PARTIAL ACTIONS OF GROUPOIDS

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Abstract. We study the notion of partial action of a locally compact groupoid on a C^*-algebra. We show that the semidirect product groupoid associated with a partial action of an amenable locally compact groupoid on a locally compact space is an amenable locally compact groupoid.

1. Introduction

Recently, we came across an example of locally compact groupoid for which we needed to show its amenability (see the proof of [4, Proposition 4.18]). It appeared that this groupoid looked like the semidirect product relative to a partial action of a groupoid on a locally compact space. Having consulted the literature we have not found trace of this notion, though well known for partial actions of groups. For groups, it was introduced by Exel and McClanahan in the more general setting of partial actions on C^*-algebras, first in the case of the group of integers [5], then for discrete groups [15], and then for any locally compact group [7]. For discrete group partial actions on C^*-algebras, the corresponding full and reduced crossed products were defined by McClanahan in [15], extending a previous construction of Exel in [5]. Since then, many C^*-algebras have been described as such crossed products, hence the importance of partial group actions.

It was observed by Exel in [7] that the notion of partial action of a locally compact group G on a C^*-algebra A is closely related to the notion of Fell bundle over the group G. To every partial action of G on A is associated in a canonical way a Fell bundle called the *semidirect product bundle of A and G*. Now, through the notion of representation of Fell bundles, one can define the full and reduced cross-sectional C^*-algebras of every Fell bundle $\mathcal{B}$ over a locally compact group (see [10, VIII.17.2], [6]). If $\mathcal{B}$ is the Fell bundle associated to a partial action of G on A, these cross-sectional C^*-algebras are respectively, by definition, the full and reduced crossed product C^*-algebras relative to this partial action. Indeed, when the group is discrete, they are canonically isomorphic to the above mentioned crossed products constructed by Exel and McClanahan (see [8]).

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In this text, we develop in a similar way the notion of partial action of a locally compact groupoid on a C^*-algebra and describe the associated Fell bundle on the groupoid. The notion of Fell bundles on groupoids appeared in [12] (see also [31]). The corresponding full and reduced cross-sectional C^*-algebras are constructed in a way similar to the case of Fell bundles over groups ([12], [16], [28, Appendix A.3], [24, 25]). We define the full and reduced crossed products relative to a partial action of a locally compact groupoid on a C^*-algebra to be respectively the full and reduced cross-sectional C^*-algebras of the associated Fell bundle.

Our motivation for this straightforward study was to show that the semidirect product groupoid relative to a partial action of a locally compact amenable groupoid on a locally compact space is an amenable groupoid. This is known to be true for genuine actions and also for partial actions of a locally compact group G. If G is discrete, this follows from [26, Proposition 9.3]. The general case of a locally compact group is a consequence of [23, Theorem 2.2].

We prove the same result for a partial action of an étale groupoid $\mathcal{G}$ on a locally compact space Y (Proposition 7.1). Our strategy is well known for partial actions of discrete groups (see for instance [14, Section 6]). We first show that the reduced C^*-algebra of the semidirect product groupoid $Y \rtimes \mathcal{G}$ is isomorphic to the cross-sectional reduced C^*-algebra of the Fell bundle over $\mathcal{G}$ associated with the partial action of $\mathcal{G}$ on the C^*-algebra $C_0(Y)$. Then we use a result of T. Takeishi saying that the cross-sectional reduced C^*-algebra of a Fell bundle over an étale locally compact amenable groupoid with nuclear fibers is nuclear [27, Theorem 4.1]. Finally we use the fact that if the reduced C^*-algebra of an étale groupoid is nuclear, then the groupoid is amenable (see [3, Corollary 6.2.14]). However, this proof is not entirely satisfactory, since it invokes the non-trivial fact that an étale groupoid is amenable whenever its reduced C^*-algebra is nuclear. It would be nice to have a direct proof in the only setting of groupoids.

Our paper is organized as follows. In Section 2 we recall some basic results about Fell bundles over groupoids. In Section 3 we describe the construction of the full and reduced cross-sectional C^*-algebras of Fell bundles over groupoids. In Section 4 we introduce the notion of partial action of a locally compact groupoid $\mathcal{G}$ on a C^*-algebra A and we describe the associated Fell bundle over $\mathcal{G}$. In Section 5 we consider the case where $A = C_0(Y)$ is commutative and we study the corresponding semidirect product groupoid $Y \rtimes \mathcal{G}$. Finally, in Section 6 we show that in this case, the full and reduced C^*-algebras of $Y \rtimes \mathcal{G}$ coincide with the full and reduced cross-sectional C^*-algebras of the corresponding Fell bundle. We conclude with the proof of our Proposition 7.1, which was our objective.

In this paper, we only consider locally compact groupoids, and our locally compact spaces are implicitly assumed to be Hausdorff and second countable.

2. Fell bundles over groupoids

Since Fell bundles are particular examples of upper semicontinuous C^*-bundles we begin by introducing this notion.

2.1. Upper semicontinuous C^*-bundles.

For a short introduction to this subject we refer to [29, Appendix C.2]. Continuous Banach bundles are comprehensively studied in the book [9]. Recently, it was realized that, in the C^*-algebra setting, it is more appropriate to replace continuity by upper semicontinuity, without significant changes in the results.

Definition 2.1. An *upper semicontinuous Banach bundle over a locally compact space* X is a topological space $\mathcal{A}$ together with a continuous open surjection $q : \mathcal{A} \to X$ and complex Banach space structures on the fibres $\mathcal{A}_x = q^{-1}(x)$ such that

(1) the map $a \mapsto \|a\|$ is upper semicontinuous;
(2) setting $\mathcal{A} * \mathcal{A} = \{(a,b) \in \mathcal{A} \times \mathcal{A} : q(a) = q(b)\}$, the map $(a,b) \mapsto a+b$ is continuous from $\mathcal{A} * \mathcal{A}$ to $\mathcal{A}$;
(3) for each $\lambda \in \mathbb{C}$, the map $a \mapsto \lambda a$ is continuous from $\mathcal{A}$ to $\mathcal{A}$;
(4) if (a_i) is a net in $\mathcal{A}$ such that $\lim_i q(a_i) = x$ and $\lim_i \|a_i\| = 0$, then $\lim_i a_i = 0_x$, where 0_x is the zero element in $\mathcal{A}_x$.

Definition 2.2. An *upper semicontinuous C^*-bundle* is an upper semicontinuous Banach bundle $q : \mathcal{A} \to X$ such that each fibre $\mathcal{A}_x$ is a C^*-algebra and such that the following additional conditions hold:

(5) the map $(a,b) \mapsto ab$ is continuous from $\mathcal{A} * \mathcal{A}$ to $\mathcal{A}$;
(6) the map $a \mapsto a^*$ is continuous from $\mathcal{A}$ to $\mathcal{A}$.

We denote by $\Gamma_0(X, \mathcal{A})$ the C^*-algebra of continuous sections of the C^*-bundle $\mathcal{A}$ that vanish at infinity and by $\Gamma_c(X, \mathcal{A})$ the $*$-algebra of compactly supported continuous sections. Note that $C_0(X)$ acts on $\Gamma_0(X, \mathcal{A})$ as follows:

$$(\varphi f)(x) = \varphi(x) f(x), \quad \text{for} \quad \varphi \in C_0(X), f \in \Gamma_0(X, \mathcal{A}).$$

This gives to $\Gamma_0(X, \mathcal{A})$ the structure of a $C_0(X)$-algebra in the following sense (see [29, Proposition C.23]).

Definition 2.3. A *$C_0(X)$-algebra* is a C^*-algebra A equipped with a homomorphism Φ from $C_0(X)$ into the center of the multiplier algebra of A, that is non-degenerated, meaning that the linear span of $\{\Phi(\varphi)a : \varphi \in C_0(X), a \in A\}$ is dense in A.

