Decoupling Unknown Input Observer for Nonlinear Quasi-LPV Systems

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Abstract— **In this paper, the problem of unknown input observer (UIO) design for nonlinear parameter varying (quasi-LPV) systems is investigated. Three main improvements of the existing UIO designs for LPV systems [11] are detailed. First, the parameter dependency of the UIO is not restricted to be the same as the one of the system, then the existing decoupling conditions are relaxed. Secondly, the class of considered systems is nonlinear which leads to the well-known quasi-LPV systems (i.e. the parameters are state dependent). This paper focuses on the case of parameters depending on unmeasured states. Finally, the proposed UIO considers the cases when only estimated time derivative of the parameters is available, and also unavailable time derivative and estimation. For these cases, the Disturbance-to-Error Stability (DES) is considered with DES-gain optimization. Examples are provided to illustrate the performances of the proposed UIO designs and highlight the improvements brought to existing ones.**

I. Introduction

Unknown Input observers have attracted a lot of attention due their central role in automatic control theory and applications. Indeed, model-based control or diagnosis need the knowledge of some state variables which are not always accessible for measure via physical sensors. In addition, systems are frequently affected to unknown inputs such as faults, disturbances, noises. These unknown inputs can also represent some modeling and parametric uncertainties, neglected dynamics, faults etc.

The research field of Unknown Input Observer design by decoupling is very active since forty years. These observers are based on some structural conditions that aim to decouple the UI from the state estimation error [4], [9], [12]. The interest of such a decoupling approach is that no assumption on the dynamics of the unknown input is needed.

The original linear UIO proposed in [4] has been extended to nonlinear polytopic systems (or Takagi-Sugeno (TS) systems [16]) by duplicating the polytopic structure of the system in the nonlinear UIO. Therefore, the structure of the UIO is fixed *a priori* and the LMI conditions have been obtained from the Lyapunov theory to ensure asymptotic convergence of the state estimation error towards zero [12], [3]. Despite the appealing simplicity of this approach, it was pointed out in our previous work [6] that the duplication of the polytopic structure of the system in the UIO reduces significantly the system class for which such an observer can be designed. For example, this UIO design may fail even if a nonlinear system is strongly algebraically observable (i.e. the state of a system affected by UI can be written using only the measured output, the known input and their time derivatives of finite orders [1]) or at least strongly detectable. In addition, all the published works on UIO design for TS or LPV systems consider linear time invariant output equations and the existing UIO designs cannot be applied to systems with LPV outputs since the UI decoupling would not be ensured anymore. In [11] a robust UIO design has been presented by considering that the first time derivatives of the parameters are available. In [13] the problem of state estimation of LPV systems with uncertain parameters is addressed in the context of bounded state estimation error by using Input-to-State Stability (ISS) concept. This result is extended for observer-based control in [5].

In this paper a new nonlinear LPV UIO is proposed, that does not need to share the same polytopic form than the one of the system (as in [11]). The class of systems under consideration are quasi-LPV systems i.e. with parameters depending on the state. Thus, in a general framework, the designed observer may depend (possibly nonlinearly) on unmeasured states. As it will be pointed out and illustrated, it allows to provide a solution to the UIO design for a larger class of systems by avoiding restrictive decoupling conditions (as those of [12], [3]). The approach is obtained by the proposition of a new, more flexible observer structure and by postponing the use of the polytopic transformation of a general quasi-LPV form into a polytopic or TS one. It results in more degrees of freedom in the UIO design, especially it avoids searching for a common solution to several equality constraints imposed by the decoupling between estimation and UI. Finally, the case of unavailable first time derivatives of the measured parameters is considered, by assuming that only an estimation of these derivatives are available. An extension to the case where the time derivatives and their estimations are unavailable is considered. The paper is organized as follows: Notations, assumptions and problem statement are provided in the section II. The UIO design ensuring a perfect decoupling of the state estimation from the UI and handling the nonlinear part is given in section III-B. Before concluding, some illustrative examples are provided in section IV.

II. Preliminaries and problem statement

This paper concerns nonlinear systems of the form

$$\begin{cases} \dot{x}(t) = A(\alpha(t))x(t) + \Upsilon(x(t), \alpha(t)) \\ \qquad + B(\alpha(t))u(t) + F(\alpha(t))d(t) \\ y(t) = C(\alpha(t))x(t) \end{cases} \tag{1}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$, $u(t) \in \mathcal{U} \subset \mathbb{R}^{n_u}$, $y(t) \in \mathcal{Y} \subset \mathbb{R}^{n_y}$ and $d(t) \in \mathcal{D} \subset \mathbb{R}^{n_d}$ represent, respectively, the state,

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the known input, the measured output and the unknown input vectors. The sets $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{D}$ are assumed to be bounded. $\alpha(t)$ is a nonlinear vector function depending only of measured variables such as $y(t)$, $u(t)$ or any external known or measured real-time signals. It is also assumed that $\alpha(t)$ is bounded ($\alpha(t) \in \Pi_\alpha$ where Π_α is a bounded set) and belongs, at least, to a class $\mathcal{C}^1$ and its time derivative is bounded and belongs to a bounded set $\Pi_{\dot{\alpha}}$. The function $\Upsilon(x,\alpha)$ is nonlinear and depend on unmeasured state variables (It represents the part of the nonlinear system that cannot be written in a quasi-LPV form depending on measured variables, i.e. the $A(\alpha(t))$ part. Therefore, it may depend on both $\alpha(t)$ and unmeasured state variables).

