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Abnormal Curves in a Zermelo Navigation Problem in the Plane and the Fan Shape of Small Time Balls

B. Bonnard*, O. Cots†, J. Gergaud‡, B Wembe§

January 13, 2020

Abstract

In this note, motivated by the Zermelo navigation problem in the flat plane, where the current field is associated to a point vortex, we discuss the role of the abnormal curves in the shape of the small time balls. Abnormal curves are occurring in the strong current domain in the vicinity of the vortex and form with the hyperbolic geodesics the boundaries of the small time accessibility sets and give to the small time balls a fan shape. Implications with the regularity of the value function are discussed.

Keywords: Helhmoltz-Kirchhoff N vortices model, Zermelo navigation problem, Geometric optimal control, Abnormal curves, Time minimal value function.

1 Introduction

Consider the Zermelo navigation problem in the plane, with a vortex singularity introduced in [5] and whose dynamics is given in cartesian coordinates $q := (x, y) \in M := \mathbb{R}^2 \setminus \{0\}$ by

$$\dot{q}(t) = F_0(q(t)) + \sum_{i=1}^{2} u_i(t)\, F_i(q(t)) \tag{1}$$

where F_0 is the current (or drift), where F_1, F_2 define the control directions associated to the heading angle α of the ship:

$$F_0(q) := \frac{\mu}{x^2+y^2}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right), \quad F_1(q) := \frac{\partial}{\partial x}, \quad F_2(q) := \frac{\partial}{\partial y}, \qquad \mu > 0,$$

and where $u := (u_1, u_2)$ is the control. The control is bounded by $\|u\| := \sqrt{u_1^2 + u_2^2} \leq 1$ and for $\|u\| = 1$, it is related to the heading angle α by $u = (\cos\alpha, \sin\alpha)$.

Following the control point of view [6], the Zermelo navigation problem is restated as a time minimal control problem to steer q_0 to q_1 for any pair q_0, $q_1 \in M$. We refer to [2] for the differential geometric frame in the case of a weak current and Randers metrics. More generally, our study will concern the local problem and we can consider the general case in an open subset Ω of $\mathbb{R}^2$ where F_0 is smooth, where the control directions are associated to $\|u\| \leq 1$ and where F_1, F_2 form an orthonormal frame for a Riemannian metric g so that $\|u\| = 1$ is the standard unit sphere. In this general frame, for $q_0 \in \Omega$ we can encountered three cases:

- **Weak current case** if $\|F_0(q_0)\|_g < 1$;
- **Strong current case** if $\|F_0(q_0)\|_g > 1$;
- **Intermediate current case** in the transitional case $\|F_0(q_0)\|_g = 1$.

*Institut de Mathématiques de Bourgogne, Dijon, France, bbonnard@u-bourgogne.fr
†ENSEEIHT, IRIT, Toulouse, France, olivier.cots@irit.fr
‡ENSEEIHT, IRIT, Toulouse, France, joseph.gergaud@irit.fr
§Université Paul Sabatier, IRIT, Toulouse, France, boris.wembe@irit.fr

The set of admissible controls $\mathcal{U}$ is the set of measurable mappings u from $[0,+\infty)$ to the unit closed Euclidian ball. For $u \in \mathcal{U}$ we denote by $q(\cdot, q_0, u)$ the trajectory of (1) associated to u, initiating at time 0 from q_0 and defined on a domain $[0, t_f(u)]$. The *accessibility set* from q_0 in time $t_f \geq 0$ is denoted by $\mathcal{A}(q_0, t_f) \coloneqq \{q(t_f, q_0, u) \mid u \in \mathcal{U}$ and $q(\cdot, q_0, u)$ is defined on $[0, t_f]\}$ and the *accessibility set* from q_0 is $\mathcal{A}(q_0) \coloneqq \bigcup_{t_f \geq 0} \mathcal{A}(q_0, t_f)$. In the case of system (1) associated to the vortex problem, from [5], for each pair (q_0, q_1) in the punctured plane there exists a time minimal solution to steer q_0 to q_1. Fixing q_0 we define the *time minimal value function*

$$T(q_0, q_1) \coloneqq \inf_{u \in \mathcal{U}} t_f \quad \text{s.t} \quad q(t_f, q_0, u) = q_1.$$

Fixing q_0, the *sphere* $\mathrm{S}(q_0, r)$ with radius r is the set of points q_1 which can be reached from q_0 in minimum time r, while the *ball* with radius r is $\mathrm{B}(q_0, r) \coloneqq \bigcup_{r' \leq r} \mathrm{S}(q_0, r')$.