Thus, an upper semicontinuous C^*-bundle gives rise to a $C_0(X)$-algebra. Conversely, every $C_0(X)$-algebra A arises in this way. Indeed, for $x \in X$, let us denote by J_x the ideal of functions in $C_0(X)$ that vanish at x and denote by I_x the closure of the linear span of $\{\Phi(\varphi)a : \varphi \in J_x, a \in A\}$. Then I_x is an ideal in A, and we set $A(x) = A/I_x$. We write $a(x)$ for the image of a in $A(x)$. Then for every $a \in A$ the function $x \mapsto \|a(x)\|$ is upper semicontinuous and vanishes at infinity. Set $\mathcal{A} = \bigsqcup_{x\in X} A(x)$ and let $q : \mathcal{A} \to X$ be the canonical map. There is a unique topology making $q : \mathcal{A} \to X$ an upper semicontinuous C^*-bundle with $\Gamma_0(X, \mathcal{A}) = \{x \mapsto a(x) : a \in A\}$ (see [29, Theorems C.25, C.26]). The open sets of $\mathcal{A}$ are the unions of sets of that form

$$W(f, U, \varepsilon) = \{a \in \mathcal{A} : q(a) \in U \text{ and } \|a - f(q(a))\| < \varepsilon\},$$

where $f \in A$, U is an open subset of X and $\varepsilon > 0$. It follows that A is $C_0(X)$-linear isomorphic to $\Gamma_0(X, \mathcal{A})$. Note that in case we had started with $A = \Gamma_0(X, \mathcal{A})$ for some upper semicontinuous C^*-bundle $\mathcal{A}$, then $A(x) = \Gamma_0(X, \mathcal{A})(x) = \mathcal{A}_x$.

Thus the notions of upper semicontinuous C^*-bundle and of $C_0(X)$-algebra are in bijective correspondence. This justifies the extension from continuity to upper semicontinuity.

Example 2.4. Let X, Y be two locally compact spaces and let $\rho : Y \to X$ be a continuous surjective map. Then $A = C_0(Y)$ is a $C_0(X)$-algebra by setting $(\varphi f)(y) = \varphi\circ\rho(y)f(y)$ for $\varphi \in C_0(X)$ and $f \in C_0(Y)$. Let $\mathcal{A}$ be the corresponding upper semicontinuous C^*-bundle over X. Then we have $\mathcal{A}_x = C_0(\rho^{-1}(x))$ for $x \in X$. Moreover the map sending $f \in C_0(Y)$ to the section $x \mapsto f_{|\rho^{-1}(x)}$ is an isomorphism of $C_0(X)$-algebras from $C_0(Y)$ onto $\Gamma_0(X, \mathcal{A})$.

Definition 2.5. Let X, Y be two locally compact spaces and let $\rho : Y \to X$ be a continuous map. Given an upper semicontinuous Banach bundle $q : \mathcal{A} \to X$, its *pullback* is defined to be the space

$$\rho^*\mathcal{A} = \mathcal{A}\,{}_q{*}_\rho Y = \{(a, y) \in \mathcal{A} \times Y : q(a) = \rho(y)\}$$

equipped with the relative topology and with the bundle map $p : \rho^*\mathcal{A} \to Y$ such that $p(y, a) = y$. It is an upper semicontinuous Banach bundle over Y and an upper semicontinuous C^*-bundle when $\mathcal{A}$ is so.

Definition 2.6. Let X, Y be two locally compact spaces and let $\rho : Y \to X$ be a continuous map. Let A be a $C_0(X)$-algebra and $\mathcal{A}$ its associated upper semicontinuous C^*-bundle. The *pullback of* A is defined to be $\rho^*A = \Gamma_0(Y, \rho^*\mathcal{A})$. It is a $C_0(Y)$-algebra and for $y \in Y$, its fiber $\rho^*A(y)$ is naturally identified with $A(\rho(y))$.

Note that ρ^*A can also be defined as a balanced tensor product of A by $C_0(Y)$ over $C_0(X)$ ([20], [11]).

2.2. **Fell bundles over groupoids.** Let $\mathcal{G}$ be a locally compact groupoid. For this notion we refer to [21], [17], [3] or [30]. We use the notation of [3]. In particular $\mathcal{G}^{(0)}$ denotes the space of units of $\mathcal{G}$, and r, s are respectively the range and source maps, $\mathcal{G}^x = r^{-1}(x)$ and $\mathcal{G}_x = s^{-1}(x)$ for $x \in X = \mathcal{G}^{(0)}$.

The notion of Banach algebraic bundle over a topological group is developed in [10]. To the author's knowledge, bundles over groupoids seem to have been considered first in [31] and then in [12].

Definition 2.7. Let $\mathcal{G}$ be a locally compact groupoid and $p : \mathcal{B} \to \mathcal{G}$ be an upper semicontinuous Banach bundle. We set

$$\mathcal{B}^{(2)} = \Big\{(a, b) \in \mathcal{B} \times \mathcal{B} : (p(a), p(b)) \in \mathcal{G}^{(2)}\Big\}.$$

We say that $p : \mathcal{B} \to \mathcal{G}$ is a *Fell bundle* if there is a continuous, bilinear, associative multiplication map $(a, b) \mapsto ab$ from $\mathcal{B}^{(2)}$ into $\mathcal{B}$ and a conjugate linear continuous involution $a \mapsto a^*$ such that,

(i) $p(ab) = p(a)p(b)$ for $(a, b) \in \mathcal{B}^{(2)}$;
(ii) $p(a^*) = p(a)^{-1}$ for $a \in \mathcal{B}$;
(iii) $(ab)^* = b^*a^*$ for $(a, b) \in \mathcal{B}^{(2)}$;
(iv) $\|ab\| \leq \|a\|\|b\|$ for $(a, b) \in \mathcal{B}^{(2)}$;
(v) $\|a^*a\| = \|a\|^2$ for $a \in \mathcal{B}$. In particular, for $x \in \mathcal{G}^{(0)}$, the Banach space $\mathcal{B}_x$ is a C^*-algebra with respect to the $*$-algebra structure induced by the multiplication and the involution;
(vi) $a^*a \geq 0$ for $a \in \mathcal{B}$.

We will give examples of Fell bundles in Section 4. For other examples we refer to [12, Section 2.5] or [16, Section 2].

We now describe the full and reduced cross-sectional C^*-algebras of Fell bundles over groupoids.

3. Full and reduced C^*-algebras of Fell bundles

Let $(\mathcal{B}, p)$ be a Fell bundle over a locally compact groupoid $\mathcal{G}$ equipped with a Haar system $\lambda = (\lambda^x)_{x\in\mathcal{G}^{(0)}}$. As usual, we set $\lambda_x = \lambda_x^{-1}$.

3.1. **Full C^*-algebra.** This construction may be found for instance in [16]. We denote by $\Gamma_c(\mathcal{G}, \mathcal{B})$ the space of compactly supported continuous sections of the Fell bundle. It is a topological $*$-algebra with respect to the inductive limit topology. The involution is given by

$$f^*(\gamma) = f(\gamma^{-1})^*$$

and the product is given by the convolution formula

$$(f * g)(\gamma) \int_{\mathcal{G}^{r(\gamma)}} f(\gamma_1)g(\gamma_1^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma_1).$$

The *full C^*-algebra of the Fell bundle* (also called the *full cross-sectional C^*-algebra*) $\mathcal{B}$ is the completion $C^*(\mathcal{G}, \mathcal{B})$ of $\Gamma_c(\mathcal{G}, \mathcal{B})$ with respect to the universal norm

$$\|f\| = \sup\{\|L(f)\| : L \text{ is an I-norm decreasing representation}\}$$

where

$$\|f\|_I = \max\big(\sup_{x\in\mathcal{G}^{(0)}}\int_{\mathcal{G}^x}\|f(\gamma)\|\,\mathrm{d}\lambda^x(\gamma), \sup_{x\in\mathcal{G}^{(0)}}\int_{\mathcal{G}_x}\|f(\gamma)\|\,\mathrm{d}\lambda_x(\gamma)\big).$$

An I-norm decreasing representation is a $*$-homomorphism L from $\Gamma_c(\mathcal{G},\mathcal{B})$ into the C^*-algebra of bounded operators on a Hilbert space H such that $\|L(f)\| \le \|f\|_I$ for all $f\in\Gamma_c(\mathcal{G},\mathcal{B})$ and such that the linear span of $\{L(f)\xi : f\in\Gamma_c(\mathcal{G},\mathcal{B}), \xi\in H\}$ is dense in H.