Notice that, the design of observers in TS or LPV polytopic forms begins by transforming the nonlinear system into polytopic form via the sector nonlinear transformation. Of course, there are other techniques aiming to approximate a nonlinear system by a TS model, such as identification or linearization around several operating points. However, the sector nonlinear transformation is more interesting since it provides an exact polytopic model, in a compact set of the state space [17], without loss of information compared to the approximation techniques. Generally, the observer is designed by using the polytopic model. In the proposed approach, the polytopic transformation is postponed at the end of the observer design (Postponing the polytopic transformation leads to avoid the conservatism related to the polytopic form as explained in our previous works [6] and [11]). In addition, the nonlinear term containing the quasi-LPV part with unmeasured state parameter dependent is handled in an efficient way in order to relax the constraint related to an admissible Lipschitz constant. The contributions of the paper are twofold:

1) Propose a novel UIO for an extended class of nonlinear systems compared to our previous work [11], [6], by taking into account nonlinear term $\Upsilon(x,\alpha)$.
2) Relax the assumption that the time derivative of $\alpha(t)$ is measured or estimated exactly. Indeed, in the present paper, the time derivative $\dot{\alpha}(t)$ is assumed to be unknown and only an estimation is available $\hat{\dot{\alpha}}(t)$. In addition, the result is extended to situations where neither $\dot{\alpha}(t)$, nor $\hat{\dot{\alpha}}(t)$ are available are available.

III. UNKNOWN INPUT OBSERVER AND LMI DESIGN

This section presents the structure of the UIO and the corresponding LMI design. Compared to our previous work [11] where the time derivative of $\alpha(t)$ is assumed available, the proposed UIO is fed by only an estimation of such a derivative. Consequently, since the time derivative of the $\alpha(t)$ is unavailable, the proposed UIO for the system (1) takes the form

$$\begin{cases} \dot{z}(t) = N(\alpha(t),\hat{\dot{\alpha}}(t))z(t) + S(\alpha(t))\Upsilon(\hat{x}(t),\alpha(t)) \\ \qquad + G(\alpha(t))u(t) + L(\alpha(t),\hat{\dot{\alpha}}(t))y(t) \\ \hat{x}(t) = z(t) - M(\alpha(t))y(t) \end{cases} \tag{2}$$

where the nonlinear matrices $N(.)$, $S(.)$, $L(.)$, $G(.)$ and $M(.)$. depend on the measured variable $\alpha(t)$ and the estimated first time derivative of $\hat{\dot{\alpha}}(t)$, and will be defined later. Let us denote the derivative estimation error given by

$$e_{\dot{\alpha}}(t) = \dot{\alpha}(t) - \hat{\dot{\alpha}}(t) \tag{3}$$

In this paper, we assume that this error is bounded (i.e. $\|e_{\dot{\alpha}}(t)\|_\infty \leq \sigma$ where $\sigma > 0$).

For clarity in the equations, in the remaining, the time dependence of $\alpha(t)$, $\dot{\alpha}(t)$ and $\hat{\dot{\alpha}}(t)$ is omitted.

A. State estimation error description

The state estimation error $e(t) = x(t) - \hat{x}(t)$ is given by

$$e(t) = P(\alpha)x(t) - z(t) \tag{4}$$

where $P(\alpha(t)) = I + M(\alpha)C(\alpha)$. By using the equation (1) and (2), the state estimation error dynamics is expressed as follows

$$\begin{aligned} \dot{e}(t) \;&=\; \left(\dot{P}(\alpha) + P(\alpha)A(\alpha) - L(\alpha,\hat{\dot{\alpha}})C(\alpha)\right)x(t) \\ &+\; P(\alpha)\Upsilon(x(t),\alpha) + (P(\alpha)B(\alpha) - G(\alpha))u(t) \\ &+\; P(\alpha)F(\alpha)d(t) - N(\alpha,\hat{\dot{\alpha}})z(t) - S(\alpha)\Upsilon(\hat{x}(t),\alpha) \end{aligned} \tag{5}$$

By selecting $S(\alpha) = P(\alpha)$, and using the fact that $z(t) = P(\alpha)x(t) - e(t)$ obtained from (4), the dynamics (5) can be written as follows

$$\begin{aligned} \dot{e}(t) \;&=\; (\dot{P}(\alpha) + P(\alpha)A(\alpha) - L(\alpha,\hat{\dot{\alpha}})C(\alpha) \\ &-\; N(\alpha,\hat{\dot{\alpha}})P(\alpha))x(t) + N(\alpha,\hat{\dot{\alpha}})e(t) \\ &+\; P(\alpha)\left(\Upsilon(x(t),\alpha) - \Upsilon(\hat{x}(t),\alpha)\right) \\ &+\; P(\alpha)F(\alpha)d(t) + (P(\alpha)B(\alpha) - G(\alpha))u(t) \end{aligned} \tag{6}$$

