Production of photon states from Λ-atoms in a cavity

B Rousseaux, A Saharyan, Z. Kis, S. Guerin

Production of photon states from Λ-atoms in a cavity

B. Rousseaux,[1] A. Saharyan,[2] Z. Kis,[3] and S. Guérin[2,*]

[1] *Department of Physics, Chalmers University of Technology, 412 96 Göteborg, Sweden*
[2] *Laboratoire Interdisciplinaire Carnot de Bourgogne, CNRS UMR 6303, Université de Bourgogne, BP 47870, 21078 Dijon, France*
[3] *Hungarian Academy of Sciences, Wigner Research Center for Physics, Konkoly-Thege Miklós út 29-33, 1121 Budapest, Hungary*

(Dated: December 27, 2019)

We analyse the system of Λ-atoms in a cavity QED of semi-transparent mirror and driven by laser fields. We derive effective models and connect concepts (photonic flux, input-output operators, photonic state) characterizing the propagation of the resulting leaking photons. We propose an atom-cavity non-resonant scheme for single- and 2-photons generation. The pulse shapes of outgoing single photons are tailored using a specifically designed driving field envelope. For the production of 2-photon states, two trapped atoms are used with two driving pulses. Their pulse shapes are characterized and it is shown that the multiphoton outgoing photonic states cannot be Fock states, since the photons are not generated strictly simultaneously.

I. INTRODUCTION

Single photons are nowadays key elements in quantum technologies, as quantum networking for distributed computation, communication and metrology [1, 2]. Sources producing single photons have been widely developed [3, 4]. Its quantization and its treatment as a wave function in connection with a corpuscular viewpoint have been debated until recently [5–7]. From a practical point of view, one can for instance mention its need in quantum cryptography [8] over the use of attenuated laser pulses for making the security of quantum key distribution device-independent or for extending quantum communication over very long distances [9]. An envisioned quantum network makes use of single photons wavepacket as carriers of quantum information (encoded for instance in the polarization state giving flying qubits) to map the states between distant quantum nodes [2], such as individual atoms in cavity QED [10–13], atomic ensembles [14, 15], trapped ions [16], or spins in quantum dots [17]. One key point is to control the node-photon interfacing in order that the node can send, receive, store and release photonic quantum information, which is in general achieved by control laser pulses. Recent studies have investigated the control of the shape of the single-photon wavepacket in Λ-atoms by a resonant stimulated Raman process [18] in order for instance to improve the impedance matching of the atom-photon interface [13]. The possible production of more complex traveling photonic states featuring $N > 1$ photons [19–21] can be envisioned for the transport of complex information. For instance, the delays and relative amplitudes between the pulse-shaped individual photons offer a large variety of encoding, which generalizes the possibility of producing train of well-separated pulses [22].

The goal of this paper is first to derive effective models for atoms driven by laser fields and cavity QED with a semi-transparent mirror and for characterizing the resulting propagating photon field. To this aim we revisit and connect concepts defined in literature, namely photon fluxes, input - output operators, effective master equation, and multiphotonic wavepackets and states. We apply the model for a non-resonant scheme in a Λ-atom trapped in a cavity QED and show that it allows a direct and simple way to design the photonic wavepacket on demand. This is extended for a two-atom scheme and the resulting photonic wavepacket is characterized and compared to an ideal traveling Fock state. We show that the resulting multiphoton outgoing photonic states cannot be Fock states, since the individual constituent photons are not generated strictly simultaneously.

This paper is organized as follows: In Section II we connect the photon flux, corresponding to the propagation of the photonic state in free space leaking from the cavity, to the quantum average of a reservoir photon number operator, in the Heisenberg representation, using the quantized Poynting vector. The condition of correspondence of this reservoir photon number operator to the standard output photon number operator is derived. We next establish that the photon flux is proportional to the quantum average of the cavity photon number operator in the condition of an initial ground state reservoir. The master equation, which allows one to determine the state of the atom-cavity-laser field system that are used to calculate the needed quantum averages, is finally derived. In section III we use the derived model to show that one can produce a single-photon wavepacket of give shape using one Λ-atom driven with a non-resonant laser pulse in a cavity mode. Section IV is devoted to the case of two single photons emission from two atoms in the cavity, where the resulting two-photon state is analyzed. We conclude in Section V.

* sguerin@u-bourgogne.fr

II. DERIVATION OF THE MODEL

In this Section, we connect the photon flux [23, 24], corresponding to the propagation of the photonic state in free space leaking from the cavity, to the quantum average of a reservoir photon number operator, in the Heisenberg representation, constructed with an integrated bath operator. We follow the formulation of Ref. [23], using the quantized Poynting vector, adapting it for the case of the presence of the cavity. We derive the condition of correspondence of this bath photon number operator to the standard output photon number operator derived in the input-output formulation [25]. We next establish that the photon flux is proportional to the quantum average of the cavity photon number operator when the reservoir is initially in the ground state [22]. We finally derive the master equation [25–27] in tracing out the bath degrees of freedom, which allows one to determine the state (and the operator density) of the atom-cavity-laser field system that are used to calculate the quantum averages needed to calculate the photon flux.

A. Hamiltonian in the Schrödinger picture

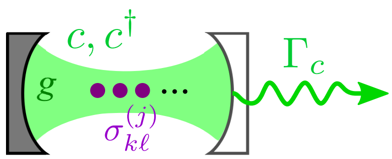

FIG. 1. Representation of the CQED system: N atoms are coupled to a single cavity mode of annihilation-creation operators $c, c^\dagger$, with the atom-cavity coupling g. Each operator $\sigma_{k\ell}^{(j)}$ corresponds to the $|k\rangle \leftrightarrow |\ell\rangle$, $k, \ell = g, e, f$ transition of the j-th atom, and photons leak through the right semitransparent mirror with decay rate Γ_c.

We consider a set $\mathcal{A}_N$ of N identical Λ-atoms of a ground $|g\rangle$, metastable $|f\rangle$ and excited $|e\rangle$ states trapped in a cavity QED. They are coupled to the (linearly polarized) cavity field, of volume V and frequency ω_c, through the atomic transition $|f\rangle \leftrightarrow |e\rangle$ of frequency ω_{ef} and dipole moment d_{fe} with the coupling factor $g = -d_{fe}\sqrt{\omega_c/2\hbar\epsilon_0 V}$ (one-photon Rabi frequency). It is assumed that g is constant for each atom. They are pumped by a (classical) laser field $\mathcal{E}_j(t)\cos(\omega_0 t + \varphi)$, with the pulse-shaped Rabi frequency $\Omega_j \equiv \Omega_j(t) = -\mathcal{E}_j(t)d_{ge}/2\hbar$ (assumed real), on the transition $|g\rangle \leftrightarrow |e\rangle$ of frequency ω_{eg} and dipole moment d_{ge}. We consider a two-photon resonance: $\omega_{gf} = \omega_c - \omega_0$. The cavity ($\mathcal{C}$) leaks into a reservoir ($\mathcal{R}$) through a semi-transparent mirror (see Fig. 1 for the schematic representation of the full system and Fig. 3 for the coupling scheme for a single atom). For brevity of the derivation, we simplify the model considering no spontaneous emission on the atomic transitions. In the rotating wave approximation (RWA) for both the modes and the driving field, the Hamiltonian of the full system $\mathcal{A}_N \oplus \mathcal{C} \oplus \mathcal{R}$ reads, in the Schrödinger picture, in a rotating frame defined by the unitary operator $\hat{U}_{\rm RW} = \exp\big[i\omega_0 t\sum_{j=1}^N \sigma_e^{(j)} + i\omega_{fg} t\sum_{j=1}^N \sigma_f^{(j)} + i\hat{H}_C t/\hbar + i\hat{H}_R t/\hbar\big]$:

$$\hat{H}(t) = \hat{H}_A(t) + \hat{H}_{AC} + \hat{H}_{RS} \tag{1a}$$

$$\hat{H}_A = \sum_{j=1}^{N}\left[\hbar\Delta\sigma_e^{(j)} + \hbar\Omega_j(\sigma_{ge}^{(j)} + \sigma_{eg}^{(j)})\right] \tag{1b}$$

$$\hat{H}_C = \hbar\omega_c c^\dagger c, \qquad \hat{H}_{AC} = \hbar g\sqrt{N}\big(c^\dagger\sigma + \sigma^\dagger c\big) \tag{1c}$$

$$\hat{H}_R = \int_0^{+\infty} d\omega\, \hbar\omega\, b_\omega^\dagger b_\omega \tag{1d}$$

$$\hat{H}_{RS} = i\hbar\int_0^{+\infty} d\omega\, \kappa(\omega)\big(b_\omega^\dagger c\, e^{-i(\omega_c-\omega)t} - \text{H.c.}\big). \tag{1e}$$