Note that, under usual separability assumptions, the universal norm can also be defined via representations of the Fell bundle $\mathcal{B}$ (for details, see [16, Section 4]).

3.2. **Reduced C^*-algebra.** (see [12], [28, Appendix A.3], [24, 25]). We set $X = \mathcal{G}^{(0)}$ and we denote by $\mathcal{B}^{(0)}$ the restriction of the Fell bundle $\mathcal{B}$ to X. It is an upper semicontinuous C^*-bundle on X. We denote by $P:\Gamma_c(\mathcal{G},\mathcal{B})\to\Gamma_c(X,\mathcal{B}^{(0)})$ the restriction map and we define on $\Gamma_c(\mathcal{G},\mathcal{B})$ the following $\Gamma_0(X,\mathcal{B}^{(0)})$-valued inner product $\langle f,g\rangle = P(f^*g)$, that is

$$\langle f,g\rangle(x) = \int_{\mathcal{G}_x} f(\gamma)^* g(\gamma)\,\mathrm{d}\lambda_x(\gamma)\in\mathcal{B}^{(0)}_x. \tag{1}$$

If $f\in\Gamma_c(\mathcal{G},\mathcal{B})$ and $h\in\Gamma_0(X,\mathcal{B}^{(0)})$, we set

$$(fh)(\gamma) = f(\gamma)h\circ s(\gamma). \tag{2}$$

This makes $\Gamma_c(\mathcal{G},\mathcal{B})$ a right pre-Hilbert $\Gamma_0(X,\mathcal{B}^{(0)})$-module and we denote by $L^2(\mathcal{G},\mathcal{B})$ its completion. We define an I-norm decreasing faithful representation π^l of $\Gamma_c(\mathcal{G},\mathcal{B})$ on $L^2(\mathcal{G},\mathcal{B})$, called the *left regular representation*, by left multiplication:

$$(\pi^l(f)\xi)(\gamma) = \int_{\mathcal{G}^{r(\gamma)}} f(\gamma_1)\xi(\gamma_1^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma_1).$$

The *reduced* (*cross-sectional*) C^*-algebra of the Fell bundle $\mathcal{B}$ is the closure $C^*_r(\mathcal{G},\mathcal{B})$ of the image of π^l in $\mathcal{B}_{\Gamma_0(X,\mathcal{B}^{(0)})}(L^2(\mathcal{G},\mathcal{B}))$, where $\mathcal{B}_{\Gamma_0(X,\mathcal{B}^{(0)})}(L^2(\mathcal{G},\mathcal{B}))$ denotes the C^*-algebra of adjoinable operators on $L^2(\mathcal{G},\mathcal{B})$.

We see the C^*-algebra $\mathcal{B}_x$ as a left $\Gamma_0(X,\mathcal{B}^{(0)})$-module by setting $fb = f(x)b$ for $f\in\Gamma_0(X,\mathcal{B}^{(0)})$ and $b\in\mathcal{B}_x = \mathcal{B}^{(0)}_x$. Then the interior tensor product

$$L^2(\mathcal{G},\mathcal{B})\otimes_{\Gamma_0(X,\mathcal{B}^{(0)})}\mathcal{B}_x$$

is isomorphic, as a Hilbert right $\mathcal{B}_x$-module, to $L^2(\mathcal{G}_x,\mathcal{B})$, where $L^2(\mathcal{G}_x,\mathcal{B})$ is defined via the formulas (1), (2) but for functions in $\Gamma_c(\mathcal{G}_x,\mathcal{B})$. The identification is made via the map

$$f\otimes b\mapsto (\gamma\in\mathcal{G}_x\mapsto f(\gamma)b)$$

for $f \in \Gamma_c(\mathcal{G},\mathcal{B})$ and $b \in \mathcal{B}_x$. We denote by q_x the canonical map from $L^2(\mathcal{G},\mathcal{B})$ onto $L^2(\mathcal{G}_x,\mathcal{B})$. We have $q_x(\xi b) = q_x(\xi)b(x)$ for $\xi \in L^2(\mathcal{G},\mathcal{B})$ and $b \in \Gamma_0(X,\mathcal{B}^{(0)})$.

By [28, Proposition A.4], there is a unique structure of upper semicontinuous bundle of Hilbert modules $\widetilde{L^2(\mathcal{G},\mathcal{B})} = \sqcup_{x\in\mathcal{G}^{(0)}} L^2(\mathcal{G}_x,\mathcal{B})$ over $\mathcal{G}^{(0)}$, such that the map $\xi \in L^2(\mathcal{G},\mathcal{B}) \mapsto (x \mapsto q_x(\xi))$ is an isomorphism from $L^2(\mathcal{G},\mathcal{B})$ onto $C_0(\mathcal{G}^{(0)}, \widetilde{L^2(\mathcal{G},\mathcal{B})})$.

The representation π^l is a field of representations $(\pi^l_x)_{x\in X}$ of $C^*_r(\mathcal{G},\mathcal{B})$, where π^l_x acts on the right Hilbert $\mathcal{B}_x$-module $L^2(\mathcal{G}_x,\mathcal{B})$ by

$$(\pi^l_x(f)\xi)(\gamma) = \int_{\mathcal{G}^x} f(\gamma\gamma_1)\xi(\gamma_1^{-1})\,\mathrm{d}\lambda^x(\gamma_1)$$

for $f \in \Gamma_c(\mathcal{G},\mathcal{B})$ and $\xi \in \mathcal{G}_x$. We have $\left\|\pi^l(f)\right\| = \sup_{x\in X}\left\|(\pi^l_x(f)\right\|$ (see [28, Proposition A.5]).

Moreover, the universal property of the full norm gives rise to a surjective map from $C^*(\mathcal{G},\mathcal{B})$ onto $C^*_r(\mathcal{G},\mathcal{B})$.

4. Partial actions of groupoids on C^*-algebras and their Fell bundles

4.1. **Definition of a partial action.**

We adapt the now classical notion of partial action of groups ([5, 7], [15], [18], [1],...) to the case of groupoids.

Definition 4.1. Let $\mathcal{G}$ be a locally compact groupoid with space of units X. Let A be a $C_0(X)$-algebra and let $q : \mathcal{A} \to X$ be its associated upper semicontinuous C^*-bundle. A *partial action of* $\mathcal{G}$ *on* A is a pair $\boldsymbol{\theta} = ((\theta_\gamma)_{\gamma\in\mathcal{G}}, (D_\gamma)_{\gamma\in\mathcal{G}})$ where for each $\gamma \in \mathcal{G}$, D_γ is a closed two-sided ideal in $A(r(\gamma))$ and θ_γ is an isomorphism from $D_{\gamma^{-1}}$ onto D_γ such that

(i) $D_x = A(x)$ and $\theta_x = \mathrm{Id}_{A(x)}$ for $x \in X$;
(ii) $\theta_{\gamma_1\gamma_2}$ is an extension of $\theta_{\gamma_1} \circ \theta_{\gamma_2}$ for $(\gamma_1,\gamma_2) \in \mathcal{G}^{(2)}$.

