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History and pedagogy of mathematics in mathematics education: History of the field, the potential of current examples, and directions for the future

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The field of history of mathematics in mathematics education—often referred to as the history and pedagogy of mathematics domain (or, HPM domain)—can be characterized by an interesting and rich past and a vibrant and promising future. In this plenary, I describe highlights from the development of the field, and in doing so, I focus on several ways in which research in the field of history of mathematics in mathematics education offers important connections to frameworks and areas of long-standing interest within mathematics education research, with a particular emphasis on student learning. I share a variety of examples to serve as cases of what has been possible in the HPM domain. Finally, I propose fruitful future directions that call for the contributions of both established and emerging scholars in the field.

Keywords: History of mathematics, mathematics education research, primary historical sources, qualitative research.

Introduction

George Sarton (1884–1956), a Belgian-born American historian of science, said:

> The main duty of the historian of mathematics, as well as his fondest privilege, is to explain the humanity of mathematics, to illustrate its greatness, beauty and dignity, and to describe how the incessant efforts and accumulated genius of many generations have built up that magnificent monument, the object of our most legitimate pride as men, and of our wonder, humility and thankfulness, as individuals. The study of the history of mathematics will not make better mathematicians but gentler ones, it will enrich their minds, mellow their hearts, and bring out their finer qualities. (Sarton, 1936, p. 28)

Putting aside that to Sarton—in this example—only men experienced this "legitimate pride" (perhaps due to the academic fabric of the 1930s), he beautifully captures one of the often-cited effects of studying the history of mathematics: that such a use of history[1] of mathematics has the ability to humanize the subject, by way of appealing on some aesthetic or non-academic level to the added value of the discipline.

I first experienced this humanistic element of studying the history of mathematics from the desire to provide a different perspective regarding mathematics for my students. At the time some 20 years ago, I was teaching mathematics to students in grades 11 and 12 at a publicly-funded residential school for academically talent students in Mississippi in the United States. I was most concerned about the content that would comprise a history of mathematics course that I was tasked to teach as part of the school's mathematics course electives. In preparation for teaching the course I became

[1] As part of the plenary talk (and paper), I plan to highlight the different notions of "use of history" —as the variants of "use" (e.g., "incorporate," "include," etc.) may be a point of contention for some.

involved with the Institute of the History of Mathematics in its Use in Teaching (IHMT), and my world of mathematics changed forever. As a result of that experience, I not only gained access to materials that would inform the first history of mathematics course I taught, I also acquired a new lens on teaching mathematics in general. Although I was teaching students who chose to attend a school focused on mathematics and science (e.g., *the* Mississippi School for Mathematics and Science in Columbus, MS), not all of the students had a positive relationship with mathematics. For many, they had become trained to view mathematics as a set of procedures to acquire a numerical answer for a "problem." As a high school mathematics teacher, I felt that I was living the embodiment of what Glaisher (1848–1928) described: "I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history" (1890, p. 466). However, continued participation in IHMT and my eventual doctoral work would provide me with new perspectives, tools, and a community with which to view, study, and teach mathematics. Therefore, in this talk, I hope to share with you a small part of the development of the community—its history, if you will—as well as its exciting present and promising future.

Plan for the plenary paper

In this paper, I will first situate the field of history of mathematics in mathematics education—often referred to as the history and pedagogy of mathematics domain (or, HPM domain) within mathematics education, with careful attention to the development leading up to establishing the International Study Group on the Relations between the History and Pedagogy of Mathematics (HPM Group) in 1976. Precipitated by the creation of the HPM Group, research in the HPM domain has continued to grow in last 40-plus years, and includes all levels of learners and teachers. Part of this growth has been marked by the creation of a thematic working group on history in mathematics education, beginning with CERME6 in 2009. Next, I will provide examples of approaches and frameworks that are useful in empirical research in the HPM domain and I will highlight a collection of specific examples in which research on the use of history of mathematics contributes to the broader landscape of research in mathematics education and will do so with respect to two frameworks useful to mathematics education research. As a first example, I will discuss contributions of working with primary historical sources on pre- and in-service teachers' mathematical knowledge for teaching. As a second example, I discuss the application of Sfard's (2008) thinking as communicating framework in research, including work by colleagues in Denmark and Brazil, as well as that currently undertaken within the *Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources* (or, TRIUMPHS) project in the United States. Finally, after a brief analysis of ongoing discussions for calls to strengthen empirical work in the HPM domain in light of certain pitfalls and dilemmas facing the field, I will propose directions for future research and the ways in which various CERME thematic working groups can contribute to bridging research in this important field with the broader mathematics education research community

Development of HPM: A study domain and a group

What is now known as HPM was established as part of the second International Congress on Mathematical Education (ICME) in 1972 in Exeter, UK, first as an official working group there (EWG 11) and then as an official study group (with the onerous original name of International Study Group on Relations between History and Pedagogy of Mathematics, cooperating with the

International Commission on Mathematical Instruction; simplified to the HPM Study Group and later as just the HPM Group) subsequent to ICME3 in Karlsruhe, Germany in 1976. The principle aims of the group were perhaps a description of the culmination of efforts focused on integrating or using history of mathematics in teaching in different contexts around the world since the end of the 19th century. In particular, the principle aims given by the HPM Study Group were:

1. To promote international contacts and exchange information concerning:
    a. Courses in History of Mathematics in Universities, Colleges and Schools.
    b. The use and relevance of History of Mathematics in mathematics teaching.
    c. Views on the relation between History of Mathematics and Mathematical Education at all levels.
2. To promote and stimulate interdisciplinary investigation by bringing together all those interested, particularly mathematicians, historians of mathematics, teachers, social scientists and other users of mathematics.
3. To further a deeper understanding of the way mathematics evolves, and the forces which contribute to this evolution.
4. To relate the teaching of mathematics and the history of mathematics teaching to the development of mathematics in ways which assist the improvement of instruction and the development of curricula.
5. To produce materials which can be used by teachers of mathematics to provide perspectives and to further the critical discussion of the teaching of mathematics.
6. To facilitate access to materials in the history of mathematics and related areas.
7. To promote awareness of the relevance of the history of mathematics for mathematics teaching in mathematicians and teachers.
8. To promote awareness of the history of mathematics as a significant part of the development of cultures. (Fasanelli & Fauvel, 2006, p. 2; originally in May, 1978, p. 76)

It is important to note that the essence of these eight aims have remained relevant and present in subsequent HPM-related meetings and remain a source of motivation for research and practice for many in the field today. It is also important to note that when appropriate, the aims apply to all levels of learners and teachers (e.g., primary (elementary), secondary, tertiary, and teacher education).