We have introduced here the collective operator $\sigma = \frac{1}{\sqrt{N}}\sum_{j=1}^N \sigma^{(j)}$ with the atomic operators $\sigma_{k\ell}^{(j)} \equiv |k\rangle\langle\ell|^{(j)}$ for the Λ-atom j, $\sigma_k^{(j)} \equiv \sigma_{kk}^{(j)}$ and $\sigma^{(j)} \equiv \sigma_{fe}^{(j)}$. The annihilation operator c corresponds to the cavity mode. The output reservoir annihilation and creation operators $b_\omega, b_\omega^\dagger$ satisfy the commutation relation:

$$[b_\omega, b_{\omega'}^\dagger] = \delta(\omega - \omega'). \tag{2}$$

The reservoir $\mathcal{R}$ couples to the cavity mode through $\kappa(\omega)$. In Eqs. (1), $\hat{H}_A \equiv \hat{H}_A(t)$ denotes the atomic RWA Hamiltonian where $\Delta = \omega_{eg} - \omega_0$ is the detuning between the frequencies of the laser driving atom j and of the transition $|e\rangle \leftrightarrow |g\rangle$, $\hat{H}_C$ is the free cavity Hamiltonian, $\hat{H}_{AC}$ describes the coupling between the atoms and the cavity, $\hat{H}_R$ is the free reservoir Hamiltonian, and $\hat{H}_{RS}$ describes the coupling between the system $\mathcal{S} = \mathcal{A}_N \oplus \mathcal{C}$ of corresponding Hamiltonian

$$\hat{H}_S(t) = \hat{H}_A(t) + \hat{H}_{AC} \tag{3}$$

and the reservoir $\mathcal{R}$.

We emphasize that this model featuring well defined inside (cavity) mode and outside modes has been well justified in [29] from the consideration of the full global modes in a coarse-graining description, when the transmission of the cavity is sufficiently weak.

B. Heisenberg-Langevin equations, Markov approximation, Poynting vector, and photon fluxes

We wish to derive the dynamics of the atoms+cavity system $\mathcal{S}$, coupled to the reservoir. Our aim is to control the production of an outgoing photon leaking from the cavity by driving specifically the atoms in the cavity by the external field. The effective model is derived in two steps: we first define an outgoing flux of photon which is connected to the quantum average of the Heisenberg evolution of the cavity operator $c^\dagger c$. Next we derive a master equation of the system $\mathcal{S}$ by eliminating the reservoir degrees of freedom, which will allow the calculation of the quantum averages.

1. Equations of motion for the operators

First, we derive the equations of motion in the Heisenberg picture for the reservoir operator $b_\omega(t) \equiv U^\dagger(t,t_0)b_\omega U(t,t_0)$ with $U(t,t_0)$ being the propagator of the total Hamiltonian $\hat{H}(t)$, whose Heisenberg representation reads $\hat{H}^{(H)}(t) = U^\dagger(t,t_0)\hat{H}(t)U(t,t_0)$. From $\dot{\mathcal{O}} = -\frac{i}{\hbar}[\mathcal{O}(t), \hat{H}^{(H)}(t)]$ for an operator $\mathcal{O}$, assumed time-independent in the Schrödinger representation, and written as $\mathcal{O}^{(H)}(t) \equiv \mathcal{O}(t) = U^\dagger(t,t_0)\mathcal{O}U(t,t_0)$ in the Heisenberg representation, we write the Heisenberg-Langevin equations:

$$\dot{b}_\omega(t) = -i\omega b_\omega(t) + \kappa(\omega)c(t), \tag{4a}$$

$$\dot{c}(t) = -i\omega_c c(t) - \int d\omega\kappa(\omega)b_\omega(t) - ig\sqrt{N}\sigma(t). \tag{4b}$$

In the following, we omit the (H) superscript for the Heisenberg picture Hamiltonian $\hat{H}^{(H)}(t) \equiv \hat{H}(t)$. The energy carried by the photons leaking from the cavity can be characterized by the Poynting vector operator in the Heisenberg representation [23], where we have assumed a propagation with increasing z and the cavity emitter at position $z = 0$ (see Fig. 2):

$$\hat{S}(z,t) = \frac{\hbar}{2\pi\mathcal{A}}\int \mathrm{d}\omega\mathrm{d}\omega'\sqrt{\omega\omega'}b^\dagger_\omega(t)b_{\omega'}(t)\mathrm{e}^{-i(\omega-\omega')\frac{z}{c}}, \tag{5}$$

with the use of the quantized fields [7, 30, 31] and $\mathcal{A}$ is the area of the free field modes propagating at the speed of light c. The range of integration for determining $\hat{S}(z,t)$ is made clearer below. We emphasize that the time dependence arises only from the Heisenberg representation of the bath operator b_ω.

2. Integrated bath operators - Input output relation

Integrating (4a) from an initial time t_0 to t, we define and calculate the integrated bath operator

$$\hat{b}(z,t) := \frac{1}{\sqrt{2\pi}}\int d\omega b_\omega(t)\mathrm{e}^{i\omega\frac{z}{c}} \tag{6a}$$

$$= b_{\mathrm{in}}\Big(t-\frac{z}{c}\Big) + \int_{t_0}^t dt'\int d\omega\frac{\kappa(\omega)}{\sqrt{2\pi}}c(t')\mathrm{e}^{-i\omega(t-t')}\mathrm{e}^{i\omega\frac{z}{c}} \tag{6b}$$

with the *input* operator

$$b_{\mathrm{in}}\Big(t-\frac{z}{c}\Big) = \frac{1}{\sqrt{2\pi}}\int d\omega b_\omega \mathrm{e}^{-i\omega(t-t_0-\frac{z}{c})}. \tag{7}$$

We can proceed with the Markov approximation, consisting in assuming the flatness of $\kappa(\omega)$ over the width of the resonance: $\kappa(\omega) \equiv \kappa(\omega_c) =: (\Gamma_c/2\pi)^{\frac{1}{2}}$, and the extension of the integral over ω on the range $]-\infty,+\infty[$. This allows one to invoke a δ-function in the time integral:

$$\delta(s-t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\,\mathrm{e}^{-i\omega(s-t')}, \tag{8a}$$

$$\int_{t_0}^t dt'\delta(t-\tfrac{z}{c}-t')c(t') = \Theta(\tfrac{z}{c})\Theta(t-\tfrac{z}{c}-t_0)c(t-\tfrac{z}{c}). \tag{8b}$$

This gives for the integrated bath operator:

$$\hat{b}(z,t) = b_{\mathrm{in}}\Big(t-\frac{z}{c}\Big) + \sqrt{\Gamma_c}\Theta\Big(\frac{z}{c}\Big)\Theta\Big(t-\frac{z}{c}-t_0\Big)c\Big(t-\frac{z}{c}\Big) \tag{9}$$

with the step function $\Theta(u) = \{0,$ for $u<0; 1/2$ for $u=0; 1$ for $u>0\}$. It takes a propagating form for $z>0$

$$b\Big(t-\frac{z}{c}\Big) \equiv \hat{b}(z>0,t) \tag{10a}$$

$$= b_{\mathrm{in}}\Big(t-\frac{z}{c}\Big) + \sqrt{\Gamma_c}\Theta\Big(t-\frac{z}{c}-t_0\Big)c\Big(t-\frac{z}{c}\Big). \tag{10b}$$

For $t > t_0 + \frac{z}{c}$ and $z>0$, we define the *output* operator

$$b_{\mathrm{out}}(t-z/c) := \hat{b}(z>0, t>t_0+z/c) = b(t-z/c>t_0) \tag{11}$$

and we obtain

$$b_{\mathrm{out}}\Big(t-\frac{z}{c}\Big) = b_{\mathrm{in}}\Big(t-\frac{z}{c}\Big)+\sqrt{\Gamma_c}\,c\Big(t-\frac{z}{c}\Big),\ t>t_0+\frac{z}{c}, \tag{12}$$

which is recognized as the input-output relation [25]. We emphasize that we have here derived the output operator and the input-output relation taking into account propagation effects. This allows one avoiding considering a (not well-defined) late time as usually done, but rather the well-defined integrated bath operator $\hat{b}(t,z)$ for $z>0$. This way of formulating allows a direct and transparent interpretation of the b_{out} operator through the Poynting vector as shown below [see Eq. (16)].

At the cavity position, $z=0$, for $t>t_0$, we obtain the integrated bath operator:

$$b_0(t) \equiv \hat{b}(z=0,t) = b_{\mathrm{in}}(t) + \frac{1}{2}\sqrt{\Gamma_c}\ c(t). \tag{13}$$

This expression (13) is used in the next subsection to derive the master equation in the cavity.

We can also simplify the Heisengerg-Langevin equation for $c(t)$ as:

$$\dot{c}(t) = -(i\omega_c+\Gamma_c/2)c(t) - \sqrt{\Gamma_c}b_{\mathrm{in}}(t) - ig\sqrt{N}\sigma(t). \tag{14}$$

This shows a fast oscillating term $\exp(-i\omega_c t)$ in $c(t)$, leading to a peaked shape of $b_\omega(t)$ as a function of ω centered at the resonance ω_c and of width Γ_c.