Lemma 1: If the UIO's matrices are selected in such a way to satisfy the following conditions:

C1. $\hat{\dot{P}}(\alpha) + P(\alpha)A(\alpha) - L(\alpha,\hat{\dot{\alpha}})C(\alpha) - N(\alpha,\hat{\dot{\alpha}})P(\alpha) = 0$
C2. $P(\alpha)F(\alpha) = 0$
C3. $P(\alpha)B(\alpha) - G(\alpha) = 0$

the state estimation error dynamics (6) become:

$$\begin{aligned} \dot{e}(t) \;&=\; N(\alpha,\hat{\dot{\alpha}})e(t) + (\dot{P}(\alpha) - \hat{\dot{P}}(\alpha))x(t) \\ &+\; P(\alpha)\left(\Upsilon(x(t),\alpha) - \Upsilon(\hat{x}(t),\alpha)\right) \end{aligned} \tag{7}$$

Proof: The condition (C2) of Lemma 1 leads to

$$M(\alpha)C(\alpha)F(\alpha) = -F(\alpha) \tag{8}$$

This equation admits a solution if and only if

$$rank(C(\alpha)F(\alpha)) = rank(F(\alpha)), \forall \alpha \in \Pi_\alpha \tag{9}$$

If this condition is satisfied, the matrix $M(\alpha)$ is computed as follows

$$M(\alpha) = -F(\alpha)\left(C(\alpha)F(\alpha)\right)^\dagger \tag{10}$$

where $X^\dagger$ is the pseudo-inverse of X. After computing the matrix $M(\alpha)$, the matrix $P(\alpha)$ is given by

$$P(\alpha(t)) = I + M(\alpha(t))C(\alpha(t)) \tag{11}$$

By using $P(\alpha)$, the condition (C1.) can be written as follows

$$N(\alpha,\hat{\dot{\alpha}}) = \hat{\dot{P}}(\alpha) + P(\alpha)A(\alpha) - K(\alpha,\hat{\dot{\alpha}})C(\alpha) \tag{12}$$

where $K(\alpha,\hat{\dot{\alpha}}) = L(\alpha,\hat{\dot{\alpha}})+N(\alpha,\hat{\dot{\alpha}})M(\alpha)$ Finally, under the conditions C1. and C2., the state estimation error dynamics (7) is obtained. ■

The state estimation error dynamics (7) depends on the nonlinear term

$$\tilde{\Upsilon}(x(t),\hat{x}(t),\alpha(t)) = \Upsilon(x(t),\alpha(t)) - \Upsilon(\hat{x}(t),\alpha(t)) \quad (13)$$

this term has been handled in many works by exploiting the Lipschitz property of $\Upsilon(x(t))$ such as

$$\|\Upsilon(x(t),\alpha(t)) - \Upsilon(\hat{x}(t),\alpha(t))\| < \gamma \|x(t) - \hat{x}(t)\| \quad (14)$$

where $\|.\|$ denotes the Euclidean norm and γ is a positive scalar representing the Lipschitz constant. This property can be local or global. However, it has been shown in many works that the Lipschitz constant can end with conservative results especially when the Lipschitz constant γ is greater than an admissible one γ^* (the word admissible is understood in the sense that the obtained LMIs have no solution when $\gamma > \gamma^*$, see for example the result of Bergsten in [2]). This same problem is encountered in observer design for nonlinear Lipschitz systems [19]. For Takagi-Sugeno observer design with unmeasurable premise variables, a less conservative solution has been proposed in [8] which consists in using the Input-to-State Stability (ISS). Indeed, the obtained LMIs are free from the Lipschitz constant but the price to pay is that the asymptotic (or exponential) convergence is no longer achieved and only bounded state estimation error is obtained. The bound of the state estimation error is optimized by minimizing the ISS-gain. In [19], an interesting approach is exploited which aims to reformulate the Lipschitz property in a polytopic form, by using the Differential Mean Value Theorem. Hence, the structure of the nonlinear term is taken into account and LMI constraints problems obtained reduce the pessimism of the results. Nevertheless, the increase of complexity of the method can bring computational issues very quickly.

Less conservative LMI conditions with respect to the Lipschitz approach are proposed therein that ensure asymptotic (or exponential) convergence of the state estimation error. The idea is based on the use of the Differential Mean Value Theorem (DMVT) in order to handle the term $\tilde{\Upsilon}(x(t),\hat{x}(t))$. Using the DMVT, the nonlinear function $\tilde{\Upsilon}(x(t),\hat{x}(t))$ can be exactly represented as

$$\Upsilon(x(t),\alpha(t)) - \Upsilon(\hat{x}(t),\alpha(t)) = \frac{\partial\Upsilon(\tilde{x}(t),\alpha(t))}{\partial x}(x(t) - \hat{x}(t)) \quad (15)$$

where $\tilde{x} \in [\min(x,\hat{x}),\max(x,\hat{x})]$ understood in a component-wise sense.

