Asymptotic Unknown Input Decoupling Observer for Discrete-Time LTI Systems

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***Abstract*—This paper addresses a new observer structure and its corresponding design for LTI discrete-time systems affected by Unknown Inputs (UI). We show that the performances of standard UIO can be enhanced, especially, in the presence of stable invariant zeros with slow dynamics (detectable systems). Under mild condition, the proposed UIO design avoids the drawback of the invariant zeros, at least in a time interval, and allows to arbitrary place the eigenvalues during that interval. The main idea is to relax the exact UI decoupling condition for an asymptotic decoupling condition. The convergence analysis is analyzed using the Lyapunov theory and the asymptotic convergence conditions of the state estimation error are expressed in LMI formalism.**

***Index Terms*—Unknown Input Observers, Asymptotic Unknown Input Decoupling, Linear Time Invariant (LTI) systems.**

I. INTRODUCTION

UNKNOWN Input Observers (UIO) have proved their effectiveness in observation, control and diagnosis of dynamical systems. Indeed, the decoupling property of the UIO provides a way to estimate the state of a system in the presence of Unknown Inputs (UI) representing faults, disturbances and modeling uncertainties [3], [6], [5], [9] for continuous-time systems and [2], [8] for discrete-time systems. It also provides a way to generate residual signals for fault isolation by decoupling a set of faults and rendering the residual signals sensitive only to another set of faults. In addition, this type of UIO allows to estimate the UI by system inversion with the estimated states.

For Linear discrete Time Invariant (LTI) systems, the UIO design has reached a certain maturity with necessary and sufficient existence conditions for UI decoupling and stabilization of the state estimation error. In addition, different design approaches have been reported in sequential way by using Linear Matrix Inequality (LMI) formalism. The decoupling condition is related to the cancellation of the UI from the state estimation error dynamics and generally expressed as a rank condition (well-known as a matching condition) [3]. The second condition ensuring the stability of the state estimation error dynamics is related to the observability or at least the detectability after decoupling [3]. An interesting extension has been reported in [4] which relaxes the matching condition (i.e. the rank condition is not satisfied). The result provides state and unknown Input estimation, however, the estimation is obtained after a finite delay with respect to the real states and unknown inputs. Concerning the second condition (Observability of at least detectability after UI decoupling), a geometric approach is provided in [7] which allows to design the UIO for all invariant zeros expect on the unit circle. This idea relaxes the detectability condition but the provided state and UI estimation is obtained after a certain delay.

After UI decoupling, if the system is observable, the dynamics of the state estimation error is constructed by arbitrary pole placement. If the observability condition is no longer satisfied (presence of invariant zeros), the system generating the state estimation error should have stable invariant zeros (detectable systems). However, in this second case, the eigenvalues of the dynamics of the state estimation error dynamics cannot be placed arbitrarily which affects the convergence rate of the state estimation error. Indeed, for detectable systems, the dynamics of the state estimation includes the stable invariant zeros which are not movable in the complex plan by output error feedback which requires their stability.

It may happen that the invariant zeros exhibit slow dynamics and then affect the convergence rate of the state estimation error. The estimated states are then obtained after a large settling time. In observer-based control systems, it is known that, unfortunatly, the observer should have a greater convergence rate than the controller in order to achieve better performances. In the field of observer-based fault diagnosis and fault tolerant control using UIO, the fault detection, isolation and estimation should be obtained quickly when the fault occurs in order to detect it and reconfigure the system if necessary. These reasons motivate the result of the present paper which provides a solution in order to enhance the performances of the UIO for LTI systems by introducing the Asymptotic Unknown Input Decoupling (AUID) notion instead of using the classical exact UI decoupling. This note presents a modification of the LTI UIO by introducing time-varying functions which enhance the convergence rate of the observer in the case of slow invariant zeros. The proposed UIO becomes then Linear Time Varying (LTV) UIO. The asymptotic convergence of the state estimation error to zero is analyzed using the Lyapunov theory and the established conditions are expressed in terms of Linear Matrix Inequalities that provide a way to design the observer gains. This UIO aims to move the invariant zeros, compared to the classical observer, in order to enhance the convergence rate. The proposed result is applicable for bounded UI and observable pair (C, A) of systems of the form (1). Notice that, the proposed approach provides an undelayed asymptotic state estimation.

The paper is organized as follows: Section II presents some preliminaries, definitions and states the problem. Section III provides the main result. Finally, section IV provides two examples illustrating the proposed approach.

