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Extant research on the teachers’ knowledge includes limited studies focusing on teachers at university level. In this work, based on the Mathematics Teacher’s Specialised Knowledge (MTSK) model and through an instrumental case study, knowledge of a lecturer in a real analysis course for prospective mathematics teachers is analyzed. We exemplified lecturer knowledge in different subdomains of the MTSK model. These results contribute to the understanding and characterization of the components of mathematics teacher’s specialised knowledge at the university level.

Keywords: lecturer knowledge, specialised knowledge, university level, real analysis, real numbers.

Introduction

The mathematics education research focusing on the teachers’ knowledge has traditionally been conducted at the elementary and secondary level. However, the study of mathematics lecturer’s knowledge has recently emerged as a line of research that seeks to understand this knowledge, its development, and how it is reflected in university teaching practices (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016). Among the scarce account of research about mathematics lecturer’s knowledge, the work of Breen, Meehan, O’Shea, and Rowland (2018) is a first approximation of teaching at the university level using Knowledge Quartet model (Rowland, Huckstep, & Thwaites, 2005). In addition, the studies conducted by Vasco, Climent, Escudero-Avila, and Flores-Medrano (2015) and Vasco and Climent (2017) highlight the utility of the Mathematics Teacher’s Specialised Knowledge model (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013) and its corresponding analytical categories for understanding mathematics lecturers’ knowledge. Other research in the same line, indicate that it is important to obtain empirical evidence pertaining to the components of specialised mathematics lecturer’s knowledge (Delgado-Rebolledo & Zakaryan, 2018) and deepening the understanding of these components in situations other than lecturers’ classroom practices (Vasco & Climent, 2018).

Taking into account this background, we propose the following research question: Which knowledge, according to the Mathematics Teacher’s Specialised Knowledge model, arises in the teaching practice of a mathematics lecturer? Teaching practice is considered in a broad sense, including lesson planning, liaising with colleagues, giving lessons, and taking time to reflect on them afterwards (Carrillo et al., 2018). In this sense, with the aim of answering the research question, we have studied one mathematics lecturer teaching a real analysis course in a mathematics teachers’ training program, and reflecting on his performance. In this paper, we present examples of the different subdomains of this lecturer’s specialised knowledge. We do not consider his characteristics as a mathematics teacher educator. A discussion about mathematics teacher educator’s knowledge is exposed in the paper of Almeida, Ribeiro and Fiorentini (in this volume).

Theoretical Framework

In the Mathematics Teacher’s Specialised Knowledge (MTSK) model, two domains of teacher knowledge are distinguished, Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK). The model also considers teacher’s beliefs about mathematics and about mathematics teaching and learning (Carrillo et al., 2018).

MK refers to teacher knowledge of mathematics within the educational context, considering some of its characteristics as a scientific discipline. MK includes the Knowledge of Topics (KoT), Knowledge of the Structure of Mathematics (KSM) and Knowledge of Practices in Mathematics (KPM) subdomains. KoT contains knowledge about definitions, properties and their foundations, procedures, registers of representation, phenomenology, and applications (e.g., knowledge of the field properties of rational numbers; knowledge of definitions of real numbers). KSM encompasses knowledge of connections among mathematical items: connections associated with an increase in complexity or with simplification, and inter-conceptual connections (e.g., knowledge of relationships between infinity and the Archimedean property of the real numbers). KPM comprises knowledge about demonstrating, justifying, defining, making deductions and inductions, giving examples, and understanding the role of counterexamples (e.g., knowledge of how to prove the density property of the rational numbers in real numbers).

On the other hand, PCK is a specific type of knowledge of pedagogy in which the mathematical content determines the teaching and learning that takes place. PCK includes the Knowledge of Mathematics Teaching (KMT), Knowledge of Features of Learning Mathematics (KFLM), and Knowledge of Mathematics Learning Standards (KMLS) subdomains. KMT includes knowledge of theories of mathematics teaching, teaching resources, and strategies, techniques, tasks, and examples (e.g., knowledge of particularities of a real analysis textbook that makes it more convenient than others for use in a course, or use of analogy to illustrate the features of the existential quantifier). KFLM comprises the knowledge of theories of mathematical learning, strengths and weaknesses associated with learning, ways in which students interact with mathematical content, and emotional aspects of learning mathematics (e.g., awareness of analysis being more difficult for students than calculus; knowledge of difficulties students encounter when working with real numbers). Finally, KMLS contains the knowledge of sequencing of topics, expected learning outcomes, and the expected level of conceptual or procedural development (e.g., the sequencing of the completeness theorem topics, characterization of the greatest element, and the Archimedean property of the real numbers).