At this stage, the term $\frac{\partial\Upsilon(\tilde{x}(t),\alpha(t))}{\partial x}$ can be handled in two ways: the first one is to bound the partial derivatives of $\frac{\partial\Upsilon(\tilde{x}(t),\alpha(t))}{\partial x}$ by considering a subset $\mathcal{D} \subseteq \mathbb{R}^n$ and transform $\frac{\partial\Upsilon(\tilde{x}(t),\alpha(t))}{\partial x}$ in a polytopic form as performed in [7]. However, this solution may lead to a huge number of vertices and increase the conservatism of the LMI constraints solutions. The second one uses classical tools of robust control and transforms the nonlinear term $\frac{\partial\Upsilon(\tilde{x}(t),\alpha(t))}{\partial x}$ as follows

$$\frac{\partial\Upsilon(\tilde{x},\alpha(t))}{\partial x} = H\Delta(\tilde{x})E(\alpha(t)), \quad \|\Delta(\tilde{x})\| \le 1 \quad (16)$$

where H is a constant known matrix and $E(\alpha(t))$ is known and depends on the measured variables $\alpha(t)$. Embedding the measured variables in the matrix $E(\alpha(t))$ aims to relax the bounds of the matrix $\frac{\partial\Upsilon(\tilde{x}(t),\alpha(t))}{\partial x}$.

Let us consider the state estimation error dynamics (7). The term $(\Upsilon(x(t),\alpha) - \Upsilon(\hat{x}(t),\alpha))$ is expressed as in (16). In order to handle the term $(\dot{P}(\alpha) - \hat{\dot{P}}(\alpha))x(t)$, let us denote by $\mathcal{P}(\dot{\alpha},x) = \dot{P}(\alpha)x(t)$, then, one obtains

$$\mathcal{P}(\dot{\alpha},x) - \mathcal{P}(\hat{\dot{\alpha}},x) = \frac{\partial\mathcal{P}(\tilde{\dot{\alpha}},x)}{\partial\dot{\alpha}}e_{\dot{\alpha}} \quad (17)$$

where $e_{\dot{\alpha}} = \dot{\alpha}(t)) - \hat{\dot{\alpha}}(t)$ and where $\tilde{\dot{\alpha}} \in [\min(\dot{\alpha},\hat{\dot{\alpha}}),\max(\dot{\alpha},\hat{\dot{\alpha}})]$ understood in a component-wise sense. The state estimation dynamics becomes

$$\dot{e}(t) = \Big(\Phi(\alpha,\hat{\dot{\alpha}}) + P(\alpha)H\Delta(\tilde{x})E(\alpha)\Big)e(t) + W\left(\tilde{\dot{\alpha}}\right)e_{\dot{\alpha}}(t) \quad (18)$$

where

$$\Phi(\alpha,\hat{\dot{\alpha}}) = P(\alpha)A(\alpha) + \hat{\dot{P}}(\alpha) - K(\alpha,\hat{\dot{\alpha}})C(\alpha) \quad (19)$$

$$W\left(\tilde{\dot{\alpha}}\right) = \frac{\partial\mathcal{P}(\tilde{\dot{\alpha}},x)}{\partial\dot{\alpha}} \quad (20)$$

For stability analysis, the Lyapunov theory is used to establish exponential stability, Disturbance-to-Error Stability (DES) or quasi-Disturbance-to-Error Stability according to, respectively, the cases available $\dot{\alpha}(t)$, only estimated version of $\alpha(t)$ and the case where neither the time derivative nor an estimation of that derivative are available. The definitions of DES, that corresponds to an ISS sense, is given in the Definition 1 (The qDES property will be given later).

Definition 1: (Disturbance-to-Error Stability (DES) [15]) Under bounded $e_{\dot{\alpha}}(t)$, the system (18) is Disturbance-to-Error Stable if

$$\|e(t)\| \le \beta\left(\|e(0)\|,t\right) + \gamma\left(\|e_{\dot{\alpha}}(t)\|_\infty\right) \quad (21)$$

with $\beta(.)$ a class $\mathcal{KL}$ function[1] and $\gamma(.)$ a class $\mathcal{K}$ function[2] and $\|.\|$ the Euclidean norm.

B. Unknown Input Observer design and stability analysis

This section provides a design approach of the observer (2) for the system (1). The objective is to design the state observer for the system (1) by decoupling the unknown input $d(t)$ from the state estimation error. Notice that at this stage, the polytopic form of the observer's matrices is not fixed. As explained in [6] and [11], postponing the polytopic transformation reduces the conservatism of observer existence compared to the classical polytopic UIOs [3]. The main result consisting in LMI conditions that ensure

[1]The function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if $\beta(.,t)$ is of class $\mathcal{K}$ for a fixed $t \ge 0$, and β(r,t) is decreasing to zero for $t \to \infty$ for each fixed r.

[2]A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if γ is continuous, strictly increasing, and $\gamma(0) = 0$. If in addition γ is unbounded, it is of class $\mathcal{K}_\infty$

Disturbance-to-Error Stability (DES) of the state estimation error dynamics for the UIO exploiting the estimated time derivative of the parameter $\alpha(t)$ and qDES for the case of unavailable time derivative and its estimation.