After the establishment of the HPM Study Group in 1976, the community continued to grow in important ways, including a focus on practitioners (e.g., school teachers) who wished to humanize mathematics in school teaching but to also engage students with historical materials, methods, and problems in their learning of mathematics. A classic example of the recommendations that were offered to teachers are given by Fauvel (1991) and which resulted from a brief historical look through curriculum documents for school mathematics teachers in the UK. In his introduction, Fauvel noted that for "decades if not centuries now, a few voices in each generation have urged the value and importance of using history in teaching mathematics—but so far without this insight taking firm and widespread root in the practice of teaching" (p. 3). National curriculum documents echoed a similar call for history of mathematics in both the UK and US in 1989:

> Pupils should develop their knowledge and understanding of the ways in which scientific ideas change through time and how the nature of these ideas and the uses to which they are put are

> affected by the social, moral, spiritual and cultural contexts in which they are developed. (*Science in the National Curriculum,* 1989; as cited in Fauvel, 1991, p. 4).
>
> Students should have numerous and varied experiences related to the cultural, historical, and scientific evolution of mathematics so that they can appreciate the role of mathematics and the disciplines it serves…. It is the intent of this goal—learning to value mathematics—to focus attention on the need for student awareness of the interaction between mathematics and the historical situations from which it has developed and the impact that interaction has on our culture and lives. (NCTM, 1989, pp. 5–6)

However, policy statements and reform efforts tell only one side of the story and to actually enable teachers with materials, techniques, skills, etc., is quite another. Still, in the decades since the HPM Group's establishment, the community grew in ways that brought together different stakeholders—mathematicians, mathematics historians, mathematics teachers, mathematics education researchers, and others—for the purpose of sharing research, historical materials, and examples of pedagogical practice in which history of mathematics informed the teaching of mathematics. The first HPM Group satellite meeting (associated with an ICME) took place in 1984 at the Stuart campus of the University of Adelaide, and the satellite meetings have taken place every four years (as with ICME) since then. Additional supports to the international community were also established. For example, also in 1984, a meeting took place at University High School in San Francisco, CA, in which the creation of an Americas Section of the HPM Group was planned. The aim for an "HPM Americas" section was to "have a more active presence in the mathematics education community than was forthcoming from the international organization" (Fasanelli & Fauvel, 2006, p. 6). In 1993, the first of regularly-occurring meetings called European Summer University (ESU), were organized by the Institutes of Research in Mathematics (IREM) and took place in Montpellier, France. The ESU was held every three years until 2010[2], when it was decided that they would occur every four years and would be staggered by two years from the quadrennial ICME/HPM satellite meeting pair.

HPM within CERME

Of course, the inclusion of a working group on history of mathematics in mathematics education at CERME may be the international venue of most interest to the present audience. The working group made its first appearance at CERME6 (Working Group 15: Theory and Research on the Role of History in Mathematics Education), and since then, it was established as Thematic Working Group (TWG) 12, History in Mathematics Education. The TWG has focused on a wide array of concerns to the field, which are represented by nine overarching themes taken from the "call for papers" since 2009:

1. Theoretical, conceptual and/or methodological frameworks for including history in mathematics education;

[2] There was a five-year gap between the 1999 ESU-3 held in Leuven and Louvain-la-Neuve, Belgium and the 2004 joint ESU-4 and ICME-10 satellite meeting of HPM held in Uppsala, Sweden.

2. Relationships between (frameworks for and empirical studies on) history in mathematics education and theories and frameworks in other parts of mathematics education [this point featured only from CERME 7 onwards];
3. The role of history of mathematics at primary, secondary, and tertiary level, both from the cognitive and affective points of view;
4. The role of history of mathematics in pre- and in-service teacher education, from cognitive, pedagogical, and/or affective points of view;
5. Possible parallelism between the historical development and the cognitive development of mathematical ideas;
6. Ways of integrating original sources in classrooms, and their educational effects, preferably with conclusions based on classroom experiments;
7. Surveys on the existing uses of history in curricula, textbooks, and/or classrooms in primary, secondary, and tertiary levels;
8. Design and/or assessment of teaching/learning materials on the history of mathematics;
9. The possible role of history of mathematics/mathematical practices in relation to more general problems and issues in mathematics education and mathematics education research. (Jankvist & van Maanen, 2018, p. 242)

Table 1 provides the titles and authors of sample papers (and the CERME meeting in which they were presented) corresponding to the nine themes given, as a way to exhibit the variety and context in which work in HPM is conducted within the CERME community.

TWG 12[3] theme	Sample paper
1	"The Teaching of Vectors in Mathematics and Physics in France During the 20th Century" (Ba & Dorier; CERME6)
2	"Uses of History in Mathematics Education: Development of Learning Strategies and Historical Awareness" (Kjeldsen; CERME7)
3	"The Role of History of Mathematics in Fostering Argumentation: Two Towers, Two Birds and a Fountain" (Gil & Martinho; CERME9)
4	"Mathematical Analysis of Informal Arguments: A Case-Study in Teacher-Training Context" (Chorlay; CERME10)
5	"Teaching the Concept of Velocity in Mathematics Classes" (Möller; CERME9)
6	"Designing Teaching Modules on the History, Application, and Philosophy of Mathematics" (Jankvist; CERME7)

[3] Again, for CERME6 only, this was Working Group 15.

7	"The Implementation of the History of Mathematics in the New Curriculum and Textbooks in Greek Secondary education" (Thomaidis & Tzanakis; CERME6)
8	"The Development of Place Value Concepts to Sixth Grade Students via the Study of the Chinese Abacus" (Tsiapou & Nikolantonakis; CERME8)
9	"Lessons from Early 17th Century for Current Mathematics Curriculum Practice" (Krüger, CERME7)

Table 1: Sample collection of CERME papers presented in TWG 12

Jankvist and van Maanen (2018) noted that TWG 12 seeks to "create a forum and a platform for fostering empirical studies in the field of history in/of mathematics education *and to also better link research in this field with research in mathematics education in general*" (p. 241, emphasis added). In the examples that follow, I especially focus on themes 2, 3, and 4 in the list summarized by Jankvist and van Maanen.

History in mathematics education: A very brief history of early research

Early scholarship[4] (e.g., conducted before 2000) in the field of history in mathematics education was primarily focused on (a) anecdotal reports of interventions used with students; (b) historical research on topics that could serve as the focus of classroom instruction; and (c) survey research, including research on students' or teachers' attitudes and beliefs related to history of mathematics. A classic example of empirical research is that of McBride and Rollins (1977), in which, as part of McBride's doctoral dissertation, they examined the effects of studying mathematics history on attitudes of college algebra students toward mathematics. The research was motivated by the lack of research reports (available at the time) that studied "the problem of determining the effectiveness" (p. 57) of "incorporating items from the history of mathematics into classroom discussions of mathematical topics" (p. 57). For McBride and Rollins, the "incorporation" of items was restricted to the use of vignette material to introduce or comment on mathematical topics in the college algebra curriculum. The authors found a significant increase in attitude (particularly since the attitudes of the treatment group increased and the those of the control group decreased); however, several limitations were identified, including the notion that the teacher effect may have been significant. Limitations aside, the McBride and Rollins contribution represented two impacts for subsequent research in the field of history in mathematics education. First, it placed the potential of history in mathematics education on the radar of future researchers (myself included). And, their use of existing research—that on attitudes towards mathematics by Aiken (such as his early work in the *Journal for Research in Mathematics Education* in 1974)—exemplified the fruitful connections for research on history in mathematics education within the broader landscape of mathematics education research. In more recent years, scholarship has begun to shift to capitalize on empirical methods that are more mainstream, and for which researchers seek a broader application of the interventions that have been the focus of their research. In the following, I discuss more recent examples of different approaches

[4] Apologies to my colleagues around the globe: for the purposes of this plenary talk (and paper), I focus on English-language scholarship.

and frameworks that are useful in empirical research in the HPM domain, including mathematical knowledge for teaching and thinking as communicating.