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II. Preliminaries and notations

The systems under consideration are expressed in the following state space form

$$x_{k+1} = Ax_k + Bu_k + Ed_k, \quad y_k = Cx_k \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $d_k \in \mathbb{R}^{n_d}$ is the unknown input vector and $y_k \in \mathbb{R}^{n_y}$ is the measured output vector. The matrices A, B, E and C are real known constant matrices with appropriate dimensions. We assume that $n_y \geq n_d$. Without loss of generality, it is assumed that C is full row rank and E is full column rank.

Classically, a full order UIO has the form

$$z_{k+1} = Nz_k + Gu_k + Ly_k, \quad \hat{x}_k = z_k - My_k \tag{2}$$

presented in [3] (and references therein) where $z_k \in \mathbb{R}^n$ and the matrices N, G, L and M are constant and designed in order to ensure the asymptotic convergence of $\hat{x}$ to x.

Lemma 1: [3] The observer (2) for the system 1 exists if and only if the following conditions are satisfied

- $rank(CE) = rank(E) = n_d$
- the invariant zeros of the triplet (C, A, E) are stable

The first condition is necessary for UI decoupling while the second one is necessary for ensuring the stability of the matrix N of the observer (2). However, if the state estimation error dynamics contains invariant stable zeros, they cannot be moved in the complex plane by the gains of the observer. Then, the matrix N may have slow dynamics due to the slow invariant zeros of the system (1). This causes slow convergence of the state estimation error which affects the quality of state estimation. A class of used time-varying functions are defined in Definition 1.

Definition 1: (Time-Decreasing Sequence) A positive Time-Decreasing Sequence (TDS) is a monotonic function $f_k : \mathbb{N} \to \mathbb{R}_+$ satisfying $\lim_{k\to+\infty} f_k = 0$.

This work assumes that the pair (A, C) of the system (1) is observable and the UI d_k is bounded at each time by the positive constant σ. Without loss of generality but for more clarity and simplicity, the used function f_k (Definition 1) is set as $f_k = \rho\alpha^k$ where $\alpha \in]0, 1[$ and ρ is an arbitrary strictly positive constant.

III. Main result

The proposed observer has the following form

$$z_{k+1} \;=\; N_k z_k + G_k u_k + L_k y_k, \quad \hat{x}_k = z_k - H_k y_k \tag{3}$$

The matrices N_k, G_k, L_k and H_k are time-dependent matrices which will be defined later in order to ensure asymptotic convergence of the state estimation error $e_k = x_k - \hat{x}_k$.

The matrix H_k is chosen in such a way that when $k \to +\infty$, it converges to M and defined by

$$H_k = (1 - f_k)\, M = \left(1 - \rho\alpha^k\right) M \tag{4}$$

For simplicity we set $\rho = 1$, and the constant matrix M is defined as follows

$$M = -E\,(CE)^{\dagger} + Y\left(I_{n_y} - (CE)\,(CE)^{\dagger}\right) \tag{5}$$

where Y is a free matrix, with appropriate dimensions, to be selected and $X^{\dagger}$ is the pseudo-inverse of the matrix X. Notice that for $f_k = 0$, the matrix M is the exact decoupling one obtained in classical approach defined by (5).

The state estimation error is defined by the equation

$$\begin{aligned} e_k &= x_k - \hat{x}_k && (6)\\ &= \underbrace{(I + H_k C)}_{P_k} x_k - z_k && (7) \end{aligned}$$

its dynamics obeys to the following difference equation

$$\begin{aligned} e_{k+1} &= (P_{k+1}A - N_k P_k - L_k C)\, x_k + (P_{k+1}B - G_k)\, u_k \\ &+ P_{k+1}Ed_k + N_k e_k && (8) \end{aligned}$$

Under the conditions

1) $P_k = I + H_k C$
2) $P_{k+1}A - N_k P_k - L_k C = 0$
3) $P_{k+1}B - G_k = 0$

are satisfied $\forall k$, the state estimation error dynamics is reduced to

$$e_{k+1} = N_k e_k + S_k d_k \tag{9}$$

where $S_k = P_{k+1}E$ and $N_k = P_{k+1}A - K_k C$ and $K_k = L_k + N_k H_k$. Notice that even if N_k and S_k depend on the sample $k + 1$, the matrix P_{k+1} can be expressed, in explicit way, only using the current sample k thanks to the function α^k. It can be expressed as follows

$$\begin{aligned} P_{k+1} &= I + H_{k+1}C && (10)\\ &= I + MC - \alpha^{k+1}MC && (11) \end{aligned}$$