3. Poynting vector

Using the definition (6a) of the bath operator and integrating over a bandwidth around the resonance ω_c from

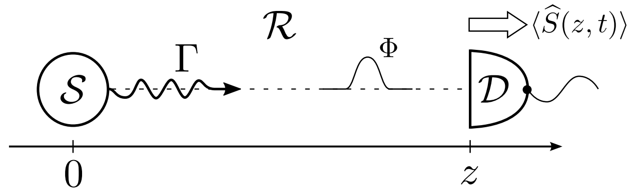

FIG. 2. Sketch of the photodetection: the source system $\mathcal{S}$ emits a photon with decay rate Γ at position 0, towards a detector $\mathcal{D}$ at a position z through the reservoir $\mathcal{R}$. The photon flux Φ is measured using the data on the averaged quantum Poynting vector $\langle\hat{S}(z,t)\rangle$.

the result of Eq. (14): $\omega_c - \Delta\omega/2 < \omega < \omega_c + \Delta\omega/2$ with $\Delta\omega \sim \Gamma_c \ll \omega_c$, the Poynting vector operator becomes $\hat{S}(z,t) = \frac{\hbar\omega_c}{\mathcal{A}}\hat{b}^\dagger(z,t)\hat{b}(z,t)$ which takes a propagating form for $z > 0$:

$$\hat{S}(z>0,t) = \frac{\hbar\omega_c}{\mathcal{A}} b^\dagger\Big(t-\frac{z}{c}\Big) b\Big(t-\frac{z}{c}\Big). \tag{15}$$

For a given state (or density matrix), the amount of energy going through the field mode area $\mathcal{A}$, during the time dt, is the quantum average of the flux of the Poynting vector through this area: $\mathcal{A}\langle\hat{S}(z,t)\rangle dt = \hbar\omega_c\langle\hat{b}^\dagger(z,t)\hat{b}(z,t)\rangle dt$. Normalizing by $\hbar\omega_c$, we get the averaged number of photons $dn(t,z) \equiv \langle\hat{b}^\dagger(z,t)\hat{b}(z,t)\rangle dt$ going through the mode area during dt, defining the photon flux (written here for $z > 0$):

$$\Phi(z,t) := \frac{dn(z,t)}{dt} = \Big\langle b^\dagger\Big(t-\frac{z}{c}\Big) b\Big(t-\frac{z}{c}\Big)\Big\rangle. \tag{16}$$

Recalling that $b(t - z/c)$ is the output operator (11) (for $t > z/c$ and $z > 0$), we emphasize that this relation gives the connection between the photon flux and this output operator.

If we choose the state of the reservoir to be initially a vacuum state: $\rho(t_0) = \rho_S(t_0) \otimes |\text{vac}\rangle\langle\text{vac}|$, the average of the terms involving $b_{\text{in}}, b_{\text{in}}^\dagger$ in the expression of the flux nullifies. This gives the expression of the outgoing photon flux through the semi-transparent mirror for $t > t_0 + \frac{z}{c}$:

$$\Phi(z,t) = \Gamma_c\Big\langle c^\dagger\Big(t-\frac{z}{c}\Big) c\Big(t-\frac{z}{c}\Big)\Big\rangle. \tag{17}$$

This key result shows that one can determine the flux from the quantum average of the dynamics of the cavity photon number in the Heisenberg representation [22].

In the following subsection, we derive the effective master equation reduced to the system $\mathcal{S}$ which is used to calculate the quantum average of (16) in order to derive the flux.

C. The master equation

We here recall for consistency a standard way to get the master equation [25, 26, 32, 33]. We need first to derive the Heisenberg equation of motion of the operators $X_S(t) = U^\dagger(t,t_0)X_S U(t,t_0)$ of the system in the Heisenberg representation. The dynamics of $X_S(t)$ is determined from the Heisenberg equation (in the Markov approximation):

$$\frac{\mathrm{d}}{\mathrm{d}t}X_S(t) = -\frac{i}{\hbar}\Big[X_S(t), \hat{H}_S^{(H)}(t)\Big] + \mathcal{D}^\dagger_{\text{in},t}\big(X_S(t)\big) \\ +\Gamma_c\big(c^\dagger(t)X_S(t)c(t) - \tfrac{1}{2}\{c^\dagger(t)c(t), X_S(t)\}\big), \tag{18}$$

where $\{A,B\} = AB + BA$ denotes the anticommutation relation, $\mathcal{D}^\dagger_{\text{in},t}(\cdot)$ is a time-dependent dissipator part involving $b_{\text{in}}(t)$, acting on $X_S(t)$, and $\hat{H}_S^{(H)}(t) = U^\dagger(t,t_0)\hat{H}_S(t)U(t,t_0)$. We have used the bath integrated operator (13) at the position $z = 0$ of the cavity.

We define the expectation value of X_S:

$$\langle X_S\rangle(t) = \mathrm{Tr}_S\{X_S\rho_S(t)\} = \mathrm{Tr}\{X_S(t)\rho(t_0)\}, \tag{19}$$

where $\rho(t_0) = \rho_S(t_0) \otimes \rho_R(t_0)$ is the complete density operator and $\rho_S(t) = \mathrm{Tr}_R\{U(t,t_0)\rho(t_0)U^\dagger(t,t_0)\}$ is the reduced density operator describing $\mathcal{S}$ with partial trace $\mathrm{Tr}_R\{\cdot\}$ eliminating the degrees of freedom corresponding to its subscript.

We here assume that the reservoir is initially a vacuum state $\rho_R(t_0) \equiv |\text{vac}\rangle\langle\text{vac}|$ such that $\mathcal{D}^\dagger_{\text{in},t}(\cdot)$ cancels out in averaging. Finally, averaging Eq. (18), using (19), the cyclic property of the trace, and the property $\forall A\ \mathrm{Tr}\{AB\} = \mathrm{Tr}\{AC\} \Leftrightarrow B = C$, we find the master equation of Lindblad form for $\rho_S(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\frac{i}{\hbar}[\hat{H}_S(t), \rho_S(t)] \\ + \Gamma_c\big(c\rho_S(t)c^\dagger - \tfrac{1}{2}\{c^\dagger c, \rho_S(t)\}\big), \tag{20}$$

where, here, all system operators σ, c are time-independent (Schrödinger representation), and the remaining time-dependence of $\hat{H}_S(t)$ is due to the driving fields $\Omega_j(t)$.

If several cavities QED are considered, where the output of one cavity is fed into that of the next cavity, the systems can be "cascaded" [25, 27, 33].

III. PRODUCTION OF A SINGLE PHOTON BY ONE DRIVEN ATOM TRAPPED IN CAVITY

We derive from the preceding analysis the model for the generation of a single photon using a leaking cavity containing one atom driven by a pulsed laser of Rabi frequency $\Omega(t)$. The production of a single photon in such a system has been demonstrated with an atom flying through the cavity in a resonant stimulated Raman adiabatic passage configuration [18, 34] and for a trapped ion in a cavity [28]. We next show that a large cavity detuning and a bad cavity allows the direct an simple control of the photon shape.

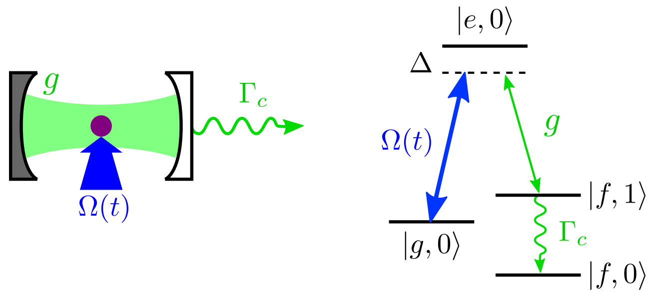

FIG. 3. Atom-field interaction in the cavity: (left panel) a single Λ-atom is driven by an external classical laser field of Rabi frequency Ω, and a quantized cavity field with coupling strength g. (Right panel) The fields are in two-photon resonance, the one-photon detuning is Δ. Initially the atom is in the ground state $|g\rangle$. In the course of the excitation process, one photon is taken from the laser field and transferred to the cavity, which eventually leaks out of the cavity through a semi-transparent mirror characterized by the decay rate Γ_c.

