The convective approximations

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Introduction

Convection can occur in a shallow layer of fluid with a small temperature contrast across the layer. Thus, it is natural to hope for a simpler description of buoyantly driven flows via an expansion of the fully compressible equations. The Boussinesq approximation performs well in modeling convection in laboratory experiments, but it is however unsatisfactory for large stratified systems, the lower part of which is compressed by the overlying material. More general approximations are then required to describe convection in natural systems like oceans, stars or planetary cores.

Conservation laws

- Mass conservation

$$\boxed{\partial_t \rho + \nabla \cdot (\rho \mathbf{V}) = 0}$$

- Momentum conservation

$$\boxed{\rho D_t \mathbf{V} = -\nabla P + \rho \mathbf{g} + \mathbf{F}^\nu}$$

$$\text{with} \quad \mathbf{g} = -\nabla U \quad \text{and} \quad \mathbf{F}^\nu = \nabla \cdot (\tau)$$

$$\text{and} \quad \tau_{ij} = 2\mu \left[\frac{1}{2} (\partial_i v_j + \partial_j v_i) - \frac{1}{3} \partial_k v_k \delta_{ij} \right]$$

where μ is the dynamic viscosity of the fluid

- Energy conservation

$$\partial_t u^{tot} + \nabla \cdot \left(\mathbf{I}^{tot} \right) = 0$$

where

$$u^{tot} = \rho \left(E^k + E^I + U \right) \quad \text{(total energy)}$$

$$\mathbf{I}^{tot} = \rho \mathbf{V} \left(E^k + E^H + U \right) + \mathbf{I}^q + \mathbf{I}^\nu \quad \text{(total energy flux)}$$

with

$$E^H = E^I + P/\rho \quad \text{(specific enthalpy)}$$

$$\mathbf{I}^q = -K \nabla T \quad \text{(heat flux)}$$

$$I_i^\nu = -V_j \tau_{ji} \quad \text{(viscous energy flux)}$$

- Conservation laws and the first principle of thermodynamics

$$\mathrm{d}\, E^I = -P \, \mathrm{d}\, \rho^{-1} + T \, \mathrm{d}\, S$$

lead to the heat transfert equation

$$\boxed{\rho T D_t S + \nabla \cdot (\mathbf{I}^q) = Q^\nu}$$

with the viscous heating $Q^\nu = \tau_{ij} \nabla_j V_i$

The reference state

Decomposition of the thermodynamics variables into the sum of a steady variable corresponding to the reference atmosphere and a convective disturbance

$$f = f_\mathsf{a} + f_\mathsf{c}$$

The reference state must be in quasiequilibrium

- mechanical quasiequilibrium: hydrostatic balance

$$\boxed{-\nabla P_a + \rho_a \mathbf{g} = 0}$$

- thermal quasiequilibrium: “well mixed” isentropic reference state

$$\boxed{\nabla S_a = 0}$$

This defines the adiabatic gradient

$$\nabla T_a = \left(\frac{\partial T}{\partial P} \right)_S \nabla P_a = \alpha^s \mathbf{g} = \frac{\alpha T_a}{C_p} \mathbf{g}$$

$$\nabla T_a = \frac{\mathbf{g}}{C_p} \quad \text{(in the case of a perfect gas)}$$

The anelastic approximation

- non-dimensionalizing the equations: a dimensionless parameter must tend to zero when the temperature contrast goes to zero
- resulting set of equations

$$\nabla \cdot (\rho_a \mathbf{V}) = 0$$

$$D_t \mathbf{V} = -\nabla \left(\frac{P_c}{\rho_a} \right) - \alpha^S S_c \mathbf{g} + \mathbf{F}^\nu / \rho_a$$

$$D_t (T_a S_c) + \nabla \cdot (\mathbf{I}_a^q + \mathbf{I}_c^q) - \rho_a \alpha^S S_c \mathbf{V} \cdot \mathbf{g} = Q^\nu$$

The Boussinesq approximation

- “thin layer approximation”: a dimensionless parameter must tend to zero when the size of the convective region tends to zero
- resulting set of equations

$$\nabla \cdot (\mathbf{V}) = 0$$

$$D_t \mathbf{V} = -\nabla \Pi - \alpha T \mathbf{g} + \frac{\mu}{\rho_0} \nabla^2 \mathbf{V}$$

$$D_t T = \kappa \nabla^2 T$$

Numerical implementation

- mean field approach for the heat transfert equation

  split the variables into a mean and fluctuating part,

$$f = \langle f \rangle + f^t$$

  so that

$$\langle \langle f \rangle \rangle = \langle f \rangle \quad \text{and} \quad \langle f^t \rangle = 0$$

  This creates a turbulent entropy flux term

$$\mathbf{I}^{St} = \rho_a \langle S^t \mathbf{V}^t \rangle \propto \nabla Sc$$

  Hypothesis: the turbulent diffusion of entropy dominates over the molecular diffusion of temperature in the heat transfert equation

- assume a polytropic reference state ($P \propto \rho^\gamma$) for a perfect gas

  Two additional non-Boussinesq control parameters:
  - the polytropic index $n = 1/(\gamma - 1)$
  - the degree of stratification $N_\rho = \ln \left(\frac{\rho_a \,\text{(bottom)}}{\rho_a \,\text{(top)}} \right)$

Issues

- use of a non steady reference state
- consistency and domain of validity of the different variants of the anelastic approximation
- averaging the heat transfert equation

$$\rho T D_t S = \nabla \cdot \left(\rho T \kappa^t \nabla S \right) + \nabla \cdot (K \nabla T) + Q^\nu$$

Conclusions

- practical interest: filter out fast processes (acoustic/seismic processes) and retain slow processes (convection)

  $\Longrightarrow$ low Mach approximation

  $\Longrightarrow$ faster numerical integration

- method: from a thermodynamical point of view, consider the convective state as a small departure from a steady reference state

References:

- Braginsky & Roberts, GAFD, 1995
- The anelastic dynamo benchmark: Jones *et al.*, Icarus, 2011
- Brown *et al.*, APJ, 2012