Of course we will require, in addition, continuity conditions for this partial action. To define them we need to introduce some notation. The pullbacks $r^*\mathcal{A} = \mathcal{A}\,{}_q{*}_r\,\mathcal{G}$ and $s^*\mathcal{A} = \mathcal{A}\,{}_q{*}_s\,\mathcal{G}$ are upper semicontinuous C^*-bundles over $\mathcal{G}$. We set

$$\mathcal{B} = \left\{(a,\gamma) : a \in D_\gamma\right\} \subset r^*\mathcal{A}, \quad \mathcal{B}^{-1} = \left\{(a,\gamma) : a \in D_{\gamma^{-1}}\right\} \subset s^*\mathcal{A},$$

and we endow $\mathcal{B}$ and $\mathcal{B}^{-1}$ with the topologies induced by those of $r^*\mathcal{A}$ and $s^*\mathcal{A}$ respectively.

Definition 4.2. We say that *the partial action* $\boldsymbol{\theta} : \mathcal{G} \curvearrowright A$ is *continuous*[a] if the following conditions hold:

[a] The continuity condition will always be implicit in the sequel.

(iii) the map $p : \mathcal{B} \to \mathcal{G}$ defined by $p(a,\gamma) = \gamma$ is open;
(iv) the map $(a,\gamma) \mapsto (\theta_\gamma(a),\gamma)$ is continuous from $\mathcal{B}^{-1}$ onto $\mathcal{B}$.

Remark 4.3. Condition (iii) implies that $\mathcal{B}$ is an upper continuous C^*-bundle over $\mathcal{G}$. This condition is equivalent to the fact that for every open subset U of $\mathcal{A}$, the set $\{\gamma \in \mathcal{G} : D_\gamma \cap U \neq \emptyset\}$ is open in $\mathcal{G}$. This follows from the equality

$$p((U \times V) \cap \mathcal{B}) = \{\gamma \in \mathcal{G} : \exists a \in D_\gamma \cap U\} \cap V,$$

where U and V are open subsets of $\mathcal{A}$ and $\mathcal{G}$ respectively.

The map defined in (iv) is a homeomorphism from $\mathcal{B}$ onto $\mathcal{B}^{-1}$.

Remark 4.4. Let $\boldsymbol{\theta} : \mathcal{G} \curvearrowright A$ be a partial action. Then we have

(1) $\theta_{\gamma^{-1}} = (\theta_\gamma)^{-1}$ for every $\gamma \in \mathcal{G}$.
(2) $\theta_\gamma(D_{\gamma^{-1}} D_{\gamma_1}) = D_\gamma D_{\gamma\gamma_1}$ if $s(\gamma) = r(\gamma_1)$;
(3) $\theta_\gamma \circ \theta_{\gamma_1}$ is an isomorphism from $D_{\gamma_1^{-1}} D_{\gamma_1^{-1}\gamma^{-1}}$ onto $D_\gamma D_{\gamma\gamma_1}$ if $s(\gamma) = r(\gamma_1)$, and $\theta_\gamma \circ \theta_{\gamma_1}(a) = \theta_{\gamma\gamma_1}(a)$ if $a \in D_{\gamma_1^{-1}} D_{\gamma_1^{-1}\gamma^{-1}}$.

The proofs are the same as in [18, Lemma 1.2].

4.2. **Fell bundle associated with a partial action.** To a partial action $\mathcal{G} \curvearrowright A$, we are going to associate a Fell bundle. It is sometimes convenient to write $a\delta_\gamma$ for $(a,\gamma) \in p^{-1}(\gamma)$. The multiplication on $\mathcal{B}$ is defined by

$$a\delta_{\gamma_1} * b\delta_{\gamma_2} = \theta_{\gamma_1}\big(\theta_{\gamma_1^{-1}}(a)b\big)\delta_{\gamma_1\gamma_2}$$

if $s(\gamma_1) = r(\gamma_2)$ and the involution is defined by

$$(a\delta_\gamma)^* = \theta_{\gamma_1^{-1}}(a^*)\delta_{\gamma_1^{-1}}.$$

It is straightforward to check that Conditions (i) to (vi) of 2.7 are fulfiled. The associativity of the product is proved as in [7, Proposition 2.4].

To show the continuity of the product, by [10, VIII.2.4] we have to show that given two continuous sections $\gamma \mapsto (a_\gamma,\gamma)$ and $\gamma \mapsto (b_\gamma,\gamma)$, then the map

$$(\gamma_1,\gamma_2) \mapsto (a_{\gamma_1},\gamma_1) * (b_{\gamma_2},\gamma_2) = \big(\theta_{\gamma_1}(\theta_{\gamma_1^{-1}}(a_{\gamma_1})b_{\gamma_2}),\gamma_1\gamma_2\big)$$

is continuous from $\mathcal{G}^{(2)}$ into $\mathcal{B}$. This follows from the continuity assumption on the partial action.

Similarly, one shows that the involution is continuous.

The Fell bundle $\mathcal{B}$ is called the *semidirect product bundle of A and $\mathcal{G}$* (after [7, Definition 2.8] and [10, Section VIII.4]).

Remark 4.5. This notion of partial action of a groupoid on a C^*-algebra extends both the notion of action of a groupoid on a C^*-algebra (see [22], [13], [19], [11]) and the notion of partial action of a group on a C^*-algebra. The full and reduced crossed product C^*-algebras for these actions coincide in these examples, respectfully, to the full and reduced C^*-algebras of the corresponding Fell bundles.

5. Partial actions of groupoids on locally compact spaces

5.1. **Definition and example.**

Definition 5.1. Let $\mathcal{G}$ be a locally groupoid with space of units X. Let Y be a locally compact space. A *partial action of* $\mathcal{G}$ *on a locally compact space* Y is the data of a continuous map $\rho : Y \to X$ and a pair $\boldsymbol{\beta} = ((\beta_\gamma)_{\gamma\in\mathcal{G}}, (Y_\gamma)_{\gamma\in\mathcal{G}})$, where for $\gamma \in \mathcal{G}$, Y_γ is an open subset of $Y^{r(\gamma)} = \rho^{-1}(r(\gamma))$ and β_γ is a homeomorphism from $Y_{\gamma^{-1}}$ onto Y_γ, such that

(i) $Y_x = Y^x$ and $\beta_x = \mathrm{Id}_{Y_x}$ for $x \in X$;
(ii) $\beta_{\gamma\gamma_1}$ is an extension of $\beta_\gamma \circ \beta_{\gamma_1}$ if $s(\gamma) = r(\gamma_1)$;
(iii) $C = \{(y, \gamma) \in Y_\rho{*_r}\,\mathcal{G} : y \in Y_\gamma\}$ is open in $Y_\rho{*_r}\,\mathcal{G}$;
(iv) the map $(y, \gamma) \mapsto \beta_{\gamma^{-1}}(y)$ is continuous from C into Y.

Remark 5.2. Let $\boldsymbol{\beta} : \mathcal{G} \curvearrowright Y$ be a partial action. Then we have

(1) $\beta_{\gamma^{-1}} = (\beta_\gamma)^{-1}$ for every $\gamma \in \mathcal{G}$.
(2) $\beta_\gamma(Y_{\gamma^{-1}} \cap Y_{\gamma_1}) = Y_\gamma \cap Y\gamma\gamma_1$ if $s(\gamma) = r(\gamma_1)$;
(3) $\beta_\gamma \circ \beta_{\gamma_1}$ is a homeomorphism from $Y_{\gamma_1^{-1}} \cap Y_{\gamma_1^{-1}\gamma^{-1}}$ onto $Y_\gamma \cap Y_{\gamma\gamma_1}$ if $s(\gamma) = r(\gamma_1)$, and $\beta_\gamma \circ \beta_{\gamma_1}(y) = \beta_{\gamma\gamma_1}(y)$ if $y \in Y_{\gamma_1^{-1}} \cap Y_{\gamma_1^{-1}\gamma^{-1}}$.

Remark 5.3. The map $(y, \gamma) \mapsto (y, \gamma^{-1})$ is a homeomorphism from $Y_\rho{*_r}\,\mathcal{G}$ onto $Y_\rho{*_s}\,\mathcal{G}$ sending C onto $C^{-1} = \{(y, \gamma) \in Y_\rho{*_s}\,\mathcal{G} : y \in Y_{\gamma^{-1}}\}$.