A. The model

In a dressed basis, one denotes states $|i\rangle|n\rangle \equiv |i,n\rangle$ with i labelling the atomic states and n is the number state in the cavity. We assume an initial condition with zero photon in the cavity, such that the basis splits into four relevant dressed states $\{|g,0\rangle,|e,0\rangle,|f,1\rangle,|f,0\rangle\}$ (see Fig. 3). Such dynamics involves the Lindblad equation derived previously (we omit the subscript S for ρ):

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -i[H_S(t),\rho(t)] + \mathcal{L}(\rho(t)), \tag{21}$$

with the dissipator $\mathcal{L}(\rho) = \Gamma_c(c\rho c^\dagger - \frac{1}{2}\{\rho, c^\dagger c\})$. Equation (21) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -i(\tilde{H}(t)\rho(t) - \rho(t)\tilde{H}^\dagger(t)) + \Gamma_c\, c\rho(t)c^\dagger, \tag{22}$$

where we introduced an anti-Hermitian dissipative Hamiltonian $\tilde{H}(t) = H_S(t) - i\frac{\Gamma_c}{2}c^\dagger c$. Expressing the Hamiltonian in a matrix form in the dressed basis

$$H_S(t) = \hbar\begin{bmatrix}\mathbf{A}(t) & [0]_{3\times1}\\ [0]_{1\times3} & 0\end{bmatrix}, \tag{23a}$$

$$\mathbf{A}(t) = \begin{bmatrix}0 & \Omega(t) & 0\\ \Omega(t) & \Delta & g\\ 0 & g & 0\end{bmatrix}, \tag{23b}$$

shows two decoupled dynamical blocks $\mathbf{A}(t)$ and $\{0\}$. From the density matrix

$$\rho(t) = \begin{bmatrix}\rho_{\mathbf{AA}}(t) & \rho_{\mathbf{A}0}(t)\\ \rho_{0\mathbf{A}}(t) & \rho_{00}(t)\end{bmatrix}, \tag{24}$$

we split Eq. (22) into two equations:

$$\dot{\rho}_{\mathbf{AA}} = -i(\tilde{\mathbf{A}}(t)\rho_{\mathbf{AA}}(t) - \rho_{\mathbf{AA}}(t)\tilde{\mathbf{A}}^\dagger(t)), \tag{25a}$$
$$\dot{\rho}_{00} = \Gamma_c\mathbf{D}\rho_{\mathbf{AA}}(t)\mathbf{D}^\dagger, \tag{25b}$$

where $\mathbf{D} = [0,0,1]$ is a block from the matrix representation $\mathbf{c}$ of the annihilation operator c, $\tilde{\mathbf{A}}(t) = \mathbf{A}(t) - \frac{i}{2}\Gamma_c\mathbf{D}^\dagger\mathbf{D}$. Choosing the initial condition in $|g,0\rangle$ makes the dynamics not involving ρ_{A0} and Eq. (25a) corresponds thus to a Schrödinger equation with losses (i.e. with a non-Hermitian Hamiltonian), i.e. $\mathrm{Tr}\rho_{AA} < 1$:

$$i\frac{\partial}{\partial t}|\psi_A\rangle = \begin{bmatrix}0 & \Omega(t) & 0\\ \Omega(t) & \Delta & g\\ 0 & g & -i\frac{\Gamma_c}{2}\end{bmatrix}|\psi_A\rangle \tag{26}$$

with $|\psi_A\rangle = c_{g,0}|g,0\rangle + c_{e,0}|e,0\rangle + c_{f,1}|f,1\rangle$. The population lost from the subspace spanned by the states $\{|g,0\rangle,|e,0\rangle,|f,1\rangle\}$ (on which the block $\mathbf{A}$ is defined) is collected in state $|f,0\rangle$ (on which the block $\{0\}$ is defined), so that the whole system is closed: $P_{g,0}(t) + P_{e,0}(t) + P_{f,1}(t) + P_{f,0}(t) = 1$ with the population $P_{i,n}(t) = \langle i,n|\rho(t)|i,n\rangle = |c_{i,n}|^2$.

Rewriting (25b) we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{f,0}(t) = \Gamma_c P_{f,1}(t). \tag{27}$$

On the other hand, from the definition of the average $\langle\mathcal{O}\rangle = \mathrm{Tr}(\rho\mathcal{O})$, one can write the photon flux (17) in terms of the populations:

$$\Phi(t) \equiv \frac{dn}{dt}(t) = \Gamma_c P_{f,1}(t). \tag{28}$$

We can then identify $P_{f,0}(t)$ as the number of the outgoing photons: $P_{f,0}(t) \equiv n(t)$. The scheme enables us to derive the shape of the leaking photon, through its flux $\Phi(t)$ from the atom-cavity dynamics, which is determined by the Schrödinger equation (26).

B. The scheme for a large detuning and a bad cavity

The direct control of production of the shape of a single leaking photon can be achieved for a large detuning $\Delta \gg \Omega, g$ (allowing the adiabatic elimination of the excited state $|e,0\rangle$ [35]) and a bad cavity regime: $\Gamma_c \gg G, g^2/\Delta$ with $G = -g\Omega/\Delta$ the (assumed positive) effective Raman coupling (allowing the adiabatic elimination of the state $|f,1\rangle$). A discussion about the characteristic atomic and cavity rates can be for instance found in Ref. [28]. In particular, g and Γ_c can be modified through the length L of the cavity for a given transmission $\underline{\mathcal{T}}(\omega_c)$ of the lossy mirror: $\Gamma_c = c\underline{\mathcal{T}}(\omega_c)/L$.

The adiabatic eliminations lead to:

$$c_{g,0}(t) = \mathrm{e}^{i\zeta(t)}\mathrm{e}^{-\frac{\theta(t)}{2}}, \tag{29a}$$

$$\zeta(t) = \int_{t_i}^{t} dt' \frac{\Omega^2(t')}{\Delta}, \tag{29b}$$

$$\theta(t) = \int_{t_i}^{t} dt' \frac{4G^2(t')}{\Gamma_c}. \tag{29c}$$

We denote the initial time t_i. From $c_{g,0}(t)$, i.e. for given g, Δ, and $\Omega(t)$, one can infer $c_{f,1}(t) = -i2(G(t)/\Gamma_c)c_{g,0}(t)$ and Eq. (28) then gives the shape of the photon flux:

$$\Phi(t) = \dot{\theta}(t)\mathrm{e}^{-\theta(t)}. \tag{30}$$

The inverse calculation allows one to tailor a desired photon flux by deriving explicitly the corresponding $\Omega(t)$ (for given g and Δ). This is achieved by determining $\theta(t)$ from (30):

$$\theta(t) = -\ln\left[1 - \int_{t_i}^{t} dt'\Phi(t')\right]. \tag{31}$$

We get the simple expression for the Rabi frequency by deriving this latter equation and from (29c):

$$\Omega(t) = \frac{\Delta\sqrt{\Gamma_c}}{2g}\sqrt{\frac{\Phi(t)}{1 - \int_0^t dt'\Phi(t')}}. \tag{32}$$

We remark that this definition of the Rabi frequency can diverge at large time. To prevent it, we introduce an efficiency parameter $\eta < 1$ which will ensure that $\Omega(t \to +\infty) = 0$ when $\Phi(t \to +\infty) = 0$ [18].

Numerical results for a chosen Gaussian probability for the single photon shape

$$\Phi(t) = \frac{\eta}{T\sqrt{\pi}}\,\mathrm{e}^{-\left(\frac{t}{T}\right)^2}, \quad \int_{-\infty}^{+\infty}\Phi(t)dt = \eta, \tag{33}$$

of width T are shown in Fig. 4a. Using $\Gamma_c = 50/T$, we obtain $\max_t G(t) \approx 5.5/T \ll \Gamma_c$. We have also checked numerically the resulting flux by determining it from the numerical solution of the Schrödinger equation (26) (without considering the adiabatic elimination) with the Rabi frequency (32). The derived photon flux closely follows the desired shape as expected.

Other more complex forms can be investigated through (32) such as the ones obtained by the resonant process with flying atoms in [18].

Figure 4b shows a different situation with a cavity of better effective quality: $\Gamma_c = 5/T$ and $\max_t G(t) = 6.25/T \approx \Gamma_c$, where the second adiabatic elimination cannot be made. In this case, the leakage of the photon occurs earlier and faster due to the earlier peak of the coupling. The better quality of the cavity leads to a deformation of the tail of the photonic shape.

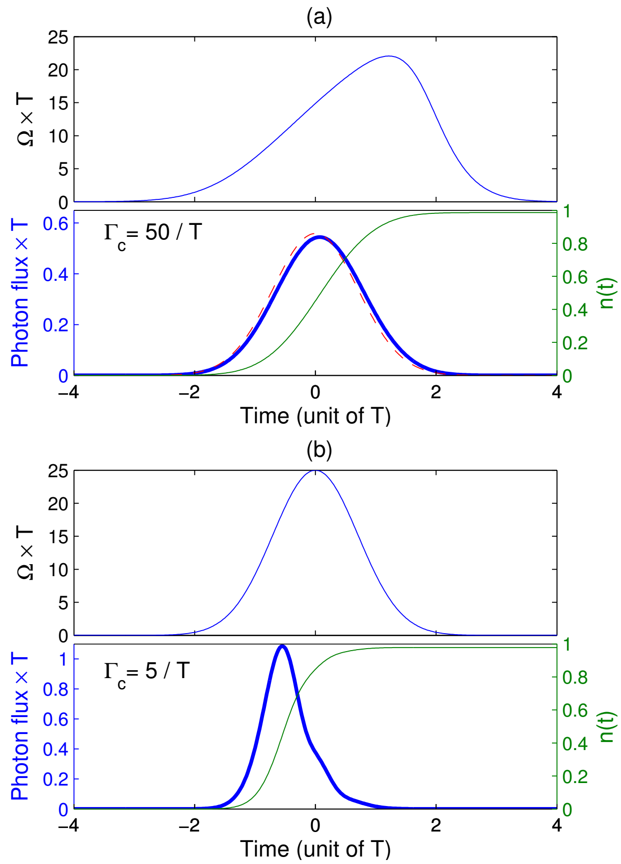

FIG. 4. (a) Rabi frequency $\Omega(t)T$ (32) with $(g, \Gamma_c, \Delta) \times T = (25, 50, 100)$, $\eta = 0.99$, determined from the desired Gaussian shape flux $\Phi(t)$ (33) [desired (dashed line) and numerical from the original model (26) (thick line)] of the single photon through the semi-transparent mirror (in units of T); number of outgoing photons $n = \int_{-\infty}^{t} dt'\Phi(t') = \Gamma_c \int_{-\infty}^{t} dt' |c_{f,1}(t')|^2$ during the process (thin line). (b) Same as above but for $\Gamma_c = 5/T$ and a chosen Gaussian Rabi frequency $\Omega(t) = 25\exp[-t^2/T^2]/T$.