Notice also that, the construction of the matrices H_k and P_{k+1} ensures that $\lim_{k\to+\infty} S_k = 0$. Of course, the matrix M defined in equation (5) ensures that $(I + MC)E = 0$, then, since $0 < \alpha < 1$, the matrix S_k satisfies the convergence property

$$\begin{aligned} \lim_{k\to+\infty} S_k &= \lim_{k\to+\infty} (I + MC)E - \alpha^{k+1}MCE && (12)\\ &= \lim_{k\to+\infty} (-\alpha^{k+1}MCE) = 0 && (13) \end{aligned}$$

Furthermore, if (13) holds and under the assumption that d_k is bounded, the term $S_k d_k$ converges towards zero when $k \to +\infty$ which ensures the asymptotic UI decoupling i.e:

$$\lim_{k\to+\infty} S_k d_k = 0 \tag{14}$$

this comes from the known basic theorem which argues that the product of bounded numerical sequence times a zero-convergent sequence converges to zero when k goes to $+\infty$. On the other hand, the Euclidean norm of the matrix S_k can be expressed as follows

$$\|S_k\| = \alpha^{k+1}\,\|MCE\| \leq \alpha^{k+1}\lambda \tag{15}$$

where $\|MCE\| \leq \lambda$. Then, the norm of S_k converges also towards zero when $k \to +\infty$ $(0 < \alpha < 1)$.

Knowing $K_k = L_k + N_k H_k$, and from the conditions 1) and 2), N_k can be rewritten as

$$N_k = P_{k+1}A - K_k C \tag{16}$$

Then, the matrix N_k becomes $N_k = \bar{\mathcal{A}}_k - K_k C$, where $\bar{\mathcal{A}}_k = (I + MC - \alpha\alpha^k MC)A$.

In order to study the stability of the system generating the state estimation error (9), the polytopic form of the system is obtained since $0 < \alpha^k \leq 1$, which is expressed as follows

$$\bar{\mathcal{A}}_k = \sum_{i=1}^{2} h_i(k)\mathcal{A}_i, \quad S_k = \sum_{i=1}^{2} h_i(k)\mathcal{S}_i \tag{17}$$

where $h_{1k} = \alpha^k$ and $h_{2k} = (1 - \alpha^k)$, and

$$\mathcal{A}_1 = A + MCA - \alpha MCA, \quad \mathcal{A}_2 = A + MCA \tag{18}$$

$$\mathcal{S}_1 = -\alpha MCE, \quad \mathcal{S}_2 = 0 \tag{19}$$

The gain matrix K_k is expressed in the same form as follows

$$K_k = \sum_{i=1}^{2} h_{ik}(k)T_i \tag{20}$$

and T_i are to be determined. Then, the state estimation error can be expressed as follows

$$e_{k+1} = \sum_{i=1}^{2} h_{ik}((\mathcal{A}_i - T_i C)e_k + \mathcal{S}_i d_k) \tag{21}$$

The following theorem provides LMI conditions that ensure the asymptotic convergence of the state estimation error towards zero and the design of the matrices T_1 and T_2 of the gain matrix K_k.

Before introducing the main contribution of this paper, given in theorem 1, let us introduce the following functions

$$\eta_k = \eta\left(1 + \mu\alpha^k\right)$$

where $0 < \eta < \frac{1}{1+\mu}$ and $\mu > 0$, satisfying

$$0 < \eta_k < 1, \quad \forall k$$

This choice is considered for simplicity. However, a general formulation using the functions f_k introduced in Definition 1 can be considered i.e. $\eta_k = \eta\left(1 + \mu f_k\right)$. For the purpose of obtaining LMI conditions, since η_k depends on $f_k = \alpha^k$, the polytopic form of η_k is expressed, with the same function h_{ik}, by

$$\eta_k = \sum_{i=1}^{2} h_i(k)\tau_i \tag{22}$$

where $\tau_1 = \eta\left(1 + \mu\right)$ and $\tau_2 = \eta$.