Methodological aspects

In this research, based on an interpretive paradigm and a qualitative methodology, an instrumental case study (Stake, 1995) was conducted. The case pertains to a lecturer and mathematics researcher that was developing a real analysis course. Real analysis is a second-year course in a mathematics teachers’ training program in a Chilean university. In the first two years, the students take calculus, algebra, and geometry courses, whereas, from the third year, the courses focus on teaching practices, general pedagogy, and didactics of mathematics (numerical systems, functions, geometry, and statistics).

The lecturer, who will be called Diego, has more than 20 years of teaching experience at the university level and this is the sixth time in recent years that he has developed the real analysis course. Diego's classes are of 90-minute duration, and each was videotaped, transcribed, and organized in class episodes according to Diego's tacit or explicit goals. For example, we consider a *class episode* the period since Diego begins until he finishes presenting a definition. In each episode, Diego's interventions that show knowledge according to the MTSK model (a knowledge that could be classified in an analytical category of some subdomain of the model) are chosen as *analysis units*. The analysis units that allow us to affirm the presence of a teacher knowledge were named *evidence*. Others, where we suspected the existence of teacher knowledge, but additional information was needed in order to confirm or refute this suspicion, were denoted as *indication*. An indication provided a reason to investigate in more detail the lecturer's knowledge. Hence, the differentiation between evidence and indication (Moriel-Junior & Carrillo, 2014) is considered with the aim to refine our interpretations and deepen the understanding of the subdomains of lecturer's knowledge.

The data obtained through video recordings was complemented with a semi-structured interview that was divided in two sessions for a total duration of three hours. The interview was audio-recorded and subsequently transcribed, reproducing Diego's speech with the highest fidelity possible. A template of questions was constructed to prompt Diego to think about some of his expressions and performances during the classes. For example, in one of Diego's classes, he referred to a YouTube video as a complementary material. Thus, in the interview, we asked Diego why he used that video and what was its intended objective. In addition, some questions were formulated with the intention to examine indications of knowledge and to explore the subdomains of specialised knowledge that has not been present in class episodes. Video clips of the classes were shown to stimulate Diego's recollection, because the interview was conducted once the course was completed. Transcriptions of the interview were analyzed in a similar manner to classes, considering each response to a particular question as an analysis unit.

Results

In this section, we exemplify with evidence the subdomains of Diego's specialised knowledge during a class session on real numbers. Diego starts this class by enunciating some concepts and properties studied in the previous class, such as the least-upper bound property and the Archimedean property of real numbers. Next, he uses these elements with the aim to construct, together with the students, the density of the rational and irrational numbers in the real numbers proof.

Diego's mathematical knowledge

As a part of his KoT, Diego knows axioms for the real numbers, constructions of this numerical system (e.g., Cauchy sequences or Dedekind cuts), definition of real numbers as a complete ordered field, and properties of real numbers. For example, Diego *knows the Archimedean property of the real numbers* (KoT) because he enunciates the property ($\forall\, x, y \in \mathbb{R}\,, x > 0, \exists n \in \mathbb{N}$ such that $nx > y$) and comments on that in the following way:

Diego: And, this property, which was not specified by Archimedes, but Euclid, is equivalent to saying that, for all small natural number ε, there exists a natural number n such that $1/n \leq \varepsilon$.

Later, Diego writes on the blackboard the following proposition: Let $a_n < b_n$, and $I_n = [a_n, b_n]$, if I_n is a set of closed and bounded intervals, there exists $x \in \mathbb{R}$, such that $a_n < x < b_n$, $\forall\, n \geq 1$. Next, he makes a comment regarding this statement.

Diego: This proposition indicates that, every time that I intercept closed and bounded intervals, that are nested… I forgot to say that they have to be nested, if not, it is not true. If the intervals are not nested, then the intersection is empty. For example, if you take the [0,1] interval and later the [10,12] interval, in the intersection you will have nothing because they are not nested. Then, to say that they are nested, I add to the proposition [writes on the blackboard, $\forall\, I_n \supset I_{n+1}$]

Diego *knows the nested intervals property* (KoT), and deepening his discussion on the property, he emphasizes that the intervals must be nested in order to have an intersection that is not empty. Also, he understands how this sufficient condition gives sense to the implication expressed in the property. This *knowledge of use of formal language* as a way of communicating the mathematical idea expressed in the nested intervals property, belongs to the KPM subdomain.