The objective of this note is to complete the results of [5] to describe the balls with a small radius. In the weak current case, the result is well known but in the strong current case, the small time accessibility set is bounded by the so-called abnormal curves and the ball has the shape of a *fan*. Our study is based on [4] which describes the shape of the accessibility set in a neighborhood of a reference trajectory under generic assumption. Following the Caratheodory-Zermelo-Goh point of view, our system is extended in the 3D-space, where the control is taken as the derivative $\dot{\alpha}$ of the heading angle and the accessibility set is described in a *conic neighborhood* of the heading angle of the reference curve. Also, in this approach, our analysis relies to consider both time minimizing and maximizing trajectories.

Combined with integrability results of the geodesic flow, due to the rotational symmetry in the vortex problem, it will allow in a forthcoming article to classify the shape of the balls for *general radii* and to classify the *singularities of the value function*.

2 Pontryagin maximum principle and geodesics classification

According to the Maximum Principle [9] and thanks to [5] every minimizing curves is a solution of the C^∞ Hamiltonian dynamics on T^*M given in canonical coordinates $z = (q, p)$ by

$$\dot{z}(t) = \overrightarrow{H}(z(t)) \tag{2}$$

where p is the nonzero adjoint, where

$$\overrightarrow{H} \coloneqq \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

is the symplectic gradient associated to the *true* (or maximized) Hamiltonian H given by

$$H(z) \coloneqq H_0(z) + \sqrt{H_1^2(z) + H_2^2(z)},$$

where $H_i(z) \coloneqq p \cdot F_i(q)$ is the Hamiltonian lift of F_i and where $\cdot$ denotes the scalar product. An *extremal* is a solution $t \mapsto z(t)$ of $\overrightarrow{H}$ and the q-projection is called a *geodesic*. Note that $H =: -p^0$ is constant along any extremal and an extremal is called *hyperbolic* if $p^0 < 0$, elliptic if $p^0 > 0$ and abnormal (or exceptional) if $p^0 = 0$. Moreover, thanks to the Maximum Principle, hyperbolic (resp. elliptic) extremals are candidates to the time minimal (resp. maximal) problem, while the time optimality status of the abnormal extremals have been clarified in [4].

A brief recap adapted to the 2-dimensional Zermelo navigation problem is presented here. The first step, using the Zermelo-Caratheodory-Goh point of view, is to parameterize the extremal controls by the derivative of the heading angle α instead of α and this allows to extend our system into the single-input (affine) system:

$$\dot{\tilde{q}} = X(\tilde{q}) + v\, Y(\tilde{q}) \tag{3}$$

with $\tilde{q} \coloneqq (q, \alpha)$, $X(\tilde{q}) \coloneqq F_0(q)$ and $Y(\tilde{q}) \coloneqq (\cos\alpha\, F_1(q) + \sin\alpha\, F_2(q)) + \frac{\partial}{\partial \alpha}$. In this prolongation, extremal curves $z = (q, p)$ extend into *singular extremal curves* $\tilde{z} \coloneqq (q, \alpha, p, p_\alpha)$ of (3) for the extended Hamiltonian $\tilde{H} = \tilde{p} \cdot (X(\tilde{q}) + v\, Y(\tilde{q})$ with the constraint $p \cdot Y(\tilde{q}) = 0$.