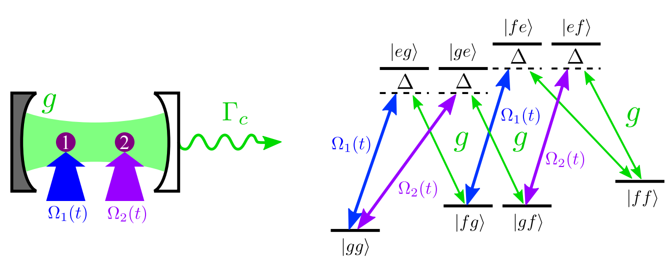

FIG. 5. 2-atom-cavity system. States $|i\rangle \otimes |j\rangle \equiv |ij\rangle$, $i, j = g, e, f$ describe Λ-atoms 1 and 2. Rabi frequencies Ω_k, $k = 1, 2$ drive the transitions $|g\rangle \to |e\rangle$.

IV. PRODUCTION OF A TWO-PHOTON STATE BY TWO DRIVEN ATOMS TRAPPED IN CAVITY

The generation of a N-photon state has been investigated using, for instance, the Zeeman sublevels of a single alkali atom [21]. Here we determine the property of the derived two-photon state when two driven Λ-atoms are in a cavity.

A. The model and the scheme

We consider the system shown in Fig. 5: We assume that each atom can be driven independently by two Rabi frequencies Ω_1 and Ω_2. The atom-cavity coupling g for the transition $e \leftrightarrow f$ allows the production of photons in the cavity mode leaking outside with the rate Γ_c. The Hamiltonian of the system is given by (3) for $N = 2$. We proceed as for the case of one atom and consider a large detuning ($\Delta \gg \Omega_i, g$). Stark shifts proportional to Ω_i^2/Δ and g^2/Δ appear from the elimination of the excited states, but the second condition of leaking cavity $\Gamma_c \gg G_i$ make them negligible in the dynamics, as they are in the same order of magnitude than the effective Raman couplings $G_i = g\Omega_i/\Delta$. The effective Hamiltonian in the dressed basis $\{|1\rangle \equiv |gg,0\rangle, |2\rangle \equiv |fg,1\rangle, |3\rangle \equiv |gf,1\rangle, |4\rangle \equiv |ff,2\rangle, |5\rangle \equiv |fg,0\rangle, |6\rangle \equiv |gf,0\rangle, |7\rangle \equiv |ff,1\rangle, |8\rangle \equiv |ff,0\rangle\}$ (we have relabeled the basis elements for convenience) writes:

$$H_{S,A.E.}(t) = \begin{bmatrix} \mathbf{A} & [0]_{4\times 3} & [0]_{4\times 1} \\ [0]_{3\times 4} & \mathbf{B} & [0]_{3\times 1} \\ [0]_{1\times 4} & [0]_{1\times 3} & 0 \end{bmatrix}, \tag{34a}$$

$$\mathbf{A} = \begin{bmatrix} 0 & G_1 & G_2 & 0 \\ G_1 & 0 & 0 & G_2 \\ G_2 & 0 & 0 & G_1 \\ 0 & G_2 & G_1 & 0 \end{bmatrix}, \tag{34b}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & G_2 \\ 0 & 0 & G_1 \\ G_2 & G_1 & 0 \end{bmatrix}. \tag{34c}$$

It features three blocks $\mathbf{A}$, $\mathbf{B}$ and $\{0\}$. Due to the strong cavity leakage, the dynamics flows from block to block, as in the preceding case starting from the initial condition $|\psi_i\rangle = |gg,0\rangle$ in the block $\mathbf{A}$. The corresponding dynamics is schematically depicted in Fig. 6. The dynamics is given by the Lindblad effective equation (22), which can be reformulated with a non-Hermitian Schrödinger equation for the block $\mathbf{A}$:

$$i\frac{\partial}{\partial t}|\psi_A\rangle = \begin{bmatrix} 0 & G_1 & G_2 & 0 \\ G_1 & -i\frac{\Gamma_c}{2} & 0 & G_2 \\ G_2 & 0 & -i\frac{\Gamma_c}{2} & G_1 \\ 0 & G_2 & G_1 & -i\Gamma_c \end{bmatrix} |\psi_A\rangle. \tag{35}$$

In the limit of strong leakage $\Gamma_c \gg G_i$, one can solve this equation. The dynamics for the block $\mathbf{B}$ features a Lindblad equation with a probability source

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\mathbf{BB}} = -i(\tilde{\mathbf{B}}\rho_{\mathbf{BB}} - \rho_{\mathbf{BB}}\tilde{\mathbf{B}}^\dagger) + \Gamma_c \mathbf{C}\rho_{\mathbf{AA}}\mathbf{C}^\dagger \tag{36}$$

with

$$\tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 & G_2 \\ 0 & 0 & G_1 \\ G_2 & G_1 & -i\frac{\Gamma_c}{2} \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \tag{37}$$

which becomes in the Redfield representation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\rho}_{\mathbf{BB}} = -i(\tilde{\mathbf{B}} \otimes \mathbb{1}_{\mathbf{B}} - \mathbb{1}_{\mathbf{B}} \otimes \tilde{\mathbf{B}}^\dagger)\vec{\rho}_{\mathbf{BB}} + \Gamma_c \vec{Y}, \tag{38}$$

that is of the form $\dot{X}(t) - M(t)X(t) = Y(t)$, where $\vec{\rho}_{\mathbf{BB}} = [\rho_{44}, \rho_{45}, \rho_{46}, \rho_{54}, \rho_{55}, \rho_{56}, \rho_{64}, \rho_{65}, \rho_{66}]^t$ corresponds to the column form of the density matrix $\rho_{\mathbf{BB}}$ associated to the block $\mathbf{B}$, and $\vec{Y}$ is the Redfield representation of the source term.

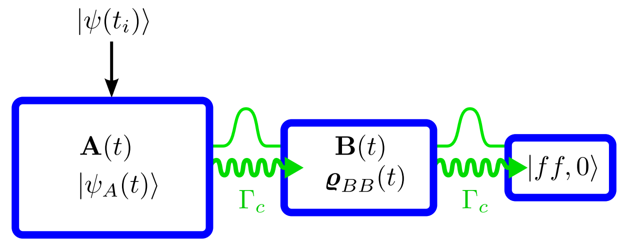

FIG. 6. Dynamical map of a two-atom system driven by two laser fields and trapped in a cavity. The dynamics splits into 3 blocks [from left to right, $\mathbf{A}$, $\mathbf{B}$, and $\{0\}$, see Eq. (34)] connected by the cavity decay rate $\Gamma_c = 2\pi\kappa^2$.

The outgoing photon flux reads

$$\Phi(t) = \Gamma_c(P_{fg;gf,1}(t) + 2P_{ff,2}(t) + P_{ff,1}(t)) \tag{39a}$$
$$= \Phi_{fg;gf,1}(t) + \Phi_{ff,2}(t) + \Phi_{ff,1}(t), \tag{39b}$$

where, here, $P_{ff,2}$ can be neglected as $\Gamma_c \gg G_i$ and the term $P_{fg;gf,1}(t) = P_{fg,1} + P_{gf,1}$ describes the emission of a single photon. One finds thus that the photon flux is a sum of partial photon fluxes: $\Phi(t) \approx \Phi_{fg;gf,1}(t) + \Phi_{ff,1}(t)$, each one corresponding to the production of a single photon.

B. Numerics

Figures 7 and 8 show the photon fluxes, for Gaussian pulse shapes of peak amplitude Ω_0: $\Omega_1(t) = \Omega_0 \exp[-(t - t_0 + \tau)^2/T^2]$, $\Omega_2(t) = \Omega_0 \exp[-(t - t_0 - \tau)^2/T^2]$, determined numerically for the following two respective cases: (i) sequence of laser pulses (the laser 1 is switched on before the laser 2), and (ii) simultaneous laser pulses. They confirm that $P_{ff,2}$ is negligible. In the first case, we obtain $\Phi_{fg;gf,1}(t) \equiv \Phi_{fg,1}(t)$ and the photons are produced one by one with the time delay as the delay between the laser pulses. Numerical results of Fig. 8 show that the partial photon fluxes overlap, but not fully: The photons are not generated separately. The resulting multiphotonic state is then not a Fock state as defined in [36, 37].