Condition (iv) means that $(y, \gamma) \mapsto (\beta_{\gamma^{-1}}(y), \gamma)$ is continuous from C onto C^{-1}. It is a homeomorphism. Indeed, the inverse map $(y, \gamma) \mapsto (\beta_\gamma(y), \gamma)$ is the composition of three continuous maps:

$$(y, \gamma) \in C^{-1} \mapsto (y, \gamma^{-1}) \in C \mapsto (\beta_\gamma(y), \gamma^{-1}) \in C^{-1} \mapsto (\beta_\gamma(y), \gamma) \in C.$$

Let us keep the notation of the above definition. We set $A = C_0(Y)$ and $D_\gamma = C_0(Y_\gamma)$, for $\gamma \in \mathcal{G}$. Then D_γ is a closed two-sided ideal of $A(r(\gamma)) = C_0(Y^{r(\gamma)})$. We denote by θ_γ the isomorphism from $D_{\gamma^{-1}}$ onto D_γ defined by

$$\theta_\gamma(f)(y) = f \circ \beta_{\gamma^{-1}}(y) \quad \text{for} \quad y \in Y_\gamma \quad \text{and} \quad f \in D_{\gamma^{-1}}.$$

Proposition 5.4. ([2, Proposition 1.5]) *Let $\boldsymbol{\beta} : \mathcal{G} \curvearrowright Y$ be a partial action of $\mathcal{G}$ on the locally compact space Y. Then $\boldsymbol{\theta} = ((\theta_\gamma)_{\gamma\in\mathcal{G}}, (D_\gamma)_{\gamma\in\mathcal{G}})$ is a partial action on the C^*-algebra $C_0(Y)$.*

Proof. The only non obvious verifications are that conditions (iii) and (iv) of Definition 4.2 hold. We keep the notation of this definition. The algebra $A = C_0(Y)$ is a $C_0(X)$-algebra and we denote by $(\mathcal{A}, q)$ the corresponding upper semicontinuous C^*-bundle. Then $r^*\mathcal{A} = \{(a,\gamma) \in \mathcal{A} \times \mathcal{G} : q(a) = r(\gamma)\}$ is an upper semicontinuous C^*-bundle over $\mathcal{G}$ and $\mathcal{A}_\gamma = C_0(Y^{r(\gamma)})$ for every $\gamma \in \mathcal{G}$. We set $C' = \big\{(y,\gamma) \in Y \times \mathcal{G} : y \in Y^{r(\gamma)}\big\} = Y_\rho {*_r}\, \mathcal{G}$. Note that $C_0(C')$ is a $C_0(\mathcal{G})$-algebra and that $C_0(C') = r^*A$ as $C_0(\mathcal{G})$-algebra. We also observe that C' is a closed subset of $Y \times \mathcal{G}$ and that $C = \big\{(y,\gamma) \in Y \times \mathcal{G} : y \in Y_{r(\gamma)}\big\}$ is an open subset of C'. Moreover, $C_0(C)$ is an ideal of $C_0(C')$ stable under multiplication by the elements of $C_0(\mathcal{G})$.

We will show that for every open subset Ω of $\mathcal{A}$, the set $\mathcal{G}_\Omega = \{\gamma \in \mathcal{G} : D_\gamma \cap \Omega \neq \emptyset\}$ is open in $\mathcal{G}$. We take $\Omega = W(f, U, \varepsilon)$ where $f \in C_0(Y)$, U is an open subset of X and $\varepsilon > 0$. We may assume that $\mathcal{G}_\Omega$ is not empty. Let γ_0 be in this set and let $a_0 \in C_c(Y_{\gamma_0})$ be such that $\sup_{y \in Y^{r(\gamma_0)}} |a_0(y) - f(y)| < \varepsilon$. Let $a \in C_c(C)$ be such that $a(y,\gamma_0) = a_0(y)$ for $y \in Y_{\gamma_0}$. The fonction $\gamma \mapsto \sup_{y \in Y^{r(\gamma)}} |a(y,\gamma) - f(y)|$ is upper semicontinuous. It follows that there is a neighborhood $V \subset r^{-1}(U)$ of γ_0 such that $\sup_{y \in Y^{r(\gamma)}} |a(y,\gamma) - f(y)| < \varepsilon$ for $\gamma \in V$. Therefore we have $V \subset \mathcal{G}_\Omega$ and $\mathcal{G}_\Omega$ is a neighborhood of γ_0.

It remains to show that the map $(a,\gamma) \mapsto (\theta_\gamma(a), \gamma)$ from $\mathcal{B}^{-1}$ onto $\mathcal{B}$ is continuous. This follows from [9, Proposition 13.16] and Condition (iv) in Definition 5.1. □

Remark 5.5. Using [9, Propositions 13.16 and 13.17], we see that the Fell bundle $\mathcal{B}$, coincides with the Fell bundle $\sqcup_{\gamma \in \mathcal{G}} C_0(Y_\gamma)$ coming from the $C_0(\mathcal{G})$-algebra $C_0(C)$. Indeed, the map sending $a \in C_0(Y_\gamma) = D_\gamma$ to $(a,\gamma) \in \mathcal{B}$ is continuous and thus an isomorphim by [9, Proposition 13.17].

Example 5.6. Let us consider $\rho : Y = Z \times X \to X$ to be the second projection, where Z and Y are locally compact spaces. In this case

$$A = C_0(Y) = C_0(Z) \otimes C_0(X)$$

and $A(x) = C_0(Z)$ for every $x \in X$. The associated C^*-bundle is $C_0(Z) \times X$ equipped with the product topology. If X is the set of units of a locally compact groupoid $\mathcal{G}$, we have $r^*\mathcal{A} = C_0(Z) \times \mathcal{G}$ and $r^*A = C_0(Z) \otimes C_0(\mathcal{G})$. A partial action of $\mathcal{G}$ on Y is a pair $\boldsymbol{\beta} = ((\beta_\gamma)_{\gamma \in \mathcal{G}}, (Y_\gamma)_{\gamma \in \mathcal{G}})$, where for $\gamma \in \mathcal{G}$, $Y_\gamma = Z_\gamma \times \{r(\gamma)\}$ with Z_γ an open subset of Z such that

(i) $Y_x = Z \times \{x\}$ and $\beta_x = \mathrm{Id}_{Y_x}$ for $x \in X$;
(ii) $\beta_{\gamma\gamma_1}$ is an extension of $\beta_\gamma \circ \beta_{\gamma_1}$ if $s(\gamma) = r(\gamma_1)$;
(iii) $C = \{(y,\gamma) \in Y_\rho {*_r}\, \mathcal{G} : y \in Y_\gamma\}$ is open in $Y_\rho {*_r}\, \mathcal{G}$;
(iv) the map $(y,\gamma) \mapsto \beta_{\gamma^{-1}}(y)$ is continuous from C into Y.

We may identify Y_γ with Z_γ and $Y_\rho *_r \mathcal{G}$ with $Z \times \mathcal{G}$. Thus Condition (iii) states that $C = \{(z,\gamma) \in Z \times \mathcal{G} : z \in Z_\gamma\}$ is open in $Z \times \mathcal{G}$ and Condition (iv) states that $(z,\gamma) \mapsto \beta_{\gamma^{-1}}(z)$ is continuous on C.

As an example of such a partial action let us consider a partial action α of a locally compact group G on a locally compact space Z. In particular α_g is a homeomorphism from $Z_{g^{-1}}$ onto Z_g where for every $g \in G$, Z_g is an open subset of Z. In addition we are given an action $(g,x) \mapsto gx$ from G on a locally compact space X. We are going to define a partial action of the transformation groupoid $\mathcal{G} = X \rtimes G$ on $Y = Z \times X$ in the following way.