They have the following interpretation: define the extremity mapping at time t_f by $E^{t_f} \colon v \mapsto \tilde{q}(t_f, \tilde{q}_0, v)$ (t_f, $\tilde{q}_0$ being fixed) and the extremity mapping $E \colon v \to \tilde{q}(\cdot, \tilde{q}_0, v)$ (only $\tilde{q}_0$ being fixed), one has the following.

Proposition 2.1 *Hyperbolic and elliptic extremals correspond to singularities of the extremity mapping (for the L^∞-norm on the set of inputs) for fixed t_f, while abnormal (or exceptional) extremals correspond to singularities of the extremity mapping.*

Moreover, a precise description of the accessibility set in time t_f can be obtained in a C^0-neighborhood of a reference singular extremal under generic assumptions that we described briefly.

Assumptions 2.1 *Take a reference extremal $t \mapsto z(t)$ on $[0, t_f]$, $z = (q, p)$ and let $\tilde{z} = (\tilde{q}, \tilde{p})$ be its extension. Dealing with the time minimal control problem, one can assume that $t \mapsto z(t)$ is a one-to-one immersion so that the extension can be identified to $\gamma\colon t \mapsto (t, 0, 0)$ and the singular control v can be taken as $v \equiv 0$, using a proper feedback. Besides, we assume the following: Along γ,*

- *X, Y are linearly independent;*
- *Y, $[X, Y]$ are linearly independent;*
- *$[Y, [X, Y]] \notin \mathrm{Span}\{Y, [X, Y]\}$.*

Then, one has:

Proposition 2.2 *In the hyperbolic (resp. elliptic) case, the reference trajectory γ is time minimizing (resp. maximizing) with respect to all trajectories contained in a tubular neighborhood if the final time t_f is less than the first conjugate time t_{1c}. A conjugate time corresponding to a singularity of the exponential mapping $\exp_{q_0}\colon (t, p_0) \mapsto \Pi(\exp t\overrightarrow{H}(q_0, p_0))$ with $\Pi(q, p) := q$. In the exceptional case, the reference extremal is time minimizing and time maximizing.*

Moreover one has a precise description of the accessibility set in the tubular neighborhood given by Figure 1. In particular for $t > t_{1c}$, the fixed time extremity mapping becomes open.

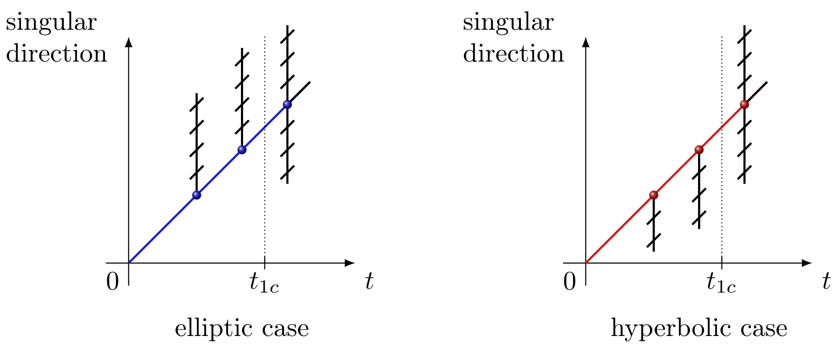

Figure 1: Projection of the fixed time accessibility set in the singular direction.

In the abnormal case, the abnormal reference trajectory corresponds to a singularity of the extremity mapping and the projection in the singular direction (note that the singular direction depends on the case) is given by Figure 2.

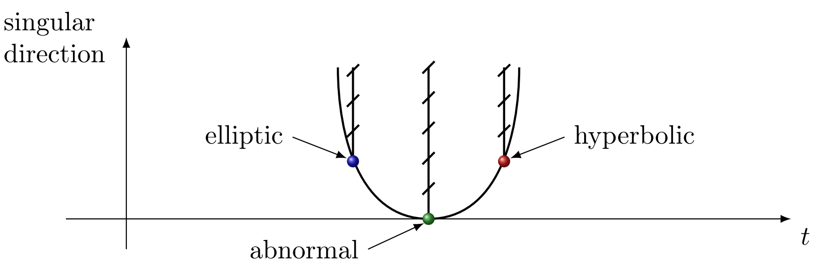

Figure 2: Projection of the accessibility set in the singular direction.