C. Characterization of the photonic state

1. *Multi-mode representation*

According to Ref. [37], general one and two-photon state $|1_\phi\rangle, |2_\Psi\rangle$ can be fully characterized from the knowledge of a function $\phi(\omega)$ for the single photon and a two-variable function $\Psi(\omega_1, \omega_2)$, both defined in the frequency

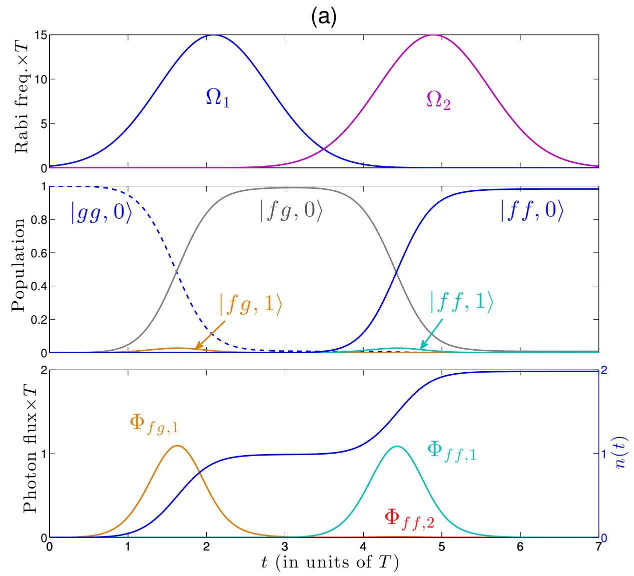

FIG. 7. (Upper panel) Rabi frequencies (in units of T) of delay $2\tau = 2.8T$. (Middle panel) Populations in the dressed basis for $(\Omega_0, \Delta, g, \Gamma_c) \times T = (15, 100, 40, 40)$. (Lower panel) Outgoing photon flux $\Phi = \Phi_{fg,1} + \Phi_{ff,1} + \Phi_{ff,2}$ (in units of T) and outgoing photon number (dark blue).

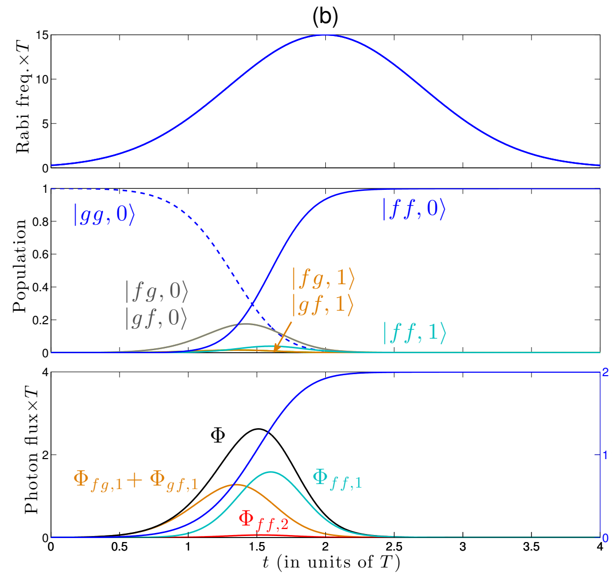

FIG. 8. Same as Fig. 7 but for $\Omega_1(t) = \Omega_2(t)$.

domain:

$$|1_\phi\rangle = \hat{a}_\phi^\dagger|\text{vac}\rangle, \quad \hat{a}_\phi^\dagger := \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \phi(\omega)\hat{b}^\dagger(\omega), \tag{40a}$$

$$|2_\Psi\rangle = \frac{1}{\mathcal{N}_2}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1\mathrm{d}\omega_2\Psi(\omega_1,\omega_2)\hat{b}^\dagger(\omega_1)\hat{b}^\dagger(\omega_2)|\text{vac}\rangle, \tag{40b}$$

where $\mathcal{N}_2$ is a normalization factor, and $\hat{b}^\dagger(\omega)$ is a creation operator for a photon in the vacuum, outside of the cavity. In the time domain, the same states write equivalently:

$$|1_\phi\rangle = \hat{a}_\phi^\dagger|\text{vac}\rangle, \quad \hat{a}_\phi^\dagger := \int_{-\infty}^{+\infty} \mathrm{d}t\, \tilde{\phi}(t)\hat{b}^\dagger(t), \tag{41a}$$

$$|2_\Psi\rangle = \frac{1}{\mathcal{N}_2}\int_{-\infty}^{+\infty} \mathrm{d}t_1\mathrm{d}t_2\tilde{\Psi}(t_1,t_2)\hat{b}^\dagger(t_1)\hat{b}^\dagger(t_2)|\text{vac}\rangle. \tag{41b}$$

where we introduce the one and two-time Fourier transforms of $\phi(\omega), \Psi(\omega_1,\omega_2)$, respectively:

$$\tilde{\phi}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega\phi(\omega)\mathrm{e}^{-i\omega t}, \tag{42a}$$

$$\tilde{\Psi}(t_1,t_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1\mathrm{d}\omega_2\Psi(\omega_1,\omega_2)\mathrm{e}^{-i(\omega_1 t_1+\omega_2 t_2)}, \tag{42b}$$

and, considering the previous functions to be square-integrable and normalized, the two-photon normalization factor is shown to be (for the time domain):

$$\mathcal{N}_2 = 1 + \int_{-\infty}^{+\infty} \mathrm{d}t_1\mathrm{d}t_2\tilde{\Psi}(t_1,t_2)\tilde{\Psi}^*(t_2,t_1). \tag{43}$$

In the following, we make the connection between $\tilde{\Psi}(t_1,t_2)$ and the photon flux in the vacuum:

$$\Phi(t) := \left\langle \hat{b}^\dagger(t)\hat{b}(t)\right\rangle = \langle 2_\Psi|\, \hat{b}^\dagger(t)\hat{b}(t)|2_\Psi\rangle. \tag{44}$$

Using equation (42b) with the expression of the flux, we show that it splits into a sum of two partial fluxes of the form:

$$\Phi(t) = \Phi_1(t) + \Phi_2(t), \tag{45a}$$

$$\Phi_1(t) = \frac{1}{\mathcal{N}_2}\int_{-\infty}^{+\infty} \mathrm{d}t' \left(\tilde{\Psi}^*(t,t') + \tilde{\Psi}^*(t',t)\right)\tilde{\Psi}(t,t'), \tag{45b}$$

$$\Phi_2(t) = \frac{1}{\mathcal{N}_2}\int_{-\infty}^{+\infty} \mathrm{d}t' \left(\tilde{\Psi}^*(t,t') + \tilde{\Psi}^*(t',t)\right)\tilde{\Psi}(t',t). \tag{45c}$$

The expression for the photon flux derived here can be used to recover the photon number, by integration over time t. We see from the latter expression that the integrated partial fluxes both provide a single photon number, that is:

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, \Phi_{1,2}(t) = 1, \tag{46}$$

which naturally brings a two-photon number for the total flux. However, we have to pay attention to the meaning of the partial fluxes $\Phi_{1,2}$. Their time integral being one does not mean that they carry one single photon, whose general state representation is given by equation (41a).

Well-separated single photon fluxes

The flux of a single photon is given, using the commutation relation $\left[\hat{b}(t), \hat{b}^\dagger(t')\right] = \delta(t-t')$ and the temporal function $\tilde{\phi}(t)$:

$$\Phi_{\rm sp}(t) = \langle 1_\phi|\, \hat{b}^\dagger(t)\hat{b}(t)|1_\phi\rangle = |\tilde{\phi}(t)|^2. \tag{47}$$

If two single photons are emitted with a time delay τ such that $\tau \gg T_{\rm sp}$ where $T_{\rm sp}$ is a characteristic pulse width for a single photon, then the two photon state function writes:

$$\tilde{\Psi}(t_1,t_2) = \tilde{\phi}_1(t_1)\tilde{\phi}_2(t_2), \tag{48}$$

where $\tilde{\phi}_{1,2}(t)$ are the temporal functions of the first and second single photons, respectively. Those two functions respect $\tilde{\phi}_1(t)\tilde{\phi}_2(t) = 0$ for all t, as the single photons are well separated. As a consequence, we have $\mathcal{N}_2 = 1$ and the partial photon fluxes (45) become simply:

$$\Phi_{1,2}(t) = |\tilde{\phi}_{1,2}(t)|^2. \tag{49}$$

As a consequence, the state (41b) writes as two orthogonal single photon states:

$$|2_\Psi\rangle \equiv |1_{\phi_1}\rangle|1_{\phi_2}\rangle, \tag{50a}$$

$$\langle 1_{\phi_1}|1_{\phi_2}\rangle = \int_{-\infty}^{+\infty} \mathrm{d}t\, \tilde{\phi}_1^*(t)\tilde{\phi}_2(t) = 0. \tag{50b}$$