Theorem 1: Given a positive scalars η and α_1. If there exist a symmetric and positive definite matrix X, $X > \alpha_1 I$, gain matrices $\bar{K}_i, i = 1, ..., r$, and positive scalars $\epsilon_i, i = 1, ..., r$ and γ such that

$$\begin{cases} \Xi_{iik} < 0, i = 1, ..., r, \quad k = 1, ..., r' \\ \frac{1}{r-1}\Xi_{iik} + \frac{1}{2}\left(\Xi_{ijk} + \Xi_{jik}\right) < 0 \\ i, j = 1, ..., r, i \neq j, k = 1, ..., r' \end{cases} \tag{22}$$

where

$$\Xi_{ijk} = \begin{bmatrix} \bar{\Gamma}_{ij} & XW_k & XP_iH & \varepsilon E_i^T \\ W_k^T X & -\gamma I & 0 & 0 \\ H^T P_i X & 0 & -\varepsilon I & 0 \\ \varepsilon E_i & 0 & 0 & -\varepsilon I \end{bmatrix} \tag{23}$$

$$\bar{\Gamma}_{ij} = XA_i - \bar{K}_i C_j + \eta X + (*)$$

Then the state estimation error is DES and satisfies

$$\|e(t)\| < \sqrt{\frac{\alpha_2}{\alpha_1}}\, \|e(0)\| \exp\left(-\eta t\right) + \sqrt{\frac{\gamma}{2\alpha_1 \eta}}\, \|e_{\dot{\alpha}}(t)\|_\infty \tag{24}$$

where α_1 and α_2 are positive scalars defined such that

$$\alpha_1 \|e(t)\|^2 \leq V(e(t)) \leq \alpha_2 \|e(t)\|^2 \tag{25}$$

Proof: The proof is divided into two parts: firstly, the Input-to-Error Stability is established with respect to the error in estimating the time derivative of $\alpha(t)$ (i.e. $e_{\dot{\alpha}}$). Secondly, LMI conditions that ensures the IES are established.

a) Input-to-Error Stability analysis: Consider the quadratic Lyapunov function

$$V(e(t)) = e^T(t) X e(t), \quad X = X^T > 0 \tag{26}$$

where (25) is satisfied. The time derivative of $V(e(t))$ along the trajectory of the state estimation error is given by

$$\begin{aligned} \dot{V}(e(t)) &= e^T(t)\left(\Omega^T X + X\Omega\right) e(t) \\ &+ e^T(t) X W_\mu e_{\dot{\alpha}} + e_{\dot{\alpha}}^T W_\mu^T X e(t) \end{aligned} \tag{27}$$

Where $\Omega = \Phi(\alpha, \hat{\dot{\alpha}}) + P(\alpha) H \Delta(\tilde{x}) E(\alpha)$. By adding and subtracting the term $-2\eta V(e(t)) + \gamma e_{\dot{\alpha}}^T(t) e_{\dot{\alpha}}(t)$ with $\eta > 0$ and $\gamma > 0$, the equation (27) is equivalent to

$$\dot{V}(e(t)) = \xi^T(t) \Xi_{\alpha,\hat{\dot{\alpha}},\tilde{\dot{\alpha}}} \xi(t) - \eta V(e) + \gamma e_{\dot{\alpha}}^T e_{\dot{\alpha}} \tag{28}$$

where

$$\xi(t) = \begin{bmatrix} e(t) \\ e_{\dot{\alpha}} \end{bmatrix}, \Xi_{\alpha,\hat{\dot{\alpha}},\tilde{\dot{\alpha}}} = \begin{bmatrix} X\Omega + \eta X + (*) & XW(\tilde{\dot{\alpha}}) \\ W(\tilde{\dot{\alpha}})^T X & -\gamma I \end{bmatrix} \tag{29}$$

Consequently, if $\Xi_{\alpha,\hat{\dot{\alpha}},\tilde{\dot{\alpha}}} < 0$, then, the time derivative of the Lyapunov function can be bounded by

$$\dot{V}(e(t)) \leq -2\eta V(e(t)) + \gamma e_{\dot{\alpha}}^T(t) e_{\dot{\alpha}}(t) \tag{30}$$

whose solution is given by

$$V(e(t)) < V(e(0)) \exp\left(-2\eta t\right) + \frac{\gamma}{2\eta} \|e_{\dot{\alpha}}(t)\|_\infty^2 \tag{31}$$

By using the definition of the Lyapunov function (26)-(25) and square root, The inequality (24) is obtained which ensures Input-to-Error Stability.

b) LMI design conditions: In the first part, it is proved that if $\Xi_{\alpha,\hat{\dot{\alpha}},\tilde{\dot{\alpha}}} < 0$ is satisfied, then the IES is ensured. Now let us consider the matrix $\Xi_{\alpha,\hat{\dot{\alpha}},\tilde{\dot{\alpha}}}$ defined in (29) with the definition of the matrices

$$\begin{bmatrix} X\Phi(\alpha, \hat{\dot{\alpha}}) + XP(\alpha) H \Delta\left(\tilde{x}\right) E(\alpha) + (*) & XW\left(\tilde{\dot{\alpha}}\right) \\ W\left(\tilde{\dot{\alpha}}\right)^T X & -\gamma I \end{bmatrix} < 0 \tag{32}$$