Note that due to the small dimension, conjugate points cannot occur and the extremity mapping is never open. In higher dimension, conjugate points can occur either if the fixed time extremity mapping becomes open or in the *generic case* if the extremity mapping becomes open, see Figure 3.

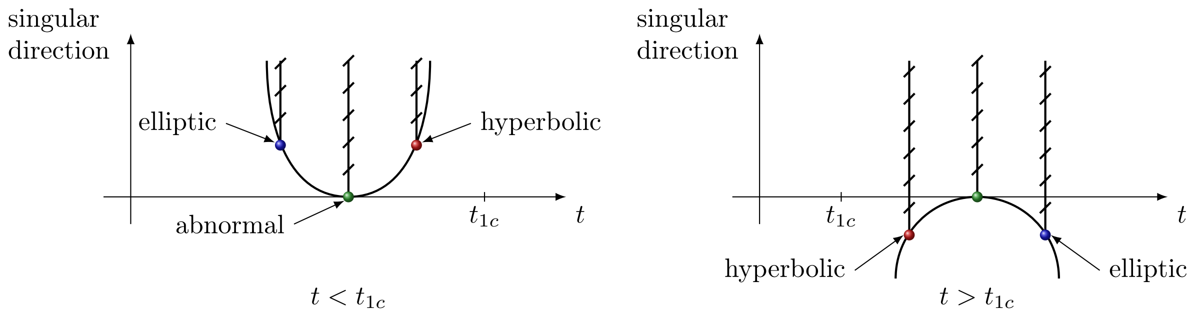

Figure 3: Case $n > 3$: generic conjugate point t_{1c} in the abnormal case.

3 Spheres with small radius

3.1 Notations and definitions

The geodesics are parameterized by t and the initial heading angle α. Fixing q_0 and $t > 0$ we denote by $\exp_{q_0,t}$ the *exponential mapping* $\exp_{q_0,t}\colon \alpha \mapsto \Pi(\exp t\overrightarrow{H}(q_0,\alpha))$ where $\Pi(q,p) = q$ is the q-projection. A *conjugate point* along a reference geodesic is a point where the exponential mapping is not an immersion and taking the set of first conjugate points they will form the *conjugate locus* $C(q_0)$. A *cut point* is the first point where the geodesic loses optimality and they will form the *cut locus* $\Sigma(q_0)$. The *separating set* $L(q_0)$ is the set of points where two minimizing geodesics starting from q_0 are intersecting.

3.2 Ball and sphere of directions

One consider the smooth Zermelo navigation problem. One can assume that g is the Euclidian metric and F_0 is vertical at the initial point q_0 which can be identified to $q_0 = 0$. The *ball of directions* at q_0 is defined by the set

$$F(q_0) \coloneqq \{F_0(q_0) + u \mid \|u\| \le 1\}\,, \tag{4}$$

whose boundary is a circle in our context. We have three cases:

Case 1: Strong current case (Figure 4). In this case the cone of directions is a translation of the unit sphere and we have two *abnormal directions* defined by $\{-\alpha_1, \alpha_1\}$ corresponding to the tangents to the circle from the initial point identified to 0. The *upper part* corresponds to the *hyperbolic directions* and the *lower part* to the *elliptic directions.*

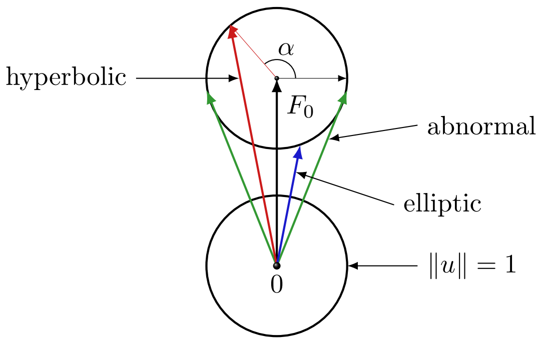

Figure 4: Strong current case.