Theorem 1: Given positive scalars μ and η such that $0 < \eta < \frac{1}{1+\mu}$. For $\tau_1 = \eta\left(1 + \mu\right)$ and $\tau_2 = \eta$, if there exist symmetric and positive definite matrices $X \in \mathbb{R}^{n\times n}$ and $G_i \in \mathbb{R}^{n\times n}$, gains matrices $\bar{T}_i \in \mathbb{R}^{n\times n_y}$ and a positive scalar γ such that the following LMIs hold (for $i = 1, 2$)

$$\begin{bmatrix} (\tau_i - 1)X & \mathcal{A}_i^T G_i^T - C^T \bar{T}_i^T & \Gamma_i \\ G_i\mathcal{A}_i - \bar{T}_i C & X - 2G_i & 0 \\ \Gamma_i^T & 0 & \Omega_i \end{bmatrix} \leq 0 \tag{23}$$

$$\Gamma_i = (\mathcal{A}_i^T G_i^T - C^T \bar{T}_i^T)S_i \tag{24}$$
$$\Omega_i = 2S_i^T G_i S_i - \gamma S_i^T S_i \tag{25}$$

then the state estimation error converges asymptotically towards zero. Then the equations $T_i = G_i^{-1}\bar{T}_i, i = 1, 2$ provide the gains of the observer.

Proof 1: The first part of the proof is similar to that proposed in [1]. Assume that the LMIs (23) are feasible, by multiplying the left side of (23) by

$$\mathcal{T}^T = \begin{bmatrix} e_k^T & \left(N_i e_k + S_i d_k\right)^T & d_k^T \end{bmatrix} \tag{26}$$

and the right side by $\mathcal{T}$, where $N_i = \mathcal{A}_i - K_i C$, one obtains

$$\begin{aligned} &\left(\tau_i - 1\right) e_k^T X e_k + \left(N_i e_k + S_i d_k\right)^T X \left(N_i e_k + S_i d_k\right) \\ - \quad &\gamma d_k^T S_i^T S_i d_k \leq 0 \end{aligned} \tag{27}$$

Using the Schur complement, it follows

$$\begin{bmatrix} \left(\tau_i - 1\right) e_k^T X e_k & \left(N_i e_k + S_i d_k\right)^T X & d_k^T S_i^T \\ X\left(N_i e_k + S_i d_k\right) & -X & 0 \\ S_i d_k & 0 & \frac{1}{\gamma} \end{bmatrix} \leq 0 \tag{28}$$

Multiplying by $h_{ik}i$, summing, and applying again the Schur complement on (28), we obtain

$$\begin{aligned} \left(N_k e + v_k\right)^T X \left(N_k e + v_k\right) - e_k^T X e_k \quad &\leq -\eta_k e_k^T X e_k \\ &+\gamma d_k^T S_k^T S_k d_k \end{aligned} \tag{29}$$

From the structure of the Lyapunov function given by $V = e_k^T X e_k$, the inequality (29) is nothing else than

$$V\left(e_{k+1}\right) - V\left(e_k\right) \leq -\eta_k V\left(e_k\right) + \gamma d_k^T S_k^T S_k d_k \tag{30}$$

which can be bounded as follows (under the definition of S_k)

$$V\left(e_{k+1}\right) \leq (1 - \eta_k)V\left(e_k\right) + \gamma\lambda^2\sigma^2\alpha^{2k} \tag{31}$$

This inequality can be seen as a non-homogeneous linear time-varying difference inequality with vanishing input. Its solution is the sum of two terms: homogeneous and steady state solutions. The homogeneous solution is given by the solution of the homogeneous inequality $\tilde{V}\left(e_{k+1}\right) < (1 - \eta_k)\tilde{V}\left(e_k\right)$, which is given by $\tilde{V}_k \leq \prod_{i=0}^{k-1}\left(1 - \eta\left(1 + \mu\alpha^i\right)\right)\tilde{V}(e_0)$. Under the assumption that $0 < (1 - \eta_k) < 1, \forall k$, the right hand side of the inequality converges to zero then $\tilde{V}_k \to 0$, when $k \to +\infty$. The global solution of the inequality is then given by

$$V(e_k) \leq \Phi(k-1,0)V(e_0) + \gamma\lambda^2\sigma^2\sum_{l=0}^{k-1}\Phi(k, l+1)\alpha^{2l} \tag{32}$$

where $\Phi(k-1,0) = \prod_{i=0}^{k-1}\left(1 - \eta\left(1 + \mu\alpha^i\right)\right)$. It is easy to see that under the fact that $\lim_{k\to+\infty}\Phi(k,0) = 0$, the right hand side of the inequality converges to zero which ensures the convergence of $V(e_k)$ towards zero and then $e_k \to 0$ when $k \to +\infty$, which ends the proof.