General two-photon Fock state

A Fock state with two photons has a temporal function which must be factorizable into two identical functions:

$$\tilde{\Psi}_{2F}(t_1,t_2) = \tilde{\phi}(t_1)\tilde{\phi}(t_2), \tag{51}$$

such that the general two-photon state (41b) can take the form:

$$|2_\phi\rangle = \frac{\left(\hat{a}_\phi^\dagger\right)^2}{\sqrt{2!}}|\text{vac}\rangle. \tag{52}$$

The criteria on producing a two-photon Fock state is then to have the partial photon fluxes (45) overlapping completely:

$$\Phi_1(t) = \Phi_2(t) = |\tilde{\phi}(t)|^2. \tag{53}$$

Outgoing two-photon state with two atoms in a cavity

We analyze the results showed in fig. 7,8: for the first case, we have two non-overlapping partial photon fluxes, each carrying one single photon. The outgoing photon state is then $|1_{\phi_1}\rangle|1_{\phi_2}\rangle$, where:

$$\tilde{\phi}_1(t) \equiv \tilde{\phi}(t), \tag{54}$$

$$\tilde{\phi}_2(t) = \tilde{\phi}(t+\tau_L), \tag{55}$$

τ_L being the delay between the two single photons, corresponding to the delay between the laser pulses. The wavefunction of this state can be fully determined from the partial fluxes:

$$|\tilde{\Psi}^{(0)}(t_1,t_2)| = \sqrt{\mathcal{N}_2}\sqrt{\Phi_1^{(0)}(t_1)\Phi_2^{(0)}(t_2)}, \tag{56}$$

where we have labelled the wavefunction and the partial fluxes with a superscript (0) to specify that they don't overlap.

We consider the intermediate situation of Fig. 8 with partially overlapping fluxes. We determine $\tilde{\Psi}(t_1,t_2)$ using the following procedure: We assume the form

$$\tilde{\Psi}(t_1,t_2) = \sqrt{\mathcal{N}_2}\sqrt{\Phi_1(t_1)\Phi_2(t_2)} \tag{57}$$

where

$$\Phi_i\Big(\frac{t}{T_i}\Big) \approx \frac{T_i^{(0)}}{T_i}\;\Phi_i^{(0)}\Big(\frac{t+\tau_i}{T_i^{(0)}}\Big), \quad i=1,2 \tag{58}$$

with $\Phi_1 \equiv \Phi_{gf;fg,1}$, $\Phi_2 \equiv \Phi_{ff,1}$ taken from Fig. 8 and $\Phi_1^{(0)} \equiv \Phi_{fg,1}$, $\Phi_2^{(0)} \equiv \Phi_{ff,1}$ from Fig. 7. The coefficients $T_i^{(0)}$, T_i and τ_i are adapted to satisfy at best (58).

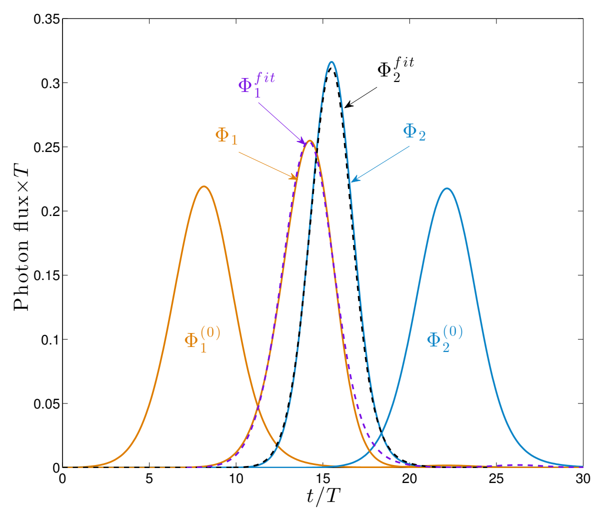

FIG. 9. Photon flux fit (dashed lines) of the partially overlapping photon fluxes ($\Phi_1 \equiv \Phi_{fg;gf,1}, \Phi_2 \equiv \Phi_{ff,1}$ in Fig. 8), using non overlapping flux shapes ($\Phi_1^{(0)} \equiv \Phi_{gf,1}, \Phi_2^{(0)} \equiv \Phi_{ff,1}$ in Fig. 7).

The result is shown in Fig. 9. We can observe very close shapes between the exact and fitted ones. This allows the characterization with a good accuracy of the two-photon state of Fig. 8 by a state of the form (41b) with (57).

2. Second-order correlation function

We study the behavior of the unnormalized second-order correlation function $G^{(2)}(t,\tau)$ associated with the outgoing field of the cavity. Based on the results of Sec. II B 3, $G^{(2)}(t,\tau)$ is defined as:

$$G^{(2)}(t,\tau) = \left\langle c^\dagger(t)c^\dagger(t+\tau)c(t+\tau)c(t)\right\rangle . \tag{59}$$

The two-time second order correlation function is not defined in the Schrödinger picture, because of the two time arguments. We apply the quantum regression theorem to compute numerically this function [38]. Using the propagator $\hat{U}(t,t_0)$ of the total system *and* environment, and the Markov assumption, one finds:

$$\begin{aligned} G^{(2)}(t,\tau) &= \mathrm{Tr}_S\{\tilde{\Lambda}(t+\tau,t)\, c\rho(t)c^\dagger\} \\ &= \mathrm{Tr}_S\{c^\dagger c\, \Lambda(t+\tau,t)\} \end{aligned} \tag{60}$$

with $\Lambda(t+\tau,t)$ and $\tilde{\Lambda}(t+\tau,t)$ being defined as follows:

$$\Lambda(t+\tau,t) := \mathrm{Tr}_R\{\hat{U}(t+\tau,t)\, c\rho(t)c^\dagger\, \rho_R\, \hat{U}^\dagger(t+\tau,t)\} \tag{61}$$
$$\tilde{\Lambda}(t+\tau,t) := \mathrm{Tr}_R\{\hat{U}(t+\tau,t)\, c^\dagger c\, \hat{U}^\dagger(t+\tau,t)\, \rho_R\}. \tag{62}$$

We see from Eq. (60) that

$$\mathrm{Tr}_S\{\tilde{\Lambda}(t+\tau,t)\, c\rho(t)c^\dagger\} = \mathrm{Tr}_S\{c^\dagger c\, \Lambda(t+\tau,t)\}, \tag{63}$$

and this equality still stands if $c\rho(t)c^\dagger$ is replaced by $\rho(t)$, leading to

$$\mathrm{Tr}_S\{\tilde{\Lambda}(t+\tau,t)\, \rho(t)\} = \mathrm{Tr}_S\{c^\dagger c\, \rho(t+\tau)\} = \langle c^\dagger c\rangle(t+\tau). \tag{64}$$

The density operator obeys the master equation

$$\frac{\partial}{\partial\tau}\rho(t+\tau) = \mathcal{L}(t+\tau)\rho(t+\tau), \tag{65}$$

where $\mathcal{L}(t)\rho(t) = -i[H_S(t),\rho(t)] + \sqrt{\Gamma_c}(c\rho(t)c^\dagger - (1/2)\{c^\dagger c,\rho(t)\})$, and the solution of this equation reads

$$\rho(t+\tau) = V(t+\tau,t)\rho(t). \tag{66}$$

According to (61) and (65), the same equation applies to $\Lambda(t+\tau,t)$:

$$\frac{\partial}{\partial\tau}\Lambda(t+\tau,t) = \mathcal{L}(t+\tau)\Lambda(t+\tau,t), \tag{67}$$

leading to

$$\begin{aligned} \Lambda(t+\tau,t) &= V(t+\tau,t)\Lambda(t,t) \\ &= V(t+\tau,t)(c\rho(t)c^\dagger). \end{aligned} \tag{68}$$

Therefore, to determine $G^{(2)}(t,\tau)$, $c\rho(t)c^\dagger$ is propagated from time t to $t+\tau$, and we finally get

$$G^{(2)}(t,\tau) = \mathrm{Tr}_S\{c^\dagger c\, V(t+\tau,t)\,(c\rho(t)c^\dagger)\}. \tag{69}$$

We show the unnormalized two-time second order correlation function in Fig. 10. In this calculation, we chose a reference time t_{peak} corresponding to the peaked value of the total photon flux, and we propagated the solution of the master equation $\Lambda(t_{\mathrm{peak}}+\tau,t_{\mathrm{peak}})$ to get the results. The figure shows a small bump due to the coincidences at zero delays ($\tau=0$), indicating that the probability of a joint generation of two photons is higher than any other delayed generation of two single photons. However, regarding the sum over all possible delays, this probability of $\tau=0$ coincidence is very small.