Mathematical knowledge for teaching and HPM

Mathematical knowledge for teaching (MKT) is a practice-based theory of the domains of knowledge that are considered necessary for the work of teaching mathematics. The framework itself (often relegated to the "egg model" diagram; see for example, Ball, Thames, and Phelps (2008)) has been applied in a variety of research contexts around the world since the early 2000s and has its foundation in the work of the Learning Mathematics for Teaching (LMT) Project. However, it is important to keep in mind that rather than taking the egg model too literally (as in, trying to situate all relevant and possible knowledge for teaching mathematics into the original six domains of knowledge), the practiced-based theory of MKT comprises two key aspects: knowledge of content in the discipline of mathematics and the recognition that teaching is at the core, and this brings with it the notion that mathematics teaching can be decomposed into tasks of teaching, of which there are many.

In my own early work with prospective mathematics teachers (PMTs), I was struck by the idea that the MKT framework could provide ways to problematize (or clarify) the ways in which studying history of mathematics informs PMTs' knowledge of topics they would soon teach. In one study (Clark, 2012), I analyzed reflection journal entries of 80 students enrolled in a "Using History in the Teaching of Mathematics" course, across four semesters. In the particular investigation, I examined 15 weeks of journal entries for each of the 80 students for their reference to solving quadratic equations using completing the square. In the course, students worked with English translations of primary source material as part of investigations designed to engage them with the historical development of a mathematical concept, and which would provide them with opportunities to expand their mathematical and pedagogical knowledge, and to consider ways in which student learning may benefit from incorporating content similar to what they worked with in the "Using History" course. The source excerpt was taken from Fauvel and Gray (1987, p. 229):

> *Roots and squares are equal to numbers*: for instance, 'one square, and ten roots of the same, amount to thirty-nine dirhems'; that is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine? The solution is this: you halve the number of the roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the roots which is five; the remainder is three. This is the root of the square you sought for; the square itself if nine. […]

When students elected[5] to discuss course tasks from al-Khwarizmi's text on solving quadratic equations in their reflection journals, they revealed what it contributed to their mathematical learning and how they would consider incorporating such content in their future teaching. Brad's reflection of

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[5] For the reflective journal assignments, students made the choice of what to include in their journals. However, regardless of content selection (typically driven by course readings, tasks, and assignments), PMTs needed to respond to at least one of six fixed reflection prompts, e.g., *In what ways has your understanding of {mathematical topic} changed as a result of considering the history of the topic?*, or, *In what ways do you envision being able to incorporate the history of {mathematical topic} in your teaching?*

his prior and current experience (with respect to his mathematical learning) was representative of the tenor of the PMTs' reflection of the al-Khwarizmi tasks:

> I remember learning the quadratic formula in 8th grade. I was in Algebra I and Mrs. Horst had *politely drilled* the "opposite of *b* plus or minus the square root of *b* squared, minus four *ac*, all over two *a*" routine into our heads. All I recall knowing is that I could apply *this formula* to solve polynomials of a 2nd degree. [T]he lesson and activity we completed...were particularly influential to my understanding. This was essentially the *first proof of any sort* that I've experienced relevant to the formula itself. It was *this part where I really had the "a-ha" feeling*. As I began to compose the area relationships using algebraic notation, I could see the beginnings of the quadratic formula; I realized that this was actually going to work! More importantly I began to view the quadratic equation as less of an algebraic equation and more of a *geometric relationship*. (Clark, 2012, p. 78)

From another view, Hillary described her idea for the ways in which she might consider history of mathematics informing her future teaching:

> If I were going to use this in my classroom I would be sure to explain to them how al Khwarizmi used his vast knowledge of many subjects to work with these numbers and develop a similar quadratic formula, one that is like that of ours today, except for the use of the negative numbers. I would show them that with a few simple manipulations and algebraic transformations, we would have the same equation and we could even have groups each try a different method but with the same equation, then compare answers; that way students can find which method suits them the best.... I feel that math has so many possibilities, so many ways in which something can be taught and or understood, so why not provide those so the students can make sense of what to them might be complicated mathematics. (Clark, 2012, p. 80)

As a result of studying PMTs' reflections of course content and engagement with materials using history of mathematics in teaching, I claimed that the work on the part of teachers to incorporate history of mathematics in teaching is a component of the "something else" that Ball and her colleagues (2008) described as knowledge for teaching beyond the obvious knowledge of "topics and procedures that [teachers] teach" (p. 395). As part of their definition of *horizon content knowledge*, Ball et al. concentrated on "how teachers need to know that content" (p. 395) and they sought to "determine what else teachers need to know about mathematics and how and where teachers might use such mathematical knowledge in practice" (p. 395). My study of PMTs' work to develop knowledge of history of mathematics—and, therefore of mathematics that they were tasked to teach—pointed to the strong potential of the history of mathematics to contribute to the "what else" described by Ball et al. and how this specialized knowledge contributes to PMTs' future practice. I also made the claim, by using Boero and Guala's (2008) component of the "cultural analysis of the content to be taught" (p. 223), that engaging in the mathematical, historical, and cultural aspects of a mathematical concept is an important way in which teachers need to know the content that they teach. Thus, although the call to "focus attention on the need for student awareness of the interaction between mathematics and the historical situations from which it has developed and the impact that interaction has on our culture and lives" (NCTM, 1989, p. 6) seems to be a distant memory[6], situating an analysis of what PMTs' claim as a contribution to their mathematical knowledge within the MKT framework enables researchers to make decisions regarding the development of prospective mathematics teachers, as well as the role that history of mathematics plays in that important work. There are still too few studies that capitalize on investigating the role that the study of history of mathematics, organized in meaningful and powerful ways to inform not only PMTs' disciplinary content knowledge but the multiple forms of tasks of teaching that they will perform in their future teaching. However, the contribution of history of mathematics on the MKT of practicing teachers is also productive for research in mathematics education.

Additional contexts for the application of MKT

There is further potential for the application of MKT in the HPM domain. Recent scholarship reveals multiple contexts and applications in which the linkages between MKT and the use of history of mathematics in the development of prospective and practicing teachers further contribute to research on teacher knowledge and the work of mathematics teachers. Two examples are worth noting here. Smestad, Janviskt, and Clark (2014) investigated components of horizon content knowledge (HCK) within MKT in relation to curricular demands that teachers experience in general, and with regard to curricular transition periods in particular; that is, when the transition taking place involves "the inclusion of elements of history of mathematics in new curricula and accompanying textbooks" (p. 180). We approached the three cases of interest with a focus on a dual aspect of HCK. For example, "concrete inclusions of history of mathematics…calls for an already developed [HCK] of a teacher" (p. 174), which can be considered "a priori HCK." Yet this inclusion of history of mathematics in a teacher's practice may itself contribute to [a] teacher's HCK—which might then be referred to as "a posteriori HCK."

The three cases (Denmark, Norway, and the US) discussed in Smestad et al. (2014) each stemmed from concrete directives (yet still considered rhetoric) calling for the inclusion of history of mathematics in school curriculum, and which represented a continuum of curricular demands for teachers in delivering their instruction while heeding the various directives. These particular transitions—for example, shifting the extent to which inclusion of elements of history of mathematics in new curricula or textbooks—impact a teacher's HCK. For example:

> In the transition phase from one curriculum not including elements of history of mathematics to another which does, in-service teachers often lack the associated CCK, KCC[7], etc. And, at this particular time, in this particular transition period while implementing the new curriculum and training in-service teachers, a priori HCK comes to play a more crucial role. (p. 180)

This example highlights the dynamic nature of a model (MKT) for understanding the nature of the practice (and perhaps, the purpose) of mathematics teaching. Furthermore, there is a synergetic

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[6] By 2000, when NCTM issued its new *Principles and Standards for School Mathematics*, what was originally recognized as a goal for learning mathematics was now reduced to identification of a feature of mathematics, *Mathematics as a part of cultural heritage*: "Mathematics is one of the greatest cultural and intellectual achievements of humankind, and citizens should develop an appreciation and understanding of that achievement…" (p. 4).