In the classical UIO, the maximal convergence rate $1-\eta$ is constrained by the maximal zeros z_i of the system as follows

$$\max_{i=1,2}\left(|z_i|\right) < 1 - \eta < 1$$

By comparison with the proposed approach, since $1-\eta_k < 1-\eta$, the convergence of the state estimation error is faster, at least in the transient phase (i.e. in a time interval $k \in [0, n]$). This convergence rate can be enhanced by increasing the parameter μ in $\eta_k = \eta(1+\mu\alpha^k)$. This is ensured, in the interval $k \in [0, n]$, thanks to the observability of the pair $(\mathcal{A}_1, C)$. When $k \to +\infty$, $\eta_k \to \eta$, then some eigenvalues of the matrix N_k convergence to the zeros of the system, but at this time the state estimation error has already converged.

Remark 1: Notice that close to the time origin $k = 0$ i.e. $k \in [0, n]$, the unknown input is not decoupled but since it is bounded, the minimization of γ aims to minimize its effect and the maximization of τ_1 allows to increase its convergence rate and minimize the effect of UI. When, $k \to +\infty$, the UI is asymptotically decoupled then τ_2 can be small. From this point of view, the LMIs of Theorem 1 can be solved by minimizing γ for given values $\tau_1 = \eta(1+\mu)$ and $\tau_2 = \eta$ where $1 > \tau_1 > \tau_2 > 0$.

Remark 2: According to the weighting functions h_{ik}, at $k = 0$ the pair $(\mathcal{A}_1, C)$ should be observable in order to be able to control, arbitrarily, the convergence rate of the state estimation error. In addition, this fact ensures the possibility to minimize the parameter γ in order to minimize the effect of the UI in the transient phase where the UI is not yet decoupled. When k becomes sufficiently large, the state estimation error dynamics becomes $e_{k+1} = (\mathcal{A}_2 - T_2 C)e_k$, which is nothing else than the error dynamics obtained by the classical UIO and given by $e_{k+1} = ((I + MC)A - KC)e_k$. The value of η is constrained by the unmovable eigenvalues of the pair $(\mathcal{A}_2, C)$ which correspond to the invariant zeros of (A, C, E).

According to remark 2 and the fact that the matrices $\mathcal{S}_1 = -\alpha MCE$ and $\mathcal{S}_2 = 0$, Theorem 1 can be expressed, equivalently, as in the following Theorem 2.

Theorem 2: Given positive scalars μ and η such that $0 < \eta < \frac{1}{1+\mu}$. For $\tau_1 = \eta(1+\mu)$ and $\tau_2 = \eta$, if there exist symmetric and positive definite matrices $X \in \mathbb{R}^{n\times n}$ and $G_i \in \mathbb{R}^{n\times n}$, gains matrices $\bar{T}_i \in \mathbb{R}^{n\times n_y}$, $i = 1, 2$ and a positive scalar γ such that the following LMIs hold

$$\begin{bmatrix} (\tau_1 - 1)X & \mathcal{A}_1^T G_1^T - C^T \bar{T}_1^T & \Gamma_1 \\ G_1\mathcal{A}_1 - \bar{T}_1 C & X - 2G_1 & 0 \\ \Gamma_1^T & 0 & \Omega_1 \end{bmatrix} < 0 \quad (33)$$

$$\begin{bmatrix} (\tau_2 - 1)X & \mathcal{A}_2^T G_2^T - C^T \bar{T}_2^T \\ G_2\mathcal{A}_2 - \bar{T}_2 C & X - 2G_2 \end{bmatrix} < 0 \quad (34)$$

$$\begin{aligned} \Gamma_1 &= (\mathcal{A}_1^T G_1^T - C^T \bar{T}_1^T) S_1 && (35) \\ \Omega_1 &= 2S_1^T G_1 S_1 - \gamma S_1^T S_1 && (36) \end{aligned}$$

then the state estimation error converges asymptotically towards zero. Then the equations $T_i = G_i^{-1}\bar{T}_i, i = 1, 2$ provide the gains of the observer.

Proof 2: The proof is similar to the proof of Theorem 1. The LMI (34) is obtained from (23) for $i = 2$ by replacing the matrix $\mathcal{S}_2 = 0$, then one obtains

$$\begin{bmatrix} (\tau_2 - 1)X & \mathcal{A}_2^T G_2^T - C^T \bar{T}_2^T & 0 \\ G_2\mathcal{A}_2 - \bar{T}_2 C & X - 2G_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq 0 \quad (37)$$

which corresponds to inequality (34) after removing the last row and column.

