīn al-Ṭūsī

Naṣīr al-Dīn al-Ṭūsī was a mathematician, astronomer, philosopher and theologian. He belongs to the last period of the Islamo-Arabic golden era in science. He was born in 1201 in the city of Tūs (North-East of Iran) and he lived essentially in Iran and Iraq. He was contemporary to the invasion by the Mongols of Russia and of large parts of Central Europe and Asia. It is generally considered that this invasion was one of the bloodiest ones in history. Baghdad and other major centers of Arabic science were ransacked during this invasion.

Al-Ṭūsī worked for the invaders. He married a Mongol woman and he became the official astrologer and scientific advisor of the ruler Hūlagū Khan, the grandson of the first conqueror, Genghis Khan. In 1259, he convinced Hūlagū, who was fond of astrology, to build a big observatory in Maragha, a city in the North-East of Iran, where the ruler used to spend his summers. The construction which took more than ten years was supervised by Naṣīr al-Dīn in person. The observatory became later a science center with a big library and a scientific community which included scholars from all over the Islamic world and China. After the death of Hūlagū, Naṣīr al-Dīn continued to occupy an important role in the Mongol court. He became the personal physician of the new ruler, Abaqa Khan, Hūlagū's son. He died in 1274 in the city of Kadhimain, near Bagdad.

In mathematics, we owe to Naṣīr al-Dīn al-Ṭūsī commented Arabic editions of several important Greek mathematical texts, including the geometrical works of Autolycus of Pitane, Euclid, Apollonius, Archimedes, Theodosius, Menelaus and Ptolemy. These publications played an important role in the transmission of the Greek works to Europe. A version of Euclid's *Elements* attributed Naṣīr al-Dīn, published in Rome in 1594, contains a commentary that was used and quoted by John Wallis and by Girolamo Saccheri in their works on the parallel problem, see [10, 12, 13]. Al-Ṭūsī also wrote original treatises on logic, astronomy, algebra, number theory and combinatorics, see [7, 5, 6, 11] and the various articles in [9]. We quoted at several places some of his comments on Menelaus' *Spherics* in the recent publication [8]. A modern complete edition of his edition of Menelaus' treatise would be a welcome event for the mathematical community. His comments on Menelaus' propositions contain a wealth of mathematical ideas on spherical geometry.

Appendix B. Alexandre Carathéodory

Alexandre Carathéodory, the translator of the *Treatise of the quadrilateral*, was born in 1838 in Constantinople and he died there in 1906. He belonged to a leading Greek Phanariote family that was close to the Sublime Porte. His father was the personal physician of the Sultan Mahmud II. Alexandre Carathéodory studied law in Paris and obtained there a doctorate, with a dissertation on the theory of error from the philosophical point of view and its applications in the sciences, especially in law. After his return to Constantinople, he became an Ottoman civil servant and started a diplomatic career. In 1874, he became ambassador of the Ottoman empire to Rome. In 1878, he was the main negotiator at the Treaty of San Stefano which ended the Russo-Turkish War (1877–78) and he participated in the Berlin Treaty which re-organized the map of Eastern Europe after the Ottoman Empire's defeat. He was appointed later Governor of Crete, then Ottoman minister of foreign affairs, then Prince of the autonomous island of Samos. His name on the cover page of his translation of Naṣīr al-Dīn al-Ṭūsī's *Treatise of the quadrilateral* is followed by the title "Ancien ministre des affaires étrangères" (Former minister of foreign affairs). Alexandre Carathéodory is the great-uncle of the mathematician Constantin Carathéodory.

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