On the other hand, starting from an indication of Diego's knowledge about the importance of the density of $\mathbb{Q}$ in $\mathbb{R}$ property, in the interview, when we asked Diego about the meaning of this property, he responded as follows:

Diego: The density is everywhere, because you cannot do anything if you do not have a dense and numerable set. The fact that the rational numbers are dense in the real numbers is very important, because you can take any real number and approximate it by a rational number. The real numbers have a cardinality greater than aleph 0, so you cannot count using them…Then, in this process of approximation, the rational numbers are important because, in practice, our calculations are limited to rational numbers.

In this excerpt, Diego demonstrates his *knowledge of a connection to items within the same topic* (KoT) because he points out that properties of $\mathbb{Q}$ such as its density in $\mathbb{R}$ and its numerability are essential to work approximation and calculation processes with real numbers. When the lecturer said, "*in practice, our calculations are limited to rational numbers*", he referred to the processes mentioned above, which are also important in applications of mathematics, such as in modelling or numerically solving differential equations. In this sense, Diego's knowledge of uses and applications of properties of $\mathbb{Q}$ is identified as a part of his KoT subdomain.

Likewise, Diego refers to the density property of the rational and irrational numbers in real numbers. Given an interval $[a, b] \in \mathbb{R}$, it holds $\mathbb{Q} \cap [a, b] \neq \emptyset$ and $\mathbb{R}/\mathbb{Q} \cap [a, b] \neq \emptyset$. Diego particularizes the proposition for the case of an interval with $a = 0$ and $\varepsilon = (b - a)/2$. Using the Archimedean property, he establishes the following lemma: Given $\varepsilon > 0$, and $n, m \in \mathbb{N}$, the interval $(0, \varepsilon)$ contains a rational number $1/n$ and an irrational number $\sqrt{2}/m$. Hence, Diego shows his *knowledge of establishing preliminary results to facilitate the development of* the density of rational numbers in real numbers proof. This knowledge of a way of proceeding in mathematics is a part of Diego's KPM. Moreover, Diego considers the case of a positive rational number a, extending his previous arguments, and demonstrates his knowledge of *process of particularization and generalization of a proposition* about real numbers (KPM) as a way of proceeding in mathematics.

Continuing with the development of the density of the irrational numbers in real numbers proof, the lecturer expresses:

Diego: Using the previous lemma, there exists a number z, real and not rational number, such that z is between 0 and ε. Then, if I add a to this inequality, I have [writes on the blackboard $a < a + z < a + \varepsilon < b$]. a is a rational number and z is an irrational number, then, where is $a + z$? . . . If I can argument that $a + z$ is not in $\mathbb{Q}$, then I can find an irrational number in $[a, b]$. Why is $a + z$ not in $\mathbb{Q}$? Because, if $a + z$ was a rational number and I add another rational number, I will have to obtain a rational, because the rational numbers are a field. So, when I add two rational numbers, the answer is a rational number. Then, what can I add to $a + z$, conveniently, to get a contradiction?

Student: $-a$

Diego: Ok, if I add $-a$ to the inequality, the addition $-a + a + z$ is equal to z. That should be a rational, but I know that this is not true, so $a + z$ cannot be a rational number. Ok, I win.

In this episode, lecturer's knowledge of ways of validating in mathematics is identified. Diego demonstrates his knowledge of *how proofs by contradiction method are done* (KPM). The lecturer understands the logic underpinning this method of proof because he exposes what should be assumed ($a + z$ is in $\mathbb{Q}$), how a contradiction is constructed (adding -a conveniently considering properties of the rational numbers) and what must be concluded ($a + z$ is not in $\mathbb{Q}$).

Diego's pedagogical content knowledge

Regarding Diego's PCK, in the class, Diego referred to the textbooks that he uses to develop the course. Later, in the interview, Diego elaborated on the reasons behind this literature selection.

Diego: In the real analysis course, I use the Spanish edition of a famous textbook; the original edition is in Portuguese, but in Spanish, it has two editions, detailed and summarized. I do not use the detailed version because that textbook has a lot of information. Instead, I use the summarized edition that provides the most essential parts and, if I am lacking something, then I complement it with other textbooks.

In the exposed fragment, we observe that Diego not only knows both editions of the analysis textbook (detailed and summarized), he also knows the specific characteristics of each one, which allows him to select the textbook that makes it more convenient to develop the real analysis course. This knowledge of the *teaching resource* belongs to Diego's KMT subdomain.