As a conclusion, the design of the LTV observer (2) for the LTI system (1) is summarized in the following procedure:

1) Check the rank condition: $rank(CE) = rank(E)$ and observability of (A, C). If satisfied go to step 2.
2) Compute the matrix M:
$$M = -E\,(CE)^{\dagger} + Y\left(I_{n_y} - (CE)\,(CE)^{\dagger}\right) \quad (38)$$
where Y is an arbitrary matrix with appropriate dimension.
3) Fix the function f_k as in Definition 1. In this paper f_k is chosen by
$$f_k = \alpha^k$$
where $\alpha \in]0, 1[$.
4) Define the matrix H_k as follows
$$H_k = \left(1 - \alpha^k\right) M \quad (39)$$
5) Construct the function η_k as follows
$$\eta_k = \eta\left(1 + \mu\alpha^k\right)$$
where $0 < \eta < \frac{1}{1+\mu}$ and $\mu > 0$, satisfying
$$0 < \eta_k < 1, \quad ,\forall k$$
6) Fix the parameters $\tau_1 = \eta(1+\mu)$ and $\tau_2 = \eta$.
7) Compute the polytopic forms as in (17).
8) Solve Theorem 2 and obtain the gains T_1 and T_2 and the gain matrix K_k given by $K_k = \sum_{i=1}^{2} h_{ik} T_i$, where $h_{1k} = \alpha^k$ and $h_{2k} = (1 - \alpha^k)$.
9) Construct the time-varying matrices of the UIO (2) as follows:
$$\begin{aligned} H_k &= \left(1 - \alpha^k\right) M && (40) \\ N_k &= P_{k+1}A - K_k C && (41) \\ &= (I + MC - \alpha^{k+1}MC)A - K_k C && (42) \\ L_k &= K_k - N_k H_k && (43) \\ G_k &= (I + MC - \alpha^{k+1}MC)B && (44) \end{aligned}$$

IV. Simulation examples and discussions

A. Example 1

Consider the linear LTI system (1) defined by the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0.99 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$E = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

For simulations, the parameters are fixed as follows: $\alpha = 0.9$, $\eta = 0.01$, $\mu = 90$. These parameters ensure that $0 < \eta_k < 1$, $\forall k$. After solving the LMIs of Theorem 1, the obtained variables are

$$T_1 = \begin{bmatrix} 0.9020 \\ 0.4923 \end{bmatrix}, T_2 = \begin{bmatrix} -0.0001 \\ 0.4944 \end{bmatrix}$$

and $\gamma = 1.3984$. These results are obtained as a feasibility problem without minimizing γ. The other matrices of the UIO are computed according to the equations $K_k = \sum_{i=1}^{2} h_{ik} T_i$, $N_k = \mathcal{A}_k - K_k C$, $L_k = K_k - N_k H_k$ and $G_k = (I + MC - \alpha^{k+1} MC)B$. For comparison, the classical UIO is designed according to the result [2]. The linear gain K is obtained by solving the corresponding LMIs. The other matrices of the UIO are obtained by $N = (I+MC)A - KC$, $L = K - NM$ and $G = (I + MC)B$. The simulations are carried out with $u_k = \sin(k)$, $d_k = \sin(2k)\cos(3k)$ and the sampling time is fixed to $T_s = 0.1$. From the figure 1, it can be seen that

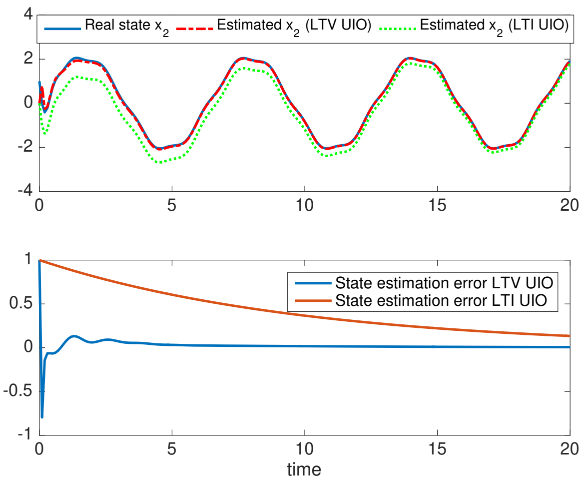

Fig. 1. Comparison of $x_2(k)$ estimations: Proposed LTV UIO and classical LTI UIO