[7] Common content knowledge (CCK), knowledge of content and curriculum (KCC), etc.

relationship between research on history of mathematics in teacher education and the evolution of models for understanding teacher knowledge. As Jankvist, Mosvold, Fauskanger, and Jakobsen (2015) observed, “the MKT framework provides a powerful language to communicate results from research on the uses of history of mathematics to researchers in other areas of mathematics education research” (p. 495).

Thinking as communicating and HPM research

In an attempt to resolve certain quandaries related to mathematical thinking and learning, Sfard (2008) operationally defined thinking as a personalized version of communication. Given the collective nature of communication, she introduced the term *commognition* to highlight the communicative nature of activities in our minds, emphasizing that individual cognitive processes (thinking) and interpersonal communication are “but different manifestations of basically the same phenomenon” (Sfard, 2008, p. 83). Using this communicative, or discursive lens, Sfard (2008) determined that “mathematics begins where the tangible real-life objects end and where reflection on our own discourse about these objects begin” (p. 129). That is, what identifies the objects of communication in mathematics is their discursive nature: they come to exist as we talk about them. Thus, taken from this viewpoint, mathematics is seen as a highly situated human activity which generates itself. As a result, the learner of mathematics faces an interesting and paradoxical situation: How can a person join a discourse for which familiarity with the discourse is a precondition for participation in that discourse?

Yet further complications exist. Sfard (2008) noted that participation in any discourse requires adopting the rules that govern that discourse, in addition to becoming familiar with the objects of the discourse. She referred to the former rules as *meta-level*, or *metadiscursive*, and the latter as *object-level*. For instance, asserting that a particular function is differentiable constitutes an object-level narrative about functions. However, a student’s method of justifying this assertion (e.g., sketching a graph versus an $\epsilon - \delta$ proof) would be indicative of the metadiscursive rules that govern her discourse about functions. Despite the usual implications of the word *rule* as being invariable and strictly deterministic, metadiscursive rules are subject to change in time and space, and they possess certain characteristics; they are tacit, contingent, constraining, flexible, value-laden, and are difficult to change. Sfard posits that these characteristics render meta-level learning possible only through direct encounters with a new discourse that is governed by meta-level rules different from those governing the learner’s current discourse (p. 256). Furthermore, such encounters generally entail a *commognitive conflict* when the discursants unknowingly operate under completely different meta-level rules.

Given their role in governing the actions of the participants in a mathematical discourse, researchers have paid particular attention to factors that affect the learning of metadiscursive rules in mathematics. In a number of these studies, the history of mathematics, and primary source readings in particular, emerged as an instructional approach with strong potential to promote such learning.

Example from Denmark

In their study of university mathematics students, Kjeldsen and Blomhøj (2012) showed that a careful selection of historical sources can help students learn about the metadiscursive rules that govern mathematicians’ discourse about functions and can allow them to recognize that these rules changed during the development of that concept. This meta-level learning, they argued, fostered students' learning of mathematics at the object-level as well. The authors shared an in-depth analysis of project reports produced by two groups of students, which were based on project work designed and carried out as part of the mathematics bachelor's and master's programs at Roskilde University. The reports result from semester-long work in which students operate within particular project constraints; in the case of the two projects described by Kjeldsen and Blomhøj, these belong to the "mathematics as a discipline" constraint. The projects exemplified were "Physics' influence on the development of differential equations and the following development of theory" and "Fourier and the concept of a function – the transition from Euler's to Dirichlet's concept of function." In their analyses of student projects, Kjeldsen and Blomhøj brought attention to the incongruent discourses of their students when compared to the primary source texts. For example, for the group whose project was "Physics' influence…," students read and studied three original sources from the 1690s:

> In order to answer their questions, the students had to read and understand the sources within the mathematical discourse of the time. On one hand, this is a difficult task because the students' points of departure in dealing with the sources are their own mathematical discourses, which are different from the discourse of the authors of the sources. On the other hand, this is exactly the reason why history, and working with original sources, can serve as an effective method for meta-level learning. (p. 336)

In their discussion of students' project work—for both project examples—Kjeldsen and Blomhøj (2012) beautifully situate the power of primary historical sources to promote students' ability to reflect upon metadiscursive rules of mathematics:

> Didactically, it is important to find and identify historical sources that are suitable for provoking discussion in classrooms among students and with their teachers about different meta-discursive rules. Likewise, it is important to perform research about how this can be done, how teaching activities that support such discussions and reflections can be designed and how the effectiveness of such teaching and learning situations can be evaluated in practice. (p. 347)

Example from Brazil

In a similar way and drawing upon the work of Kjeldsen and Blomhøj (2012) and Kjeldsen and Petersen (2014), as well as Sfard's theory of thinking as communicating, Bernardes and Roque (2018) conducted two experiments with a small group of undergraduate students in a mini-course focused on the topic of "Different roles of the notion of matrix in two episodes of the history of matrices" (p. 219). The course included two teaching modules which introduced students to original source materials from J. J. Sylvester (1814–1897) and Arthur Cayley (1821–1895). In a similar way to Kjeldsen's empirical work with colleagues, Bernardes and Roque's research consisted of three goals, in which they sought to investigate:

(1) how historical sources encourage reflections about metarules related to matrices and determinants;
(2) how reflections about metarules impact students' conceptions about matrices and determinants; and
(3) the development of a historical consciousness in the students. (p. 211)

In their analysis, Bernardes and Roque (2018) found that students were able to discuss and reflect on the historical metarules present in the primary source texts; as well, they identified three metarules in the student participants' discourse. Furthermore, Bernardes and Roque highlighted the occurrence of commognitive conflicts – that is, "conflicting narratives in which the…participants and the historical sources were guided by different metarules" (p. 224). The combination of these outcomes prompted the researchers to question the order in which they teach topics in a Linear Algebra course. For example, they questioned whether it is appropriate to begin such a course with the "concept of a matrix as an object in itself" (p. 226). In their proposal for a future instructional sequence, Bernardes and Roque observed that historical episodes showed that the introduction and development of the concept of matrix was driven by a need for such a representation (e.g., introduction of multiplication of matrices in conjunction with composition of linear transformations). Thus, in addition to the use of primary sources promoting students' reflection of metalevel rules governing their mathematical discourse, such an innovation has the potential for guiding instructional changes which can serve to impact students' mathematical learning.

Example from the United States: The TRIUMPHS Project

As part of a large grant project, several colleagues and I have begun a study to further investigate the potential that "history can have a profound, perhaps even indispensable, role to play in teaching and learning mathematics from the point of view of learning proper meta-discursive rules" (Kjeldsen and Blomhøj, 2012, p. 328). Before describing features of that work, it is perhaps helpful to describe the greater context in which the research is taking place.