For $(x,g) \in X \rtimes G$ we set $Y_{(x,g)} = Z_g \times \{x\}$ and

$$\beta_{(x,g)} : Y_{(x,g)^{-1}} = Z_{g^{-1}} \times \left\{g^{-1}x\right\} \to Z_g \times \{x\}$$

is defined by

$$\beta_{(x,g)}(z, g^{-1}x) = (\alpha_g(z), x).$$

One immediately check that we obtain in this way a partial action of $\mathcal{G}$. This is the example we have to consider in the proof of [4, Proposition 4.13].

5.2. **Semidirect product groupoid.** Let $\boldsymbol{\beta} : \mathcal{G} \curvearrowright Y$ be a partial action. For simplicity of notation, we will write γy instead of $\beta_\gamma(y)$ when $y \in Y_\gamma$. We define a new locally compact groupoid $Y \rtimes \mathcal{G}$, called the *semidirect product groupoid* or the *transformation groupoid*. As a topological space, it is $C = \{(y,\gamma) \in Y_\rho *_r \mathcal{G} : y \in Y_\gamma\}$. The range of (y,γ) is $(y, r(\gamma)) = (y, \rho(y))$ and its source is $(\gamma^{-1}y, s(\gamma))$. The product is given by

$$(y,\gamma)(\gamma^{-1}y, \gamma_1) = (y, \gamma\gamma_1).$$

Note that $\gamma^{-1}y, \in Y_{\gamma^{-1}} \cap Y_{\gamma_1}$ and therefore $y \in Y_{\gamma\gamma_1}$. The inverse is given by

$$(y,\gamma)^{-1} = (\gamma^{-1}y, \gamma^{-1}).$$

Observe that $(y, \rho(y)) \mapsto y$ is a homeomorphism from $(Y \rtimes \mathcal{G})^{(0)}$ onto Y. Therefore we will identify these two spaces. Sometimes we will denote by the boldface letter $\mathbf{r}$ the range map of $Y \rtimes \mathcal{G}$ to distinguish it from the range map r of $\mathcal{G}$ (and similarly for s).

The range map $\mathbf{r} : Y \rtimes \mathcal{G} \to Y$ is open. Indeed, since C is open in $Y_\rho *_r \mathcal{G}$, it suffices to show that the map $(y,\gamma) \mapsto y$ from $Y_\rho *_r \mathcal{G}$ to Y is open. Let Ω be an open subset of $Y_\rho *_r \mathcal{G}$. Let $y_0 \in \mathbf{r}(\Omega)$ and let γ_0 be such that $(y_0, \gamma_0) \in \Omega$. There exist open neighborhoods U of y_0 and V of γ_0 such that $U_\rho *_r V \subset \Omega$. Then we have $\rho^{-1}(r(V)) \cap U \subset \mathbf{r}(\Omega)$ and $\rho^{-1}(r(V)) \cap U$ is an open neighborhood of y_0 since r is open.

In the rest of this paper we assume that $\mathcal{G}$ has a left Haar system $\lambda = (\lambda^x)_{x \in X}$ where $X = \mathcal{G}^{(0)}$. We are now going to construct a Haar system for $Y \rtimes \mathcal{G}$. We set

$C_y = \{\gamma \in \mathcal{G} : (y,\gamma) \in C\} = \{\gamma \in \mathcal{G} : y \in Y_\gamma\}$. Then we have $(Y \rtimes \mathcal{G})^y = \{y\} \times C_y$. Since C is a Borel subset of $Y \times \mathcal{G}$, we see that C_y is a Borel subset of $\mathcal{G}$.

When $B = \{y\} \times B_1$ is a Borel subset of $\{y\} \times C_y$, we set $\tilde{\lambda}^y(B) = \lambda^{\rho(y)}(B_1)$.

Proposition 5.7. *The family $\tilde{\lambda} = (\tilde{\lambda}^y)_{y\in Y}$ is a left Haar system on $Y \rtimes \mathcal{G}$.*

Proof. Let us first observe that the support of $\tilde{\lambda}^y$ is $\{y\} \times C_y$. This is obvious because C_y is open in $\mathcal{G}^{\rho(y)}$ since C is open in $Y_\rho *_r \mathcal{G}$.

Next, let us show the left invariance, meaning that for every $f \in C_c(Y \rtimes \mathcal{G})$ and for every $(y,\gamma) \in Y \rtimes \mathcal{G}$ we have

$$\int_{(Y\rtimes\mathcal{G})^{\gamma^{-1}y}} f((y,\gamma)(y_1,\gamma_1))\,\mathrm{d}\tilde{\lambda}^{\gamma^{-1}y}(y_1,\gamma_1) = \int_{(Y\rtimes\mathcal{G})^y} f(y_1,\gamma_1)\,\mathrm{d}\tilde{\lambda}^y(y_1,\gamma_1). \tag{3}$$

The right-hand side is

$$\int_{(Y\rtimes\mathcal{G})^y} f(y_1,\gamma_1)\,\mathrm{d}\tilde{\lambda}^y(y_1,\gamma_1) = \int_{\mathcal{G}^{\rho(y)}} \mathbb{1}_{C_y}(\gamma_1) f(y,\gamma_1)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma_1).$$

As for the left-side integral in (3), we integrate on the set of (y_1,γ_1) such that $y_1 = \gamma^{-1}y \in Y_{\gamma_1}$, that is on the set of $(\gamma^{-1}y,\gamma_1)$ with

$$y \in \gamma(Y_{\gamma^{-1}} \cap Y_{\gamma_1}) = Y_\gamma \cap Y_{\gamma\gamma_1}.$$

It follows that

$$\int_{(Y\rtimes\mathcal{G})^{\gamma^{-1}y}} f((y,\gamma)(y_1,\gamma_1))\,\mathrm{d}\tilde{\lambda}^{\gamma^{-1}y}(y_1,\gamma_1) = \int_{\mathcal{G}^{s(\gamma)}} \mathbb{1}_{C_y}(\gamma\gamma_1) f(y,\gamma\gamma_1)\,\mathrm{d}\lambda^{s(\gamma)}(\gamma_1).$$

This proves the invariance.

It remains to show that for $f \in C_c(Y \rtimes \mathcal{G})$ the map

$$y \mapsto \int_{(Y\rtimes\mathcal{G})^y} f(y,\gamma)\,\mathrm{d}\tilde{\lambda}^y(y,\gamma) = \int_{\mathcal{G}^{\rho(y)}} \mathbb{1}_{C_y}(\gamma) f(y,\gamma)\,\mathrm{d}\lambda^{\rho(y)}(\gamma)$$

is continuous.

We have

$$\int_{\mathcal{G}^{\rho(y)}} \mathbb{1}_{C_y}(\gamma) f(y,\gamma)\,\mathrm{d}\lambda^{\rho(y)}(\gamma) = \int_{\mathcal{G}^{\rho(y)}} f(y,\gamma)\,\mathrm{d}\lambda^{\rho(y)}(\gamma)$$

since the support of f is contained in C.

We will show that for every $f \in C_c(Y \times \mathcal{G})$, the function

$$y \mapsto \int_{\mathcal{G}^{\rho(y)}} f(y,\gamma)\,\mathrm{d}\lambda^{\rho(y)}(\gamma)$$

is continuous. This will conclude the proof, since that by the Tietze extension theorem, every function in $C_c(C) \subset C_c(C')$ can be extended as an element of $C_c(Y \times \mathcal{G})$.