Cases 2-3: Weak and intermediate current cases (Figure 5). In the intermediate current case the abnormal directions degenerate into the single point 0 and we have only an hyperbolic sector. In the weak drift case, we have only hyperbolic directions.

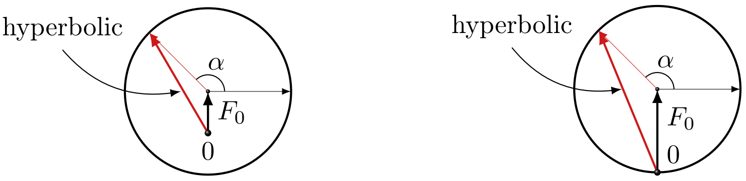

Figure 5: Weak (left subgraph) and intermediate (right subgraph) current cases.

3.3 Small balls and spheres

For small times, the ball of directions $F(q_0)$ gives the shape of the small balls and spheres and it is in accordance with the results of Section 2 about the properties of the extremity mapping.

First case: weak current drift. It corresponds to a Randers problem in the plane, in the frame of the Finsler geometry, see [1].

Proposition 3.1 *In the weak current case, the exponential mapping for small time t is a diffeomorphism from the unit circle to the sphere with radius r and it is represented on Figure 6.*

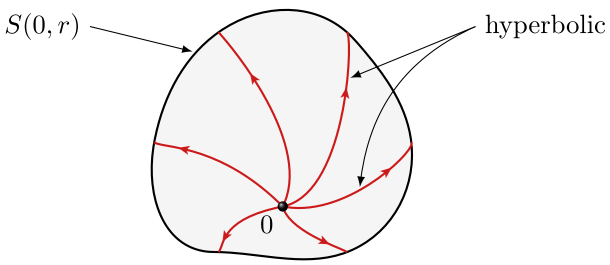

Figure 6: Small sphere and ball for Randers metrics.

Second case: strong drift case. In this case, according to the cone of directions, the small balls have a *fan shape* represented on Figure 7.

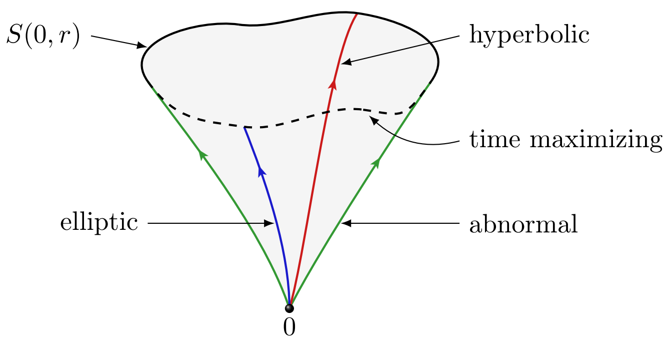

Figure 7: Small sphere and ball for the strong current case.

Proposition 3.2 *In the strong current case, the exponential mapping for a small time is a homeomorphism from the unit circle onto its image, which is formed on the the upper part by the extremities of the hyperbolic trajectories, the lower part being the extremities of the elliptic trajectories, the two parts being separated by the two points corresponding to the abnormal trajectories. Hyperbolic and abnormal points correspond to the time minimizing trajectories, while elliptic points correspond to time maximizing trajectories. The exponential mapping is a diffeomorphism in the hyperbolic conic sector.*

Remark 3.3 *This result is in accordance with the geometric analysis of the accessibility set in time t near the reference extremal and corresponds to the three cases presented on Figure 8.*

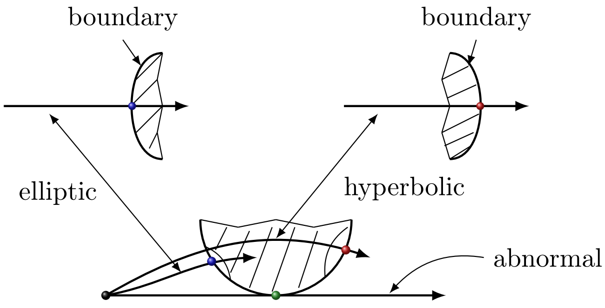

Figure 8: Boundary of the small time accessibility set in time t.