In 2015, the National Science Foundation (NSF) in the United States funded a seven-institution collaborative project to design, test, and evaluate curricular materials for teaching standard topics in the university mathematics curriculum via the use of primary historical sources. The goal of the project is to assist students in learning and developing a deeper interest in and appreciation of mathematical concepts by creating educational materials in the form of Primary Source Projects (PSPs) based on original historical sources written by mathematicians involved in the discovery and development of the topics being studied. The project, *Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources*, or TRIUMPHS, is developing PSPs which contain (1) excerpts from one or several historical sources, (2) a discussion of the mathematical significance of each selection, and (3) student tasks designed to illuminate the mathematical concepts that form the focus of the sources. PSPs are designed to guide students in their explorations of these original texts in order to promote their own understanding of those ideas.

The numerous PSPs are indeed the life force of the TRIUMPHS project. During the grant-funded effort, the principal investigators (PIs) promised that some 50 PSPs (which span the undergraduate mathematics curriculum, from basic statistics and trigonometry, to real analysis, abstract algebra, and topology) will be developed, tested, and evaluated. Of the 50 PSPs, 20 are planned to be "full-length" and 30 are what we refer to as "mini-PSPs." Full-length PSPs are designed to typically encompass at least two to four class sessions, which represents the same amount of time that it normally takes to teach the mathematical topic of focus within the PSP. However, among the full-length PSPS there are also longer ones that could be used by instructors to comprise an entire course's content[8]. Alternatively, "mini-PSPs" can be completed in one to two class sessions and each of the mini-PSPs have been developed to teach a particular topic or concept in mathematics that would normally be addressed in a single class session, but which will be done via a primary historical source. To date, 33 full-length PSPs and 28 mini-PSPs have been developed. Though we have exceeded our commitment to develop 20 full-length PSPs, there are additional full-length PSPs in development, as well as the remaining, promised mini-PSPs.

In Fall 2015 the first PSPs were tested[9] in two undergraduate mathematics classrooms in the United States; in Year 3 (academic year 2017–18), 46 distinct site testers tested one or more PSPs in undergraduate mathematic classrooms. However, in total, by the end of Year 3, 53 instructors have site tested PSPs as part of the TRIUMPHS project, with some one-third of those serving as repeat testers. In the first semester of Year 4, we have 20 student data collection site testers; again, of these, we have several repeat site testers, where 13 are new to site testing TRIUMPHS PSPs.

Whereas this progress across almost four years of a large NSF grant project may seem to some as modest, it is important to note that as only one of three such grants ever funded on this level in the United States, this represents significant progress with regard to efforts designed to promote the teaching of undergraduate mathematics via primary historical sources. The two previous grants – to which three of the seven TRIUMPHS PIs and one of the advisory board members contributed – also produced a number of primary source projects. Table 2 summarizes the origin and availability of these projects.

Years	Name of funded project (URL)	Number of projects developed (and tested in classrooms)
2003–2006	Teaching Discrete Mathematics via Primary Historical Sources; Pilot Grant (https://www.math.nmsu.edu/hist_projects/)	14
2008–2012	Learning Discrete Mathematics (LDM) and Computer Science via Primary Historical Sources; Expansion Grant (https://www.cs.nmsu.edu/historical-projects/)	20
2015–present	Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources; Collaborative Grant (http://webpages.ursinus.edu/nscoville/TRIUMPHS.html)	61, to date

[8] In fact, this was done recently (Spring 2018) by Janet H. Barnett, in an Abstract Algebra course.

[9] By "tested" we mean that we collected student data from the implementation of PSPs in these classrooms.

Table 2: PSP development, 2003–2018

There are two features unique to TRIUMPHS when compared to the two previous grants. First, it was difficult to recruit site testers for the projects that were developed. The Pilot Grant (2003–2006) involved only five total site testers and in the Expansion Grant (2008–2012), a total of 10 site testers participated. However, in the TRIUMPHS project we have developed mechanisms to recruit site testers through a variety of outreach efforts, including presentations at conferences, three-day TRIUMPHS-focused workshops (particularly focused on instructors who might not have previously included primary sources in their teaching), mini-courses and short workshops at conferences, and listserv announcements via professional organizations with members who may share interests in history of mathematics and its use in teaching[10]. In advance of each autumn and spring semester we advertise the site testing opportunity and accept applications for two different site tester streams: those who will serve as student data collection site testers and those who will serve as instructor-only data collection sites.

The second feature unique to TRIUMPHS (when compared to the previous grant efforts) is the evaluation-with-research (EwR) component of the grant project. The two previous grants included only an evaluation component, which resulted from the analysis of surveys completed by students at the beginning and end of courses at the particular participating institutions "in any course in which a historical project could be used, regardless of whether a project [was] actually used or not" ("Instructions for testers," LDM, n.d.). The surveys asked students to respond to approximately 30 questions: 18 "Need for Cognition Scale" items, 10 "Understanding Computer Science Scale" (and/or Mathematics, depending upon the course) items, and two open-ended items:

> In your opinion, what are the benefits of learning Mathematics (and/or Computer Science) from historical sources?
>
> In your opinion, what are the drawbacks of learning Mathematics (and/or Computer Science) from historical sources?

Thus, in the evaluation of the funded grant projects just prior to TRIUMPHS, assertions were made only with regard to students' self-reported understanding of mathematics or computer science concepts broadly and beliefs about their problem solving and cognitive efforts. That said, student responses to the two open-ended items did provide the PI team with confirmation that the use of primary historical projects was a worthwhile inclusion in the teaching of mathematics and computer science courses. Typical student responses (LDM, n.d.) include:

> I really enjoyed it. I found it to be very intriguing.
>
> As a student you get to see where the math we do today came from and engage in the kind of thinking that was necessary to create it.
>
> It's a perfect way to given math some context in the world. Historical sources teach you the math while simultaneously fitting math into history and give you meaning for why the math was and is

[10] We have also advertised training opportunities and site testing via inquiry-based learning (IBL) audiences, since mathematics faculty interested in active learning strategies in undergraduate mathematics courses may find PSPs provide useful materials for promoting active learning, particularly in upper-division mathematics courses.

important. Historical sources also break up the monotony of textbooks, making math more accessible for a wider variety of students.

I think that it gives the student a [deeper] understanding of the subject.

The TRIUMPHS Project: Metadiscursive rules investigation (MDRI)

It is important to point out that the EwR component of the TRIUMPHS project was built after the formulation of the main focus; that is, the development and dissemination of the PSPs was the primary focus of the grant effort. Therefore, evaluation questions were constructed that would enable the PI team to report several metrics to the funding agency (in this case, the NSF in the US) regarding the successful completion of the goals and sub-goals of the project. However, developing research questions about what can be learned from TRIUMPHS—given that the design of the development of the PSPs was fixed first—proved difficult in both the development of the grant proposal and the "pilot year" of the project. In particular, the PIs working on the EwR component[11] strongly believe that TRIUMPHS provides a unique opportunity to contribute to mathematics education research more broadly, especially given the potential that a large collaborative grant project affords, including working with a variety of university teaching contexts (e.g., two-year colleges and four-year colleges and universities) and student populations. And, in response to the NSF prior to receiving funding, we highlighted the importance of the development of communication skills—and all modes of this: written, verbal, reading—as part of students' mathematical learning was an area of potential impact. After several iterations, the EwR working group decided that the thinking as communicating framework (Sfard, 2008) would provide the most fruitful lens for our research.