By using the square completion and the fact that $\Delta^T\left(\tilde{x}\right)\Delta\left(\tilde{x}\right) \leq I$, the term $XP(\alpha) H \Delta\left(\tilde{x}\right) E(\alpha) + (*)$ can ve bounded by

$$\begin{aligned} XP(\alpha) H \Delta\left(\tilde{x}\right) E(\alpha) + (*) < \varepsilon^{-1}(\alpha) XP(\alpha) H H^T P(\alpha) X \\ + \varepsilon(\alpha) E^T(\alpha) E(\alpha) \end{aligned} \tag{33}$$

with $\varepsilon(\alpha) > 0$, $\forall \alpha \in \Pi_\alpha$. Consequently, the Matrix Inequality (32) is bounded as follows

$$\begin{bmatrix} \Gamma(\alpha, \hat{\dot{\alpha}}) & XW\left(\tilde{\dot{\alpha}}\right) & XP(\alpha) H & \varepsilon(\alpha) E^T(\alpha) \\ W\left(\tilde{\dot{\alpha}}\right)^T X & -\gamma I & 0 & 0 \\ H^T P(\alpha) X & 0 & -\varepsilon(\alpha) I & 0 \\ \varepsilon(\alpha) E(\alpha) & 0 & 0 & -\varepsilon(\alpha) I \end{bmatrix} < 0 \tag{34}$$

where $\Gamma(\alpha, \hat{\dot{\alpha}}) = X\Phi(\alpha, \hat{\dot{\alpha}}) + \eta X + (*)$. In order to derive LMI conditions aiming to determine the different matrices of the UIO, the polytopic form is exploited. The polytopic form is obtained by the sector nonlinear transformation [17] and given as follows

$$\hat{\dot{P}}(\alpha) + P(\alpha) A(\alpha) \quad = \quad \sum_{i=1}^{r} h_i(\alpha, \hat{\dot{\alpha}}) \mathcal{A}_i$$

$$\begin{aligned} E(\alpha) &= \sum_{i=1}^{r} h_i(\alpha, \hat{\dot{\alpha}}) \mathcal{E}_i, \quad P(\alpha) = \sum_{i=1}^{r} h_i(\alpha, \hat{\dot{\alpha}}) \mathcal{P}_i \\ K(\alpha, \hat{\dot{\alpha}}) &= \sum_{i=1}^{r} h_i(\alpha, \hat{\dot{\alpha}}) \mathcal{K}_i, \quad C(\alpha) = \sum_{i=1}^{r} h_i(\alpha, \hat{\dot{\alpha}}) \mathcal{C}_i \\ \varepsilon(\alpha) &= \sum_{i=1}^{r} h_i(\alpha, \hat{\dot{\alpha}}) \epsilon_i, \; W\left(\tilde{\dot{\alpha}}\right) = \sum_{i=1}^{r'} \mu_i(\tilde{\dot{\alpha}}) \mathcal{W}_i \end{aligned} \tag{35}$$

Consequently, the inequality (34) is expressed as follows

$$\sum_{i=}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r'} h_i(\alpha, \hat{\dot{\alpha}}) h_j(\alpha, \hat{\dot{\alpha}}) \mu_i(\tilde{\dot{\alpha}}) \Xi_{ijk} < 0$$

where

$$\Xi_{ijk} = \begin{bmatrix} \Gamma_{ij} & X\mathcal{W}_k & X\mathcal{P}_i H & \epsilon_i \mathcal{E}_j^T \\ \mathcal{W}_k^T X & -\gamma I & 0 & 0 \\ H^T \mathcal{P}_i X & 0 & -\epsilon_i I & 0 \\ \epsilon_i \mathcal{E}_j & 0 & 0 & -\epsilon_i I \end{bmatrix}$$

where

$$\Gamma_{ij} = X\left(\mathcal{A}_i - \mathcal{K}_i \mathcal{C}_j\right) + \eta X + (*)$$

Now, by using the Tuan's Lemma [18] and a change of variables $\bar{K}_i = XK_i$, the LMIs given in the Theorem 1 are obtained which ensures Disturbance-to-Error Stability. ■

The Theorem 1 presents LMI conditions that ensure Disturbance-to-Error Stability. In order to enhance the performances of the UIO, it is interesting to minimize the DES-gain expressed by the term $\sqrt{\frac{\gamma}{2\alpha_1 \eta}}$ in the bound of the state estimation error (24). For that a purpose, the following corollary provides results that optimized the DES-gain under LMI conditions.

Corollary 1: Given a positive scalars η and α_1. If there exist a symmetric and positive definite matrix $X > \alpha_1 I$, gain matrices $\bar{K}_i, i = 1, ..., r$, and positive scalars $\epsilon_i, i = 1, ..., r$, γ and ξ solution to the following optimization problem

$$\min_{X, \bar{K}_j, \gamma, \varepsilon} \quad \zeta \quad \text{s.t. (22) and} \quad \gamma - 2\alpha_1 \eta \zeta \leq 0$$

Then the state estimation error is DES and satisfies the inequality (24) with minimal DES-gain.