Let f be in $C_c(Y \times \mathcal{G})$ and let $\Omega_1 \times \Omega_2$ be an open neighborhood of the support of f, where Ω_1 and Ω_2 are relatively compact. Then we have

$$M = \sup_{y \in \Omega_1} \lambda^{\rho(y)}(\Omega_2) < +\infty.$$

Let us consider the algebra $\mathcal{C}$ linearly generated by the functions of the form

$$(y, \gamma) \in Y \times \mathcal{G} \mapsto (h \otimes k)(y, \gamma) = h(y)k(\gamma)$$

where h, k are continuous with compact support in Ω_1 and Ω_2 respectively. Using the Stone-Weierstrass theorem, given $\varepsilon > 0$ there is $g \in \mathcal{C}$ such that $\|f - g\|_\infty < \varepsilon$. It follows that

$$\left| \int_{\mathcal{G}^{\rho(y)}} f(y,\gamma)\, \mathrm{d}\lambda^{\rho(y)}(\gamma) - \int_{\mathcal{G}^{\rho(y)}} g(y,\gamma)\, \mathrm{d}\lambda^{\rho(y)}(\gamma) \right| \leq M\varepsilon$$

for every $y \in Y$. Therefore, it suffices to show that

$$y \mapsto \int_{\mathcal{G}^{\rho(y)}} h(y)k(\gamma)\, \mathrm{d}\lambda^{\rho(y)}(\gamma)$$

is continuous for $h \in C_c(Y)$ and $k \in C_c(\mathcal{G})$. But this is obvious. □

6. C^*-algebras of partial actions of groupoids on locally compact spaces

6.1. **Full C^*-algebra.** Let $\boldsymbol{\beta} : \mathcal{G} \curvearrowright Y$ be a partial action as above and let $\mathcal{B}$ be the semidirect product bundle of $A = C_0(Y)$ and $\mathcal{G}$. We study the relation between $C^*(Y \rtimes \mathcal{G})$ and $C^*(\mathcal{G}, \mathcal{B})$. On one hand $C^*(Y \rtimes \mathcal{G})$ is the enveloping C^*-algebra of the completion of $C_c(C)$ with respect to the norm

$$\|f\|_I = \max\big(\sup_{y \in Y} \int_{C_y} |f(y,\gamma)|\, \mathrm{d}\lambda^{\rho(y)}(\gamma),\ \sup_{y \in Y} \int_{C_y} \left| f(\gamma^{-1}y, \gamma^{-1}) \right| \mathrm{d}\lambda^{\rho(y)}(\gamma) \big).$$

Note that for a second countable groupoid $Y \rtimes \mathcal{G}$, the C^*-algebra $C^*(Y \rtimes \mathcal{G})$ is also the completion of $C_c(C)$ relative to the norm $\|f\| = \sup_L \|L(f)\|$ where L ranges over all the nondegenerate representations of $C_c(Y \rtimes \mathcal{G})$ into a Hilbert space H that are continuous when $C_c(Y \rtimes \mathcal{G})$ is equipped the inductive limit topology and $\mathcal{B}(H)$ is equipped with the weak operator topology (see [22, Corollaire 4.8], or [16, Remark 4.14]).

On the other hand $C^*(\mathcal{G}, \mathcal{B})$ is the completion of $\Gamma_c(\mathcal{G}, \mathcal{B})$ with respect to the universal norm $\|f\| = \sup_L \|L(f)\|$ where L ranges over all I-norm decreasing representations, the I-norm being defined as

$$\|f\|_I = \max\big(\sup_{y \in Y} \int_{C_y} \|f(\gamma)\|\, \mathrm{d}\lambda^{\rho(y)}(\gamma), \sup_{y \in Y} \int_{C_y} \left\| f(\gamma^{-1}) \right\| \mathrm{d}\lambda^{\rho(y)}(\gamma) \big).$$

Equivalently $C^*(\mathcal{G},\mathcal{B})$ is the completion of $\Gamma_c(\mathcal{G},\mathcal{B})$ with respect to the universal norm $\|f\| = \sup_L \|L(f)\|$ where L ranges over all the representations of $\mathcal{B}$ in the sense of [16, Definition 4.7] (see [16, Remark 4.14].

We have $\mathcal{B} = \sqcup_{\gamma\in\mathcal{G}} C_0(Y_\gamma)$. For $f \in C_c(Y \rtimes \mathcal{G})$ we denote by $\hat{f}$ the element of $\Gamma_c(\mathcal{G},\mathcal{B})$ such that $\hat{f}(\gamma)(y) = f(y,\gamma)$. The map $\Psi : f \mapsto \hat{f}$ is an injective homomorphism of $*$-algebras from $C_c(Y \rtimes \mathcal{G})$ into $\Gamma_c(\mathcal{G},\mathcal{B})$. Its image is dense in $\Gamma_0(\mathcal{G},\mathcal{B})$ (see [16, Lemma A.4]). It is also dense in the inductive limit topology (see [2, Proposition 3.1] or [9, Proposition 14.6]).

If π is a representation of $\Gamma_c(\mathcal{G},\mathcal{B})$, then $\pi\circ\Psi$ is a representation of $C_c(Y \rtimes \mathcal{G})$. This map is injective. Let us show that it is surjective. We follow the proof of [2, Theorem 3.3]. Let $\tilde{\pi} : C_c(Y \rtimes \mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be a representation of $Y \rtimes \mathcal{G}$ and define $\pi : \Psi(C_c(Y \rtimes \mathcal{G})) \to \mathcal{B}(\mathcal{H})$ by $\pi(\Psi(f)) = \tilde{\pi}(f)$. We have $\left\|\pi(\hat{f})\right\| \leq \|f\|_I$. If $\hat{f}$ is supported into a symmetric compact subset K of $\mathcal{G}$, then

$$\left\|\pi(\hat{f})\right\| \leq \|f\|_I \leq \sup_{x\in X} \lambda^x(K)\left\|\hat{f}\right\|_\infty.$$

Then, as in the proof of [2, Theorem 3.3], we see that π extends to a linear map from $\Gamma_c(\mathcal{G},\mathcal{B})$ into $\mathcal{B}(\mathcal{H})$: if $f \in \Gamma_K(\mathcal{G},\mathcal{B})$ is the limit in norm $\|\cdot\|_\infty$ of a sequence $\Psi(f'_n)$ with support in K then $\pi(f)$ is defined as $\lim_n \tilde{\pi}(f'_n)$. We check as in [2] that π is a well defined linear map on $\Gamma_c(\mathcal{G},\mathcal{B})$, continuous in the inductive limit topology, and that π preserves the involution and the product of $\Gamma_c(\mathcal{G},\mathcal{B})$. Of course, we have $\tilde{\pi} = \pi\circ\Psi$.

It follows that the map Ψ extends to an isomorphism from the C^*-algebra $C^*(Y \rtimes \mathcal{G})$ onto the C^*-algebra $C^*(\mathcal{G},\mathcal{B})$.

6.2. **Reduced C^*-algebra.** We keep the same notation as in the previous subsection as well as the notation of Section 5.1. We want to show that the C^*-algebras $C^*_r(Y \rtimes \mathcal{G})$ and $C^*_r(\mathcal{G},\mathcal{B})$ are canonically isomorphic. For the definition of the reduced C^*-algebra of a locally compact groupoid, in the way it is presented here, we refer to [4, Section 6.1]. Recall that $C^*_r(Y \rtimes \mathcal{G})$ acts on $L^2_{C_0(Y)}(Y \rtimes \mathcal{G}, \tilde{\lambda})$. This Hilbert $C_0(Y)$-module is the completion of $C_c(Y \rtimes \mathcal{G})$ with respect to the inner product

$$\langle \xi,\eta\rangle(y) = \int_{C_y} \overline{\xi(y,\gamma)}\eta(y,\gamma)\,\mathrm{d}\lambda^{\rho(y)}(\gamma).$$

The right action of $C_0(Y)$ is given by

$$(\xi f)(y,\gamma) = \xi(y,\gamma)f(y).$$

The left regular representation Λ is defined by

$$(\Lambda(f)\xi)(y,\gamma) = \int_{C_y} f(\gamma^{-1}y,\gamma^{-1}\gamma_1)\xi(y,\gamma_1)\,\mathrm{d}\lambda^{\rho(y)}(\gamma_1).$$

On the other hand, $C_r^*(\mathcal{G},\mathcal{B})$ acts on $L^2(\mathcal{G},\mathcal{B})$, which is the Hilbert $\Gamma_0(X,\mathcal{B}^{(0)})$-module obtained by completion of $\Gamma_c(\mathcal{G},\mathcal{B})$ with respect to the inner product

$$\langle \xi,\eta\rangle(x)=\int_{\mathcal{G}^x}\theta_\gamma\big(\xi(\gamma^{-1})^*\eta(\gamma^{-1})\big)\,\mathrm{d}\lambda^x(\gamma).$$