Our investigation builds on prior research concerning the potential of primary source readings for mathematics education (e.g., Bernardes & Roque, 2018; Kjeldsen & Blomhøj, 2012) that has been conducted within the framework of Sfard's participationist theory of "learning as discourse" (Sfard, 2008). In particular, we focus on the role played within that framework by the metadiscursive rules which govern the actions of the participants in a mathematical discourse. When considering the research literature available which contains similar emphases with regard to using primary historical sources in the teaching and learning of mathematics, we have been empowered with a strong conviction—as have others—that, under the right conditions, the use of history does promote the learning of metadiscursive rules in mathematics. Our goal, within the EwR component of the TRIUMPHS project, is to contribute to the important work of identifying what occurs for student learning under what conditions, work which is important to both the researcher who is interested in, for instance, how the learner is thinking along the way, and to the practitioner, for whom the educational setting that motivates meta-level learning opportunities for students is of paramount importance.

Thus, we launched a metadiscursive rules investigation (MDRI), in which we posed the following research questions:

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[11] There are actually three foci of the EwR component of TRIUMPHS: "student change," "faculty expertise," and "development cycle."

> What is the evidence of students' progress in "figuring out" (Sfard, 2014, p. 201) the meta-level rules that govern a new mathematical discourse as a result of studying specific mathematical concepts when using primary source projects?
>
> To what extent do students' actions (e.g., verbal, written), both during and after engagement with the primary source projects, provide evidence of their acceptance of a new discourse?

The construction of our research questions was heavily influenced by Sfard's (2014) observation that university mathematical discourse is "far removed from what the student knows from school as a discourse can be" (p. 200). Thus, in our work, we investigate an alternative to lecturing that makes use of the history of mathematics (e.g., via PSPs) in order to provide a learner with the opportunities for "watching a mathematician in action and imitating his moves while also trying to *figure out* the reasons for the strange things he is doing" that Sfard suggests "may be the only way to come to grips with [the] objects [that she is supposed to operate upon]" (Sfard, 2014, p. 202; emphasis added).

We are also strongly influenced by the agenda articulated by Kjeldsen and Blomhøj (2012):

> Didactically, it is important to find and identify historical sources that are suitable for provoking discussion in classrooms among students and with their teachers about different meta-discursive rules. Likewise, it is important to perform research about how this can be done, how teaching activities that support such discussions and reflections can be designed and how the effectiveness of such teaching and learning situations can be evaluated in practice. (p. 347)[12]

In particular, our MDRI research draws upon three semesters of undergraduate mathematics instruction that took place at one institution during the Autumn 2016 (Introduction to Analysis; 11 consenting students), Spring 2017 (Number Theory; 8 consenting students), and Spring 2018 (Abstract Algebra; 15 consenting students). The contexts, student populations, and PSPs used are somewhat different across the three semesters[13]. However, the data sources were similar in each instance, and included:

- Video recordings of all class sessions;
- Audio recordings of each group during small group work;
- Students' written work on all PSPs implemented during the courses and related "Reading and Study Guides" (RSGs);
- Instructor class notes;
- Pre- and post-PSP student interviews[14]; and
- Responses to four surveys per student (pre- and post-course surveys, and two post-PSP surveys).

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[12] This passage was previously quoted in this paper, but it bears repeating here because it is particularly critical for the MDRI work as part of TRIUMPHS.

[13] However, there was a small subset of students (8) who participated in two of the three courses.

[14] Not all consenting students were interviewed pre- and post-PSP, due to students' class and work schedules.

An example from the Autumn 2016 course (*Introduction to Analysis*), which contained source material "suitable for provoking discussion in classrooms among students and with their teachers about different meta-discursive rules" (Kjeldsen & Blomhøj, 2012, p. 347), was the first PSP of the semester (and which students met during the second week of the course): *Why Be So Critical: Nineteenth Century Mathematics and the Origins of Analysis* (Barnett, 2017a). This project explored the question of: *Why, after nearly 200 years of success in the development and application of calculus techniques, did 19th-century mathematicians feel the need to bring a more critical perspective to the study of calculus?* – and did so through selected excerpts from the writings of the nineteenth century mathematicians who led the initiative to raise the level of rigor in the field of analysis (Barnett, 2017a, p. 1).

The project includes excerpts from four mathematicians: Bolzano, Cauchy, Dedekind, and Abel. In the PSP, Barnett (2017a) provided oriented students to the various primary sources with: "…these mathematicians expressed their concerns about the relation of calculus (analysis) to geometry, and also about the state of calculus (analysis) in general. As you read what they each had to say, consider how their concerns seem to be the same or different" (p. 1). For example, Figure 1 displays an excerpt from Dedekind.

Richard Dedekind, 1872, Stetigkeit und irrationale Zahlen (Continuity of irrational numbers)

> My attention was first directed toward the considerations which form the subject of this pamphlet in the autumn of 1858. As professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic[7]. In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question until I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis.

Figure 1: Excerpt from Barnett, 2017a (source: Dedekind, 1901)

In keeping this excerpt in mind, we found evidence that such sources can prompt classroom discussion around the metadiscursive rules that we see at the time of Dedekind—which constituted a shift from what had been in place—and that are different still from what students were trying to reconcile.

Some weeks later—approximately seven weeks into the course—students spent approximately two weeks working on the second PSP in this analysis course: *Rigorous Debates over Debatable Rigor: Monster Functions in Introductory Analysis* (Barnett, 2017b). In the project, students are introduced to the correspondence between Gaston Darboux (1843–1917) and Jules Houël (1823–1886), in which

Houël has requested feedback on early drafts of his intended textbook on differential calculus. Throughout the correspondence, however, Darboux "offered various counterexamples in a (vain) attempt to convince Houël of the need for greater care in certain of his (Houël's) proofs" (Barnett, 2017b, pp. 2–3). Examples of this correspondence are provided in Figures 2 and 3.

> Here is what I reproach in your reasoning which no one would now find rigorous. When we have
>
> $$\frac{f(x+h)-f(x)}{h} - f'(x) = \epsilon,$$
>
> ϵ is a function of two variables x and h that approaches zero when, x remaining fixed, h approaches zero. But if x and h [both] vary as they do in your proof, or worse yet, if to each new subdivision of the intervals $x_1 - x_0$ there arise new quantities ϵ, then I find it altogether unclear and your proof has nothing but the appearance of rigor. [Darboux, as quoted in (Gispert, 1983, p. 99)]

Figure 2: Excerpt (A): Darboux correspondence with Houël (Barnett, 2017b, p. 6)

> Yes, I admit as a fact of experience (without looking to prove it in general, which might be difficult) that in the functions that I treat, one can always find h satisfying the inequality $\frac{f(x+h)-f(x)}{h} - f'(x) < \epsilon$, no matter what the value of x, and I avow to you that I am ignorant of what the word derivative would mean if it is not this. ...I believe this hypothesis is identical with that of the existence of a derivative. [Houël, as quoted in Gispert 1987, pp. 56 – 57].

Figure 3: Excerpt (B): Houël correspondence with Darboux (Barnett, 2017b, p. 6)

Two project tasks related to these excerpts were:

> Do you agree with Houël about this being what the word 'derivative' means? Why or why not?
>
> How does what Darboux said in the excerpt (A) seem to be different from what Houël is saying here [excerpt (B)]?