4 Conclusion

This description of the small time spheres is the first step to make a global description. Our contribution is to make a neat interpretation of the abnormal curves in the problem, in relation with the description of the accessibility set in a conic neighborhood of a reference extremal for single input affine control systems. In the context of Zermelo geometry, this gives an interpretation of the micro-local spheres in the abnormal directions. Note that to understand the problem one must consider both hyperbolic and elliptic extremals.

To make the global analysis of the spheres, besides the standard phenomena about cut points in Riemannian geometry related to conjugate and separating points, one must analyze new phenomena related to intersection between abnormal and hyperbolic geodesics (Figure 9) and to singularities of the geodesics in relation with the collinearity locus and cusp points along abnormals (Figure 10).

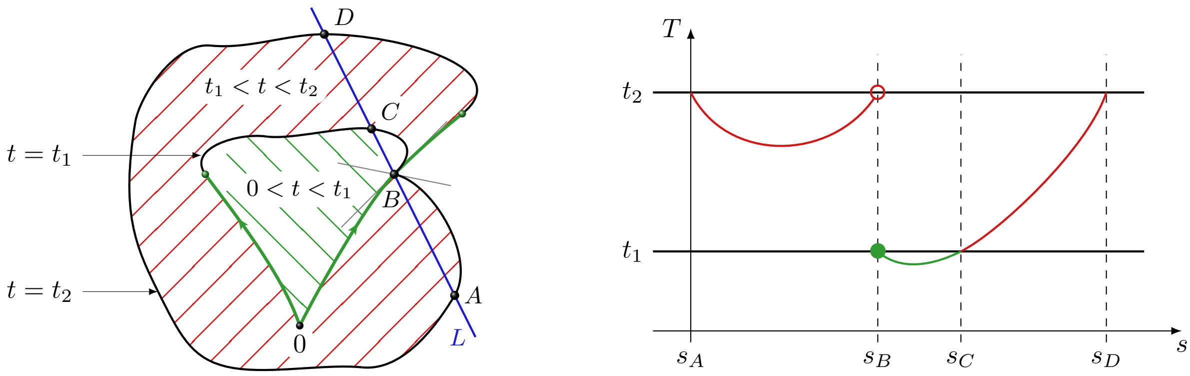

Figure 9: Discontinuity of the time minimal value function T. Let consider two times $0 < t_1 < t_2$, with t_1 small and t_2 large enough. (Left) Balls of radii t_1 and t_2 with the corresponding spheres. The value function T is discontinuous at B, at the intersection of the right abnormal of length t_1 and a hyperbolic extremal of length t_2. (Right) The time minimal value function T along the line L parameterized by s and such that the coordinates value s_A, s_B, s_C and s_D correspond respectively to the points A, B, C and D. One can see the discontinuity of T at s_B.

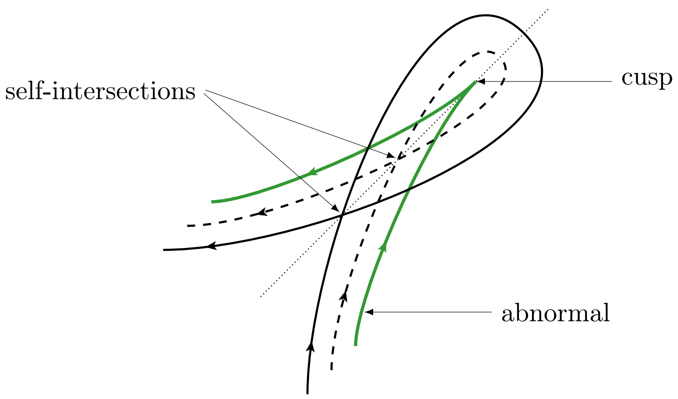

Figure 10: Abnormal with cusp singularity as limit case of self-intersecting normal extremals.

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