Proof: In order to minimize the DES-gain, let us consider the inequality $\sqrt{\frac{\gamma}{2\alpha_1 \eta}} \leq \sqrt{\zeta}$, where ζ is a positive scalar. Then minimizing ζ will minimize the DES-gain. The inequality is equivalent to $\gamma - 2\alpha_1 \eta \zeta \leq 0$, which is linear with respect to γ and ζ. Then minimizing ζ under the LMIs (22) will ensure minimal DES-gain. ■

C. Case of known or measured time derivative of $\alpha(t)$

The theorem 1 considers the case only an estimation of the first time derivative of $\alpha(t)$. If this time derivative is available in real-time (i.e. $\hat{\dot{\alpha}}(t) = \dot{\alpha}(t)$, the Theorem can be reduced to the result of the following Corollary.

Corollary 2: Given a positive scalars η and α_1. If there exist a symmetric and positive definite matrix $X > \alpha_1 I$, gain matrices $\bar{K}_i, i = 1, ..., r$, and a positive scalar $\epsilon_i, i = 1, ..., r$ such that

$$\begin{cases} \Xi_{ii} < 0, i = 1, ..., r \\ \frac{1}{r-1}\Xi_{ii} + \frac{1}{2}\left(\Xi_{ij} + \Xi_{ji}\right) < 0, i, j = 1, ..., r, i \neq j \end{cases} \quad (36)$$

where Ξ_{ij} is similair to Ξ_{ijk} in (23) without the second column and line. Then the state estimation error is exponentially stable and satisfies: $\|e(t)\| < \sqrt{\frac{\alpha_2}{\alpha_1}} \|e(0)\| \exp(-\eta t)$.

Proof: The proof is similar to the one given for Theorem 1. It takes into account the fact that $\hat{\dot{\alpha}}(t) = \dot{\alpha}(t)$. Then, the state estimation error dynamics (18) becomes

$$\dot{e}(t) = \Big(\Phi(\alpha, \hat{\dot{\alpha}}) + P(\alpha) H \Delta(\tilde{x}) E(\alpha)\Big) e(t) \quad (37)$$

With this state estimation error, the Corollary 1 is obtained which ensures exponential stability. ■

D. Case of neither time derivative nor its estimation are available

If neither the time derivative of $\alpha(t)$ nor its estimation are unavailable, the proposed UIO (2) is expressed as follows

$$\begin{cases} \dot{z}(t) = N(\alpha(t))z(t) + P(\alpha(t))\Upsilon(\hat{x}(t), \alpha(t)) \\ \qquad + G(\alpha(t))u(t) + L(\alpha(t))y(t) \\ \hat{x}(t) = z(t) - M(\alpha(t))y(t) \end{cases} \quad (38)$$

where the matrices depend only on the measured parameter $\alpha(t)$. Following the same steps as before with the conditions $P(\alpha) = I + M(\alpha)C(\alpha)$, $N(\alpha) = P(\alpha)A(\alpha) - K(\alpha)C(\alpha)$, $G(\alpha) = P(\alpha)B(\alpha)$, $K(\alpha) = L(\alpha) + N(\alpha)M(\alpha)$, The state estimation error between (1) and the observer (38) is given by

$$\dot{e}(t) = \left(N(\alpha) + P(\alpha) H \Delta(\tilde{x}) E(\alpha)\right) e(t) + \dot{P}(\alpha)\, x(t) \quad (39)$$

The difference with the equation (18) is in the last term $\dot{P}(\alpha)\, x(t)$ and the fact that $N(\alpha)$ depends only on the measured parameter $\alpha(t)$. Under boundedness of $x(t)$ and $\dot{\alpha}(t)$, the Theorem 1 can be used to ensure quasi-Disturbance-to-Error Stability (qDES) with respect to bounded $x(t)$ (i.e. $\|x(t)\| \leq K$). The qDES property is given in the following definition 2.

Definition 2: (quasi-Disturbance-to-Error Stability (qDES) [15]) Under $\|x(t)\| \leq K$, the system (39) is quasi-Disturbance-to-Error Stable if

$$\|e(t)\| \leq \beta_K\left(\|e(0)\|, t\right) + \gamma_K\left(\|x(t)\|_\infty\right) \quad (40)$$

with $\beta_K(.)$ a class $\mathcal{KL}$ function and $\gamma_K(.)$ a class $\mathcal{K}$ function and $\|.\|$ the Euclidean norm.

Remark 1: Notice that in all the cases dealt with above, the unknown input $d(t)$ is completely decoupled. However, if the time derivative $\alpha(t)$ is unavailable for measure, the state estimation error is DES if Theorem 1 is satisfied for available estimation of the time derivative of $\alpha(t)$. If this estimation is unavailable too, the state estimation error can be qDES with respect to $x(t)$ when $\|x(t)\| \leq K$.