The intensive work on how to analyze the data in order to address our original research questions is just beginning. We produced a preliminary report on an initial discussion of methodological issues that we experienced in our first review of the Autumn 2016 data. In our report (Can, Barnett, & Clark, 2018), we addressed two questions:

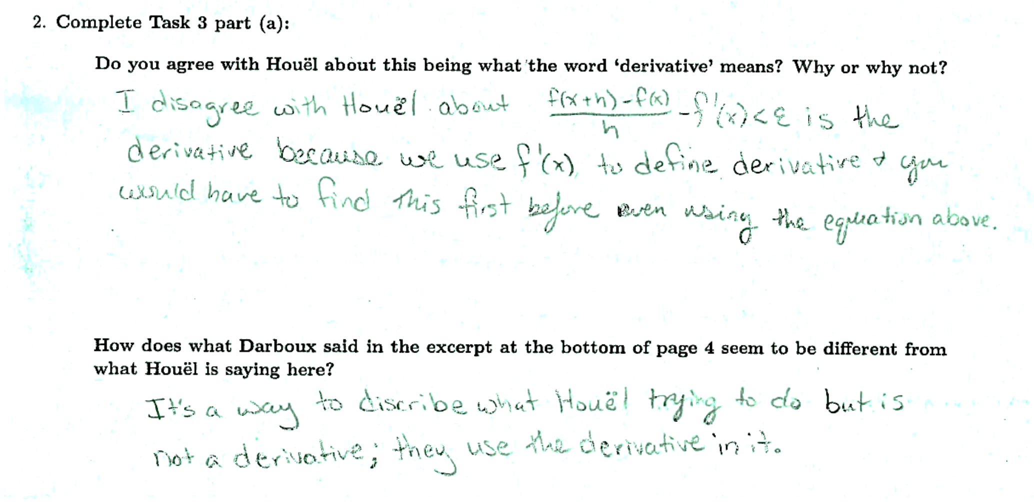
1. How can we characterize the nature of students' participation in mathematical discourse in their written work related to primary source projects?
2. What constitutes evidence of students' noticing of meta-level rules in this written work?

Since our research report was quite preliminary (and page-limited), we focused on analyzing students' written work from just one PSP[15] (and associated RSGs) that was implemented in the *Introduction to Analysis* course we studied in Autumn 2016. We sought to document evidence of

---

[15] For the purposes of the preliminary report, we focused on the PSP, *Rigorous Debates over Debatable Rigor: Monster Functions in Introductory Analysis* (Barnett, 2017b).

students' noticing of metadiscursive rules in the form of meta-level reflections of two kinds: on either the mathematical objects under discussion (what we called *object-reflection*) or the discourse itself (what we called *discourse-reflection*). In the sample student's reflection (given in Figures 4a, 4b, 5a, and 5b), we identified the student's 'talk' as object-reflection, in which she provided meta-level narrative about mathematical objects (i.e., derivative).


2. Complete Task 3 part (a):

Do you agree with Houël about this being what the word 'derivative' means? Why or why not?

I disagree with Houël about $\frac{f(x+h)-f(x)}{h} - f'(x) < \epsilon$ is the derivative because we use $f'(x)$ to define derivative & you would have to find this first before even using the equation above.

How does what Darboux said in the excerpt at the bottom of page 4 seem to be different from what Houël is saying here?

It's a way to discribe what Houël trying to do but is not a derivative; they use the derivative in it.

Figure 4a: Student object-reflection (meta-level)

Do you agree with Houël about this being what the word 'derivative' means? Why or why not?

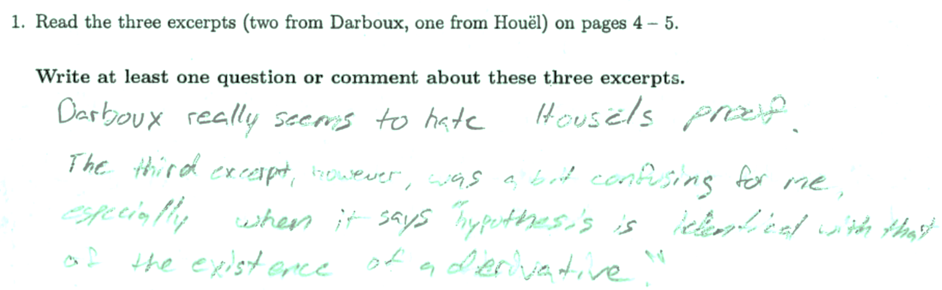
I disagree with Houël about $\frac{f(x+h)-f(x)}{h} - f'(x) < \epsilon$ *is the derivative because we use* $f'(x)$ *to define the derivative & you would have to find this first before even using the equation above.*

How does what Darboux said in the excerpt at the bottom of page 4 seem to be different from what Houël is saying here?

It's a way to [describe] *what Houël* [is] *trying to do but is*

not a derivative; they use the derivative in it.

Figure 4b: Transcription of task and student response given in Figure 4a


1. Read the three excerpts (two from Darboux, one from Houël) on pages 4 – 5.

Write at least one question or comment about these three excerpts.

Darboux really seems to hate Houël's proof.
The third excerpt, however, was a bit confusing for me, especially when it says "hypothesis is identified with that of the existence of a derivative."

Figure 5a. Student discourse-reflection (meta-level)

Write at least one question or comment about these three excerpts.

Darboux really seems to hate Houël's proof.

The third excerpt, however, was a bit confusing to me,

Especially when it says "hypothesis is identified with that

of the existence of a derivative."

Figure 5b: Transcription of task and student response given in Figure 5a

We intended the further application of our definitions for object-reflection and discourse-reflection to serve as tools to characterize the nature of students' participation in mathematical discourse in other course artifacts (e.g., small group work, whole-class discussion, interviews), but we have since found that a we need an analytical framework that includes at least two components. One component of the analytical framework is the set of metadiscursive rules that we have identified for each of the PSPs that students used in a given course; it is imperative that our analysis attends to and is informed by the relevant metadiscursive rules present in the PSPs used. The second component is the criteria for evidence of students' "figuring out" the meta-level rules governing a new mathematical discourse. We believe these two components in tandem will enable us to capture a critical perspective in the work surrounding the role of primary sources (and in this case, specifically, the PSPs); that is, the implemented PSPs were intended to promote the mathematical learning goals of a given course. And, given the nature of the *Introduction to Analysis* course curriculum, these included both object- and meta-level learning goals. That said, the data analysis for the research described here is ongoing, and our particular struggle at the moment is figuring out what we mean by "figuring out," as suggested by Sfard (2014, p. 202), in order to move forward through our data to determine not only the progress made by students but what such progress could signal for changing instructional practice at the undergraduate level. This work is complex and draws upon multiple perspectives from mathematics, history, and mathematics education, and consequently, possesses ample opportunity for future collaborations and contexts in mathematics education research.

Calls for the future: Future contributions

Need for collaboration

There has been exciting progress in research conducted in the HPM domain in the last 40 years, and in the last 20 years, this is particularly true. As I shared earlier in this paper, at some point not so many years ago, empirical work (available in the English language) in the field of history in mathematics education was predominantly anecdotal in nature. With the growth of professional conferences—HPM, ICME, ESU, and now CERME—collaboration with colleagues around the world has not only afforded but has increased the demand for ways in which research on history in mathematics education can inform and be informed by research in mathematics education more broadly.

To this end, I would like to end with proposing two areas of research that I believe are particularly important and interesting (and necessary?), which draw upon the themes of different CERME thematic working groups and which present opportunities for fruitful collaboration in the future.