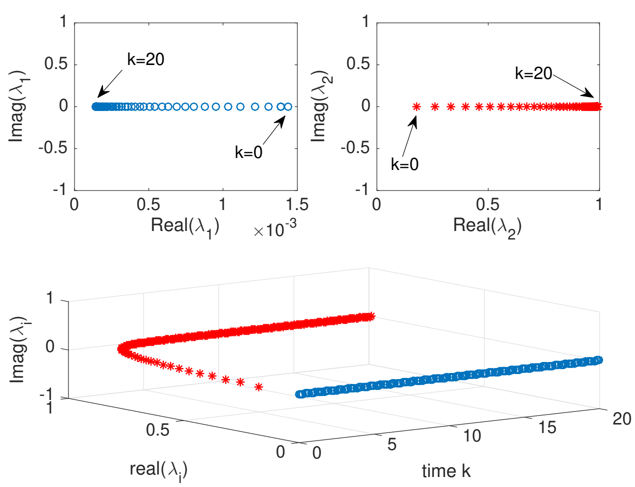

Fig. 2. Time evolution of the two eigenvalues of the matrix N_k

the proposed LTV UIO presents better performances than the classical LTI UIO. Since the UI is decoupled asymptotically, its effect is visible in the transient phase but vanishes as time increases, but the convergence rate is enhanced considerably. This is due to the fact that the proposed approach allows to avoid, at least in a short interval time, the restriction caused by the slow dynamics due to the zero $\lambda_2 = 0.99$. Figure 2 shows the time evolution of the two eigenvalues of the matrix N_k according to time k. It can be seen then $\lambda_2 = 0.99$ is moved to 0.14 at $k = 0$ and when $k \to +\infty$, this zero converges to λ_2. This explains the enhancement in the convergence rate of the state estimation error. In addition, the choice of the bounds η_1 and η_2 are directly related to the properties of the two vertexes i.e. since the pair $(\mathcal{A}_1, C)$ is observable, it is possible to place the poles arbitrarily, then a good choice for η_1 close to 1 in order to enhance the convergence rate. However, for the second vertex, the pair $(\mathcal{A}_2, C)$ is only detectable and presents a slow invariant zero dynamics, η_2 should be less or equal to the absolute value of the invariant zero.

B. Example 2

In this example, let us consider the third order system of the form (1) defined by the matrices

$$A = \begin{bmatrix} 0.5 & -0.3847 & 0.7036 \\ 0 & 0.7 & 0.5468 \\ 0 & 0 & -0.8 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.3518 & -0.2734 & 0.5 \end{bmatrix}$$

The pair (A, C) is observable and the invariant zeros of the system are located at $z_1 = 0.995$ and $z_2 = 0.999$. It can be seen that their dynamics are very slow. By applying, the proposed procedure, the matrix M is given by

$$M = \begin{bmatrix} 0 & 0 & -2 \end{bmatrix}^T$$

The function f_k is defined by $f_k = 0.993^k$ according to Definition 1. The parameter η should be chosen such that

$$\max_{i=1,2}(|z_i|) < 1 - \eta < 1$$

in order to ensure a solution for the LMI (34) . By solving the LMIs in Theorem 2 iteratively, the maximal value of μ is obtained satisfying the LMIs and ensuring that $\tau_1 = \eta(1+\mu)$ satisfies $0 < \tau_1 < 1$. For the present example, $\eta = 7 \times 10^{-4}$, the maximal value obtained for μ is $\mu = 15$. Consequently, the values of $\tau_1 = \eta(1+\mu) = 0.0112$ and $\tau_2 = \eta = 7 \times 10^{-4}$ are obtained. After solving the LMIs in Theorem 2, the following results are obtained:

$$T_1 = \begin{bmatrix} 1.4037 \\ -1.0108 \\ -0.6693 \end{bmatrix}, T_2 = \begin{bmatrix} 1.4037 \\ -1 \\ 0.4405 \end{bmatrix}$$

With these values the gains of the observer (2) are obtained according to the procedure given above. Indeed, according to the step 8), one obtains the gain matrix $K_k = \sum_{i=1}^{2} h_{ik} T_i$ and the other matrices are derived from equations (40)-(44). Figures 3 and 4 illustrate the state estimation and estimation errors with both the proposed approach and the classical approach. It can be seen that the performances of the LTV observer provide better results. Figure 5 illustrates the evolution of the eigenvalues of the matrix N_k with respect to time. It can be seen that the invariant zeros are moved in the complex plan in the transient phase which enhances the performances of the proposed UIO.