The right action of $\Gamma_0(X,\mathcal{B}^{(0)})$ is given by

$$(\xi f)(\gamma)=\theta_\gamma\big(\theta_{\gamma^{-1}}\xi(\gamma)f\circ s(\gamma)\big).$$

The representation π^l is given, for $f\in\Gamma_c(\mathcal{G},\mathcal{B})$, by

$$(\pi^l(f)\xi)(\gamma)=\int_{\mathcal{G}^x}\theta_{\gamma_1}\big(\theta_{\gamma_1^{-1}}(f(\gamma_1))\xi(\gamma_1^{-1}\gamma)\big)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma_1).$$

But $\Gamma_0(X,\mathcal{B}^{(0)})=C_0(Y)$ and therefore $L^2(\mathcal{G},\mathcal{B})$ is also a continuous field of Hilbert spaces over Y. The structure of right Hilbert $C_0(Y)$-module is given by the following formulas, for $\xi,\eta\in\Gamma_c(\mathcal{G},\mathcal{B})$, $f\in C_0(Y)$ and $y\in Y^x$:

$$\begin{aligned}\langle\langle \xi,\eta\rangle\rangle(y)=\langle \xi,\eta\rangle(x)(y)&=\int_{\mathcal{G}^x}\theta_\gamma\big(\xi(\gamma^{-1})^*\eta(\gamma^{-1})\big)(y)\,\mathrm{d}\lambda^x(\gamma)\\&=\int_{\mathcal{G}^x}\overline{\xi(\gamma^{-1})(\gamma^{-1}y)}\eta(\gamma^{-1})(\gamma^{-1}y)\,\mathrm{d}\lambda^x(\gamma)\\&=\int_{C_y}\overline{\xi(\gamma^{-1}y,\gamma^{-1})}\eta(\gamma^{-1}y,\gamma^{-1})\,\mathrm{d}\lambda^{\rho(y)}(\gamma),\end{aligned}$$

$$(\xi\cdot h)(y,\gamma)=(\xi h)(\gamma)(y)=0\quad\text{if}\quad y\in Y^{r(\gamma)}\setminus Y_\gamma$$

and

$$\begin{aligned}(\xi\cdot h)(y,\gamma)&=\theta_\gamma\big(\theta_{\gamma^{-1}}(\xi(\gamma))h\circ s(\gamma)\big)(y)\\&=\xi(y,\gamma)h(\gamma^{-1}y)\end{aligned}$$

otherwise.

Finally,

$$(\pi^l(f)\xi)(y,\gamma)=\int_{C_y}f(y,\gamma_1)\xi(\gamma_1^{-1}y,\gamma_1^{-1}\gamma)\,\mathrm{d}\lambda^{\rho(y)}(\gamma_1).$$

Let us denote by U the map from $C_c(Y\rtimes\mathcal{G})$ onto $C_c(Y\rtimes\mathcal{G})$ defined by $(U\xi)(y,\gamma)=\xi(\gamma^{-1}y,\gamma^{-1})$. One immediately check that $\langle\langle U\xi,U\eta\rangle\rangle=\langle\xi,\eta\rangle$ and that $U(\xi h)=U(\xi)\cdot h$.

Let us show that the range of U is dense in $L^2(\mathcal{G},\mathcal{B})$. Let

$$\xi\in\Gamma_c(\mathcal{G},\mathcal{B})\subset\Gamma_0(\mathcal{G},\mathcal{B})=C_0(C)$$

and let K be a compact subset of $\mathcal{G}$ such that $\xi(y,\gamma)=0$ if $\gamma\notin K$. Set $M=\sup_{y\in Y}\int_{C_y}\mathbb{1}_K(\gamma^{-1})\,\mathrm{d}\lambda^{\rho(y)}(\gamma)$. Given $\varepsilon>0$ let $f\in C_c(C)$ such that $\|\xi-f\|_\infty<\varepsilon M^{-1/2}$. We may take f such that $f(y,\gamma)=0$ if $\gamma\notin K$. We have

$$\begin{aligned}\|\langle\langle\xi - f, \xi - f\rangle\rangle\| &= \sup_{y\in Y}\int_{C_y}\left|\xi(\gamma^{-1}y,\gamma^{-1}) - f(\gamma^{-1}y,\gamma^{-1})\right|^2 \mathrm{d}\lambda^{\rho(y)}(\gamma)\\ &\le \varepsilon^2 \sup_{y\in Y}\int_{C_y}\mathbb{1}_K(\gamma^{-1})\,\mathrm{d}\lambda^{\rho(y)}(\gamma) \le \varepsilon^2.\end{aligned}$$

It follows that U extends to an isomorphism of Hilbert $C_0(Y)$-module from $L^2_{C_0(Y)}(Y \rtimes \mathcal{G}, \tilde{\lambda})$ onto $L^2(\mathcal{G}, \mathcal{B})$.

Let us check that $(U\Lambda(f)\xi = (\pi^l(f)U)\xi$. We have

$$\begin{aligned}(U\Lambda(f)\xi)(y,\gamma) &= (\Lambda(f)\xi)(\gamma^{-1}y,\gamma^{-1})\\ &= \int_{C_{\gamma^{-1}y}} f(y,\gamma\gamma_1)\xi(\gamma^{-1}y,\gamma_1)\,\mathrm{d}\lambda^{s(\gamma)}(\gamma_1)\\ &= \int_{C_y} f(y,\gamma_1)\xi(\gamma^{-1}y,\gamma^{-1}\gamma_1)\,\mathrm{d}\lambda^{\rho(y)}(\gamma_1),\end{aligned}$$

and

$$\begin{aligned}(\pi^l(f)U\xi)(y,\gamma) &= \int_{C_y} f(y,\gamma_1)(U\xi)(\gamma_1^{-1}y,\gamma_1^{-1}\gamma)\,\mathrm{d}\lambda^{\rho(y)}(\gamma_1)\\ &= \int_{C_y} f(y,\gamma_1))\xi(\gamma^{-1}y,\gamma^{-1}\gamma_1)\,\mathrm{d}\lambda^{\rho(y)}(\gamma_1).\end{aligned}$$

Thus the map $\Psi_r : \Lambda(f) \mapsto U\Lambda(f)U^*$ induces an isomorphism from $C^*_r(Y \rtimes \mathcal{G})$ onto a C^*-subalgebra of $C^*_r(\mathcal{G}, \mathcal{B})$. It remains to show that Ψ_r is surjective, but this holds true since we have a commutative diagram where Ψ is an isomorphism and the vertical arrows are surjective:

$$\begin{array}{ccc} C^*(Y \rtimes \mathcal{G}) & \xrightarrow{\Psi} & C^*(\mathcal{G},\mathcal{B})\\ \downarrow & & \downarrow\\ C^*_r(Y \rtimes \mathcal{G}) & \xrightarrow{\Psi_r} & C^*_r(\mathcal{G},\mathcal{B})\end{array}$$

7. Partial actions of amenable étale groupoids

Proposition 7.1. *Let $\beta : \mathcal{G} \curvearrowright Y$ be a partial action where $\mathcal{G}$ is an étale locally compact amenable groupoid. Then the groupoid $Y \rtimes \mathcal{G}$ is amenable.*

Proof. Let $(\mathcal{B}, p)$ be the corresponding Fell bundle. Then, the reduced C^*-algebra $C^*_r(\mathcal{G}, \mathcal{B})$ is nuclear by [27, Theorem 4.1]. It follows that the groupoid reduced C^*-algebra $C^*_r(Y \rtimes \mathcal{G})$ is nuclear and so the groupoid $Y \rtimes \mathcal{G}$ is amenable by [3, Corollary 6.2.14]. □

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