History of mathematics in mathematics teacher education

An overarching question that requires careful and thorough study is:

How does a historical perspective contribute to the mathematical and pedagogical development of mathematics teachers (at all levels, and both pre-service and in-service)?

And, there are numerous questions it motivates, including:

What are the different ways in which history of mathematics is used in the education of teachers?

What are the different challenges (e.g., historical, mathematical, attitudinal, philosophical, methodological, institutional) for each?

Are the outcomes of teachers' study of history of mathematics seen in their classroom practice in explicit ways, and if so how? If not, why? Are the implicit ways equally meaningful?

Although these questions represent only a small sample of what is yet unknown to any extent in the HPM domain, the collaboration with other domains represented by CERME TWGs could provide the means to increase efforts to conduct research in concerted ways. Some of the TWGs, in addition to contributions from TWG 12, well poised to do so include:

TWG 18: Mathematics Teacher Education and Professional Development

TWG 19: Mathematics Teaching and Teacher Practice(s)

TWG 20: Mathematics Teacher Knowledge, Beliefs and Identify

TWG 22: Curricular Resources and Task Design in Mathematics Education

TWG 23: Implementation of Research Findings in Mathematics Education

History of mathematics in the teaching and learning of mathematics

Research on the many ways in which history of mathematics can be used in the teaching and learning of mathematics seems boundless. There are many open questions in the field, yet when considering educational standards set by different countries around the globe and the persistent (primary and secondary) teacher lament that "there is not enough time to teach history of mathematics"[16] in mathematics lessons, addressing the questions through different research efforts can be problematic. Part of the issue with conducting research on the use of history in teaching and learning mathematics is the need to make clear the potential for student learning that research in the HPM domain has shown. Thus, a question of particular interest is: *How do we encourage, enable, and enlighten large-scale research on the ways in which using history of mathematics in teaching impacts learning?*

Similar to proposing research on history of mathematics in mathematics teacher education, inquiry on teaching and learning of mathematics informed by history of mathematics can be addressed by the expertise represented within several CERME TWGs. For example, for research focused on how

[16] Many classroom teachers perceive of "using history" as being equivalent to teaching the history of mathematics. However, this is often a misguided idea. That is, many who propose to use history of mathematics call for robust ways to do so, such as using historical methods to solve problems and to help make sense of the procedures many students find difficult, or reading, interpreting, and applying historical sources in learning mathematics, and not to simply teach who was the first to discover a particular mathematical concept.

history of mathematics can contribute to student learning and engagement with mathematics, collaboration amongst the following TWGs (again, in addition to TWG 12) is valuable:

TWG 1: Argumentation and Proof

TWG 2: Arithmetic and Number Systems

TWG 3: Algebraic Thinking

TWG 4: Geometry Teaching and Learning

TWG 8: Affect and the Teaching and Learning of Mathematics

TWG 14: Undergraduate Mathematics Education

Furthermore, investigating ways in which history of mathematics can connect tools and technologies in the teaching and learning of mathematics may benefit from the expertise of:

TWG 6: Applications and Modelling

TWG 9: Mathematics and Language

TWG 15: Teaching Mathematics with Technology and Other Resources

TWG 16: Learning Mathematics with Technology and Other Resources

And, finally, under the purview of TWG 21 (Assessment in Mathematics Education), there are considerable opportunities to address concerns held by many regarding whether teaching mathematics informed by history of mathematics *actually* contributes to student learning.

An additional consideration: Flipping the research perspective

A large proportion of research conducted on questions regarding the use and impact of history of mathematics has been focused either pre-service or in-service teachers (at the elementary and secondary level) and their students. However, investigating another population of teachers is a promising direction to pursue: teachers of teachers (e.g., university educators). Povey (2014) conducted research conversations with four university instructors designed to address the following research question: What can studying the history of mathematics with initial teacher education students offer us? (p. 148). In her analysis, Povey determined four broad themes, two which seem aligned with affective dimensions and two which are situated more with content and mathematical understanding. The thematic categorizations of the instructors' responses were:

- to deepen mathematical understanding;
- to broaden and humanize mathematics;
- to develop critical thinking; and
- to provide motivation and fun for learners. (Povey, 2014, p. 148)

Povey (2014) provided the foundation for what I believe could promote research that focuses on educators of teachers, who can provide opportunities "for studying history of mathematics [that] sets up a productive relationship with the subject and deepens mathematical understanding" (p. 154). From application and further extensions of existing frameworks such as MKT and Mathematical Knowledge for Teaching Teachers (MKTT; see for example Jankvist, Clark, & Mosvold, 2019), mathematics education research and history in mathematics education can both capitalize on what many in the field *know* (as in, instinctually know): that historical content, problems, and perspectives in mathematics teaching "requires the development of such critical skills and can develop disposition towards enquiry based on questions posing and evidence" (Povey, p. 155). However, it is imperative that researchers seek to legitimize what has been known for decades, and to do so in concrete and robust ways so that mathematics educators, teachers, and learners can benefit from ways of knowing mathematics to which history of mathematics uniquely contributes. An interesting "flipping" of the research perspective begins with extending research that Povey began with teacher educators, and from what is learned, it is possible to build on these new perspectives coupled with guidance from existing frameworks to develop new knowledge. We may find, as Jo (one of the teacher educators) did:

> [finding out about the history of mathematics] has made me realise that there are many more questions to ask than I ever thought about before and there's probably no end to that, and I think that's a good thing for maths teachers to know. (Povey, 2014, p. 155)

More importantly, once teacher educators experience this shift, it can permeate their practice with pre-service teachers, which can in turn be impactful in *their* future practice. The greatest imperative, however, is that we must make research in history in mathematics education part of the research landscape, as much as say, how the field has investigated the educational benefit of use of technology in teaching and learning mathematics, or ways to improve concept building in learning algebra.

Wanted: A few good researchers

As previously described, there are numerous perspectives from which researchers can approach important questions regarding teaching and learning of mathematics, of which an historical perspective is just one. Though empirical literature in the field of history of mathematics in mathematics education is much more prevalent today than some 40 years ago, there are several approaches, frameworks, and methodological lenses that are can and should be employed in order to strengthen and expand current examples.

In addition to the examples I have provided, there are other calls for research that have recently been issued and investigated to some degree. However, the potential for future research is significant and importantly, within the field of history in mathematics education, there are applications across age of learners, level of teachers, and mathematical concepts. For example, there are examples for which design-based research seems well-suited, as in the study described by Wang, Wang, Li, and Rugh (2018), in which they proposed a framework to make sense of "how to help teachers…who lack experience in [integrating the history of mathematics in teaching] IHT, use historical materials in their teaching" (p. 135). Wang et al. concluded that

> Although the framework provides a new pathway for teachers' professional development in IHT and a new opportunity for the theoretical development of IHT, *it still requires further empirical studies to confirm its educational value in the future. It is also essential that researchers closely collaborate with teachers and historians* as suggested by the dynamic pyramid model. (p. 153, emphasis added)

In closing, I have attempted to provide a broad landscape of questions, approaches, and collaborative contexts in which educational research of interest to the CERME community is possible and motivated by the field of history of mathematics in mathematics education. I challenge mathematics education researchers to embrace and pursue these questions, approaches, and contexts and to contribute to the expanding perspectives that move teaching and learning of mathematics forward.

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