Let us focus on the function $f_k = \alpha^k$. In the previous simulations, the parameter α is fixed to 0.993. Some simulations

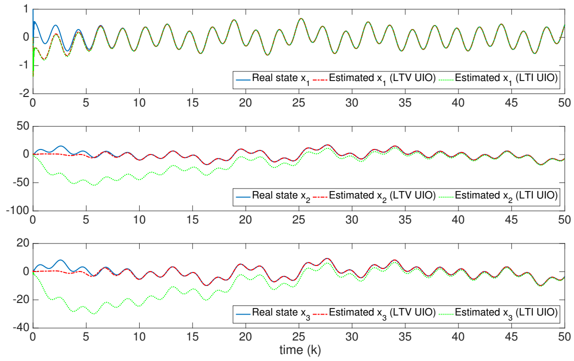

Fig. 3. Comparison of state estimations: Proposed LTV UIO and classical LTI UIO

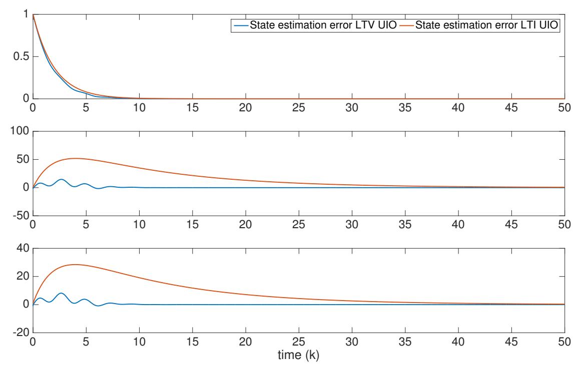

Fig. 4. Comparison of state estimation errors: Proposed LTV UIO and classical LTI UIO

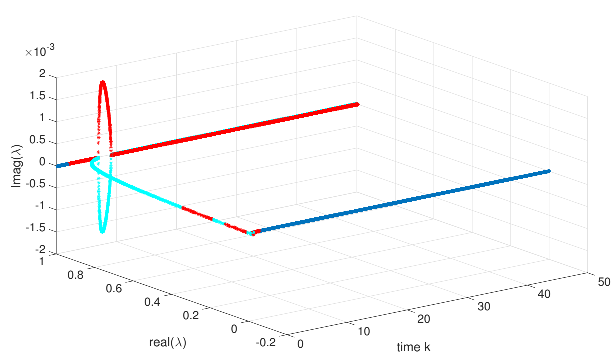

Fig. 5. Time evolution of the eigenvalues of the matrix N_k

are conducted for different values of α in order to show how this parameter affects the state estimation. For this purpose, let us take 4 different values of α: 0.9, 0.98, 0.993 and 0.99. Figure 6 illustrates the state estimation performances for each value of α. From the figure 6, it can be seen that if $\alpha = 0.9$, the UIO provides similar performances than the classical LTI UIO. When $\alpha = 0.99$, the estimation function $f_k = \alpha^k$ converges to zero slowly which affects the estimation since the unknown input is slowly decoupled. Then, the value $\alpha = 0.993$ provides the best result for this example. Consequently, α should be chosen as a compromise between convergence rate and UI decoupling rate.

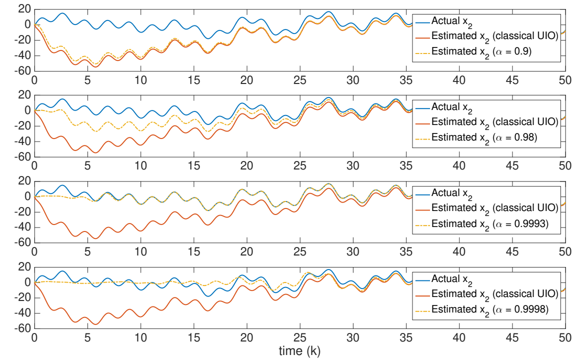

Fig. 6. Effect of the parameter α on the state estimation error convergence

V. Conclusion

This paper shows that it is possible to improve Unknown Input Observers (UIO) for LTI systems in the presence of slow dynamics invariant zeros. The approach consists in relaxing the exact decoupling condition by an asymptotic one under mild conditions: observable pair (A, C) and bounded unknown inputs. The approach is based on introducing time-varying power function which allows to enhance the convergence rate of the state estimation instead of undergoing the slow stable invariant zeros. Future work will concern a depth study of the different choices of the parameter η_k and the value of α.

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