Likewise, Diego exposes the reasons behind the use of YouTube videos as a complementary material to the classes.

Diego: I like these videos because they are produced by the author of the textbook that we use in the course. The teacher worked in a prestigious university in Brazil and these videos are from a course intended for postgraduate students. The course duration is two months. I present the course [the same content] in a semester, with more time and I share the videos with the students, although the videos are in Portuguese and the language could be a problem.

Diego highlights the advantages of YouTube videos (they are created by the textbooks' author and he can develop the content in more detail). However, Diego knows that this digital resource has the limitation of language that is unfamiliar for some students. In this sense, Diego shows his knowledge of the advantages and limitations associated with the videos as a *digital teaching resource*. This knowledge is included in Diego's KMT subdomain.

On the other hand, an indication of Diego's knowledge was confirmed in the interview when Diego talked about students' understanding of the real numbers.

Diego: The trouble with real numbers is the least-upper bound property, a historical difficulty that comes from Greeks. To humanity, it took more than 2000 years to comprehend the continuum . . . and maybe more, because I say until Newton, but actually, the formulation comes from Weierstrass. Then, this difficulty, this epistemological obstacle, inevitably emerges when you talk about real numbers.

Diego knows *students' weaknesses pertaining to the understanding of real numbers* (KFLM). He explains this issue considering historically intrinsic difficulties in conceptualizing real numbers due to the presence of the continuum as an epistemological obstacle.

Linked with the above, Diego comments:

Diego: The other issue is that, in high school, real numbers are reduced to an algorithmic point of view only. I do not claim that this is wrong, but it should not be restricted to only that.

In the previous statement, Diego shows his knowledge of how are taught real numbers in high school. He refers to "*an algorithmic point of view*" highlighting deep level regarding the approach given to real numbers in learning standards in some Chilean high schools. This knowledge of *the expected level of procedural development* belongs to Diego's KMLS subdomain.

Final remarks

In this research, we have shown some examples of how the mathematics teacher's specialised knowledge could be identified and characterized in the case of a mathematics lecturer. In the pedagogical content knowledge, we exemplified the three subdomains (KMT, KFLM, and KMLS) and in mathematical knowledge we exemplified the KoT and KPM subdomains. Evidence of KSM was not supported by our findings in line with the results reported by Vasco et al. (2015). However, we obtained some indications of knowledge which supply several ideas to deepen in the KSM in the case of mathematics lecturers. Furthermore, descriptors of KPM regarding ways of communicating and ways of validating in mathematics are exposed in this work. Descriptors about ways of proceeding and ways of validating were reported by Delgado-Rebolledo and Zakaryan (2018), although the descriptors' grouping into categories is still under study. Establishing categories of KPM is an important topic of research in the MTSK model, and descriptors obtained in this study focusing on one mathematics lecturer are a good starting point for investigating components of this subdomain in teachers at other educational levels, given that scarce evidence of KPM has been reported in the research literature (Zakaryan & Sosa, 2019).

Additionally, in some examples, we observed possible relationships among the subdomains of knowledge. For instance, when the lecturer exposes the nested intervals property, his KPM regarding use of formal language allows him to understand this property in his KoT. Also, in the lemma used to develop the density of the irrational numbers in the real numbers proof (KPM), the lecturer relies on his knowledge of the Archimedean property (KoT). In a similar manner, relationships within pedagogical content knowledge domain could be established. For example, the lecturer ascribes students' weaknesses to the epistemological obstacle associated to real numbers, which allows him to understand the procedural approach to this notion in high school. Then, when the lecturer refers to the approach to real numbers adopted in high school (KMLS), he also demonstrates his awareness of students' weaknesses when working with real numbers (KFLM).

Thus far, relationships between disciplinary knowledge and pedagogical content knowledge have been reported in studies of secondary teachers (e.g., Sherin, 2002). We propose that this type of relationships could also be established in the knowledge of mathematics lecturer. These relationships are an opportunity to investigate the development of lecturer's mathematical and pedagogical knowledge.

In line with the above, the identification of the mathematics lecturer's specialised knowledge contributes to understand mathematics lecturers' knowledge and how this knowledge is reflected in their teaching practice (Biza et al., 2016). However, more research is necessary to deepen the understanding of the nature and the components of mathematics lecturer's knowledge. In this sense, we propose to explore the relationships between the different subdomains of mathematics lecturer's knowledge as a topic for further research.

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