IV. Simulation Example

Let us consider the nonlinear system

$$\begin{cases} \dot{x}_1 = x_2 - k\sin(x_2) \\ \dot{x}_2 = x_3 + (1 + x_1^2)d \\ \dot{x}_3 = -2x_1 - 3x_2 + (x_1^2 + 1)x_3 - 4x_3 + d \\ y_1 = x_1, \quad y_2 = x_3 \end{cases} \quad (41)$$

where k is a positive constant and d represents the unknown input. Notice that the nonlinearity $\sin(x_2)$ depends on unmeasured state which leads to a quasi-LPV nonlinear system with unmeasured parameters. After rewriting the system into the form (1), one obtains the matrices

$$A(\alpha(t)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4 + \alpha(t) \end{bmatrix}, F(\alpha(t)) = \begin{bmatrix} 0 \\ \alpha(t) \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Upsilon(x) = \begin{bmatrix} -k\sin(x_2) \\ 0 \\ 0 \end{bmatrix}$$

where $\alpha(t) = (1 + x_1^2) = (1 + y_1^2)$. The nonlinear function $\Upsilon(x)$ is handled as in (16) with the matrices

$$H = \begin{bmatrix} -k \\ 0 \\ 0 \end{bmatrix}, \Delta(\tilde{x}_2) = \cos(\tilde{x}_2), E = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

After computations, the matrices of the observer in LPV form are given by

$$M(\alpha(t)) = \begin{bmatrix} 0 & 0 \\ 0 & -\alpha(t) \\ 0 & -1 \end{bmatrix}, P(\alpha(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha(t) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{A}(\alpha(t), \dot{\alpha}(t)) = \begin{bmatrix} 0 & 1 & 0 \\ 2\alpha(t) & 3\alpha(t) & 1 - \alpha(t)(\alpha(t) - 4) - \hat{\dot{\alpha}}(t) \\ 0 & 0 & 0 \end{bmatrix}$$

The objective of this example is to show both the decoupling possibility offered by the proposed flexible design and the approach that handles the nonlinear part. Indeed, since the nonlinear term $\Upsilon(x)$ is Lipschitz with a Lipschitz constant k, the LMIs presented in (22) provide a solution for values of k around 10^6, while the classical approach (see for example Theorem 1 in [14]) used for Lipschitz systems exploiting the equation (14) provides a solution only for $k \leq 0.9995$. For the approach using the alternative Lipschitz condition $\|\Upsilon(x(t), \alpha(t)) - \Upsilon(\hat{x}(t), \alpha(t))\| < \|G(x(t) - \hat{x}(t))\|$, where G is a square matrix, solutions exists for $k \leq 48$. By comparison with the approach of [19], a solution exists for high values of the Lipschitz constant, but the number of LMIs to solve is 9 while in the proposed approach (Theorem 1) it is reduced to 5. Now let us fix $k = 0.5$, the proposed observer provides the results depicted in the Figure 1. The time derivative of $\alpha(t)$ is obtained by a High Order Sliding Mode Differentiator (HOSMD) [10] with order 4. Notice that the parameters of this differentiators are chosen in such a way to have a degraded estimation of the time derivative $\dot{\alpha}(t)$ in order to highlight the capabilities of the proposed design approach (see Figure 1(bottom)). As can be seen, the estimation of x_2 obtained by the result of Theorem 1 without optimization of DES-gain is acceptable, while the result obtained by using the Corollary 1 is better and robust against errors in the estimation of the time derivative $\dot{\alpha}(t)$.

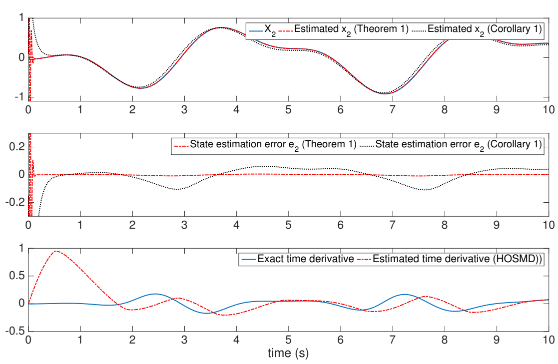

Fig. 1. UIO (Top) x_2 and its estimation with both Theorem 1 and Corollary 1. (Middle) State estimation error $x_2 - \hat{x}_2$ (Theorem 1 vs Corollary 1). (Bottom) Exact time derivative of $\alpha(t)$ and its estimation by a HOSMD

V. Conclusion

In this paper, a new observation approach for a class of nonlinear systems affected by unknown inputs is addressed. This class of systems is known as quasi-LPV systems (i.e. parameters are state dependent). The contributions of the proposed approach can be summarized as follows: New structure of unknown input observer which relaxes the classical rank condition ensuring UI decoupling. Secondly, handling the nonlinear terms in order to relax the constraint related to the Lipschitz approach. Then, thanks to the DMVT and robust stability analysis, LMI conditions are established in order to ensure asymptotic convergence of the state estimation error. It is shown that the proposed approach is more efficient and general compared to the existing results based on Lipschitz condition (admissible Lipschitz constant), ISS (bounded error) or classical DMVT (may leads to huge number of LMIs which render the LMI problem intractable). In addition, it is shown that postponing the polytopic transformation avoids the conservatism introduced by the polytopic form. Finally, the problem of availability of the time derivatives of the parameters is handled by using the Disturbance-to-Error Stability with optimized DES-gain.

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