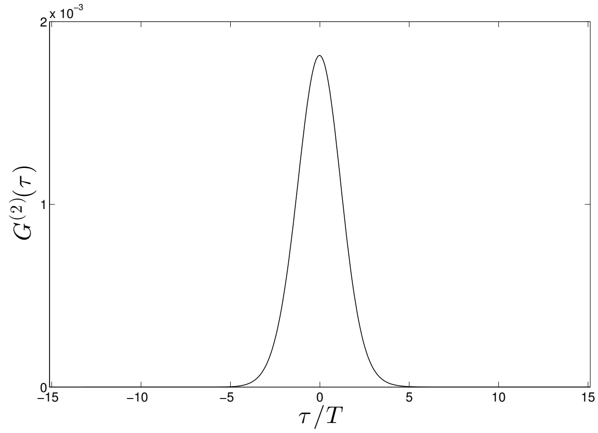

FIG. 10. Unnormalized two-time second order correlation function $G^{(2)}(t_{\mathrm{peak}},\tau) \equiv G^{(2)}(\tau)$, with respect to the reference time t_{peak} corresponding to the maximum of $\Phi(t)$.

V. CONCLUSION

In this paper, we have derived and analyzed models for a system of Λ-atoms trapped in a cavity QED, featuring a semi-transparent mirror, and driven by laser pulses. Concepts, such as Poynting vector, photon flux, input-output operators, photon state, that characterize the propagation of the resulting leaking photons, have been connected: We have formulated an input-output relation taking into account the propagating effects, which allows a direct interpretation of the b_{out} operator through the Poynting vector and the photon flux. The generated flux is then determined from the quantum average of the dynamics of the photon number in cavity, which results from a standard master equation that we have derived using the operators at $z=0$.

Two particular systems have been analyzed: A single atom or a two-atom system trapped in the cavity. In the case of a single atom, the master equation can be reformulated by a Schrödinger equation with a non-Hermitian Hamiltonian. For the problem with two driven atoms, the formulation leads to a Schrödinger equation with a non-Hermitian Hamiltonian whose probability is a source to a reduced Lindblad equation. We have considered the simplest situation with a large detuning and a bad cavity. In the case of a single trapped atom, one can directly link the envelop of the driving field to the pulse shape of the outgoing single photon which can be tailored at will. The use of two driven atoms allows the production of a propagating two-photon state. We have characterized such generated states using a second-order correlation function and a multi-mode representation. We have shown that, whatever the shape of the driving fields, the resulting two-photon outgoing photonic state cannot be a Fock state, since the two photons cannot be generated strictly simultaneously.

The production of multiphoton states may find applications for quantum algorithms processing and transmission of quantum information, as e.g. dense coding. In view of these applications, generating propagating multiphoton Fock states is of interest. We envision the simultaneous use of a ion trap and a cavity QED to achieve producing such states.

ACKNOWLEDGMENTS

We acknowledge support from the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No. 765075 (LIMQUET). Z.K. acknowledges the support of the Hungarian Research Fund (OTKA K83858).

[1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, UK, 2000); D. Bouwmeester, A. Ekert, A. Zeilinger, *The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation* (Springer-Verlag, Berlin, 2000).
[2] J.I. Cirac, P. Zoller, H.J. Kimble, and H. Mabuchi, Phys. Rev. Lett. 78, 3221 (1997); H.J. Kimble, Nature 453, 1023 (2008).
[3] B. Lounis, M. Orrit, Rep. Prog. Phys. **68**, 1129 (2005).
[4] C. Santori, D. Fattal, and Y. Yamamoto, Single-Photon Devices and Applications (Wiley-VCH, New York, 2010).
[5] I. Bialynicki-Birula in Optics XXXVI ed. E. Wolf (Elsevier, Amsterdam, 1996) p. 245.
[6] J.E. Sipe Phys. Rev. A **52** 1875 (1995).
[7] B.J. Smith and M.G. Raymer, New J. Phys. **9** 414 (2007).
[8] N. Gisin, G. Ribordy, W. Tittel and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002).
[9] N. Sangouard and H. Zbinden, J. Mod. Opt. **59**, 1458 (2012).
[10] X. Maître, E. Hagley, G. Nogues, C. Wunderlich, P. Goy, M. Brune, J.-M. Raimond, and S. Haroche, Phys. Rev. Lett. **79**, 769 (1997).
[11] A. Kuhn, M. Hennrich, and G. Rempe, Phys. Rev. Lett. **89**, 67901 (2002); J. McKeever, A. Boca, A. D. Boozer, R. Miller, J.R. Buck, A. Kuzmich, H.J. Kimble, Science **303**, 1992 (2004); T. Wilk, S. C. Webster, A. Kuhn, and G. Rempe, Science **317**, 488 (2007); S. Ritter, C. Nölleke, C. Hahn, A. Reiserer, A. Neuzner, M. Uphoff, M. Mücke, E. Figueroa, J. Bochmann, G. Rempe, Nature **484**, 195 (2012); M. Mücke, J. Bochmann, C. Hahn, A. Neuzner, C. Nölleke, A. Reiserer, G. Rempe, S. Ritter, Phys. Rev. A **87**, 063805 (2013).
[12] A.D. Boozer, A. Boca, R. Miller, T.E. Northup, and H.J. Kimble, Phys. Rev. Lett. **98**, 193601 (2007).
[13] A. Kuhn and D. Ljunggren, Contemp. Phys. **51**, 289 (2010); J. Dilley, P. Nisbet-Jones, B.W. Shore, and A. Kuhn, Phys. Rev. A **85**, 023834 (2012).
[14] A.V. Gorshkov, A. André, M.D. Lukin, and A.S. Sørensen, Phys. Rev. A **76**, 033804 (2007).
[15] K.S. Choi, H. Deng, J. Laurat, and H.J. Kimble, Nature **452**, 67 (2008).
[16] M. Keller, B. Lange, K. Hayasaka, W. Lange, and H. Walther, Nature **431**, 1075 (2004); N. Piro, F. Rohde,C. Schuck, M. Almendros, J. Huwer, J. Ghosh, A. Haase, M. Hennrich, F. Dubin, and J. Eschner, Nat. Phys. **7**, 17 (2010).
[17] W. Yao, R.-B. Liu, and L.J. Sham, Phys. Rev. Lett. **95**, 030504 (2005).
[18] G.S. Vasilev, D. Ljunggren, and A. Kuhn, New J. Phys. 12, 063024 (2010); P. Nisbet-Jones, J. Dilley, D. Ljunggren, and A. Kuhn, *ibid* 13, 103036 (2011).
[19] K.R. Brown, K.M. Dani, D.M. Stamper-Kurn, and K.B. Whaley, Phys. rev. A **67**, 043818 (2003).
[20] H. Eleuch, S. Guérin, and H.R. Jauslin, Phys. Rev. A **85**, 013830 (2012); M. Amniat-Talab, S. Lagrange, S. Guérin, and H. R. Jauslin, ibid. **70**, 013807 (2004).
[21] A. Gogyan, S. Guérin, C. Leroy and Yu. Malakyan, Phys. rev. A **86**, 063801 (2012).
[22] A. Gogyan, S. Guérin, H.-R. Jauslin, and Yu. Malakyan, Phys. Rev. A **82**, 023821 (2010).
[23] K.J. Blow, R. Loudon, S.J.D. Phoenix and T.J. Shepherd, Phys. Rev. A **42**, 4102 (1990).
[24] C.K. Law and H.J. Kimble, J. Mod. Opt. **44**, 2067 (1997).
[25] C.W. Gardiner and P. Zoller, *Quantum Noise*, 3rd edition (Springer-Verlag, Berlin, 2004).
[26] H.P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford university press, New York, 2002).
[27] K.M. Gheri, K. Ellinger, T. Pellizzari and P. Zoller, Fortschr. Phys. **46** 401-415 (1998).
[28] M. Keller, B. Lange, K. Hayasaka, W. Lange and H. Walther, New J. Phys. **6**, 95 (2004).
[29] L. Knöll, W. Vogel, and D.-G. Welsch, Phys. Rev. A **43**, 543 (1991).
[30] R. Loudon, *The quantum theory of light*, 3rd edition (Oxford University press, Oxford, 2000).
[31] G. Grynberg, A. Aspect, and C. Fabre, *Introduction to Quantum Optics: From the Semi-classical Approach to Quantized Light* (Cambridge University Press, 2010).
[32] W.H. Louisell, *Quantum statistical properties of radiation* (Wiley, 1973).
[33] H.J. Carmichael, *Statistical methods in quantum optics* (Springer-Verlag Berlin Heidelberg, 1999).
[34] A. Kuhn, M. Hennrich, T. Bondo and G. Rempe, Appl. Phys. B **69**, 373 (1999).
[35] B. W. Shore, *Manipulating quantum structures using laser pulses* (Cambridge University Press, UK, 2011).
[36] P.P. Rohde, W. Mauerer and C. Silberhorn, New J. Phys. **9**, 91 (2007).

[37] Z.Y. Ou, Phys. Rev. A **74**, 063808 (2006); arXiv:quant-ph/0601118v3 (2008).

[38] J. Keeling, *Light-matter interactions and quantum optics*, http://www.st-andrews.ac.uk/ jmjk/keeling/teaching/quantum-optics.pdf