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Modified Double Displacement and Modified Medial Laws; A New Approach

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Abstract:
In this article we introduced new structures that are modifications in medial, LDD-semigroup and RDD-semigroup and discussed that on what conditions a non commutative groupoid S which is medial becomes RDD-semigroup and LDD-semigroup and we also explained the different conditions on which a groupoid S becomes commutative.

Introduction: N. Kimura [1], explained the idea of idempotent semigroups in their research and explained details about left regular, right regular and regular semigroups. Yamada and Kimura [2], discussed the left (right) normality of semigroup and proved that if semigroup S is left (right) normal then S is left (right) regular. Clifford and Preston [3], elaborated the idea that “if I is an ideal of a semigroup S, then I and S/I are regular (inverse) if and only if S is regular (inverse)”. Clifford and Preston also proved the result that if R is some zero minimal right ideal of S then either rS = R for each r belongs to R\{0} or else rS = 0 is satisfied with R = {0, r}. T.E. Hall [4], presented idea of such structures which are regular semigroups and their idempotents also form regular semigroup. T.E. Hall discussed the concept of commutativity, left normality, right normality and normality and achieved his results. Kazim and Naseeruddin [5], introduced the concept of left almost semigroup in 1971 during their Ph.D studies they proved that every commutative semigroup is LA-Semigroup as well as RA-Semigroup. Mushtaq and Yousaf [6], extended the work of Kazim and Naseeruddin in their research and developed the idea of LA-Semigroups which are locally associative. Q. Mushtaq [7], extended these results and introduced the concept that on what conditions left almost semigroup becomes commutative monoid and becomes group. N. Kehayopulu [8], used specific multiplication and ordering of elements and constructed semigroups which are completely regular. N. Kehayopulu also explained a result that “An ordered semigroup S is completely regular if and only if the bi-ideals of S are semiprime”. Commutative LA-Semigroup work was extended by Mushtaq and Kamran [9] and they proved that left almost semigroup S = S^2 is commutative if and only if (ab)c = a(cb) for all a, b and c being inputs of set S. M. Khan [10], constructed Cayley diagram for LA-Semigroups constructed from finite fields. M. Khan et al [11], explained that if S is LA-Semigroup with left identity then S holds paramedial law i.e. (ab)(cd) = (db)(ca) holds. N. Ahmad et al [12] introduced the idea of LDD-semigroup and RDD-semigroup and put open problem for RDD-semigroup. J.L. Chrislock [13] elaborated in detail that a semigroup S is medial is for all inputs a, b, c and d in S, cabd = cbad. Cho et al [14] explained and proved the conditions for which a groupoid S becomes paramedial.

In the Preliminaries section we put basic definitions so readers can grasp idea in painfree way and in New-Structures section we are introducing new structures with definitions. We used the concepts discussed in [1, 14] and by using new definitions, we solved open problem put by N. Ahmad et al [12] and achieved results.

Preliminaries:
1.1: In literature a groupoid $S$ is called left almost semigroup abbreviated as LA-Semigroup if for all inputs $a$, $b$, and $c \in S$ the condition $(ab)c = (cb)a$ holds.

1.2: A groupoid $S$ is called right almost semigroup abbreviated as RA-Semigroup if for all $a$, $b$, and $c \in S$, $a(bc) = c(ba)$.

1.3: A groupoid $S$ is called medial if for all $a$, $b$, $c$, and $d \in S$, $(ab)(cd) = (ac)(bd)$. This property $(ab)(cd) = (ac)(bd)$ is called medial law or bisymmetry law.

1.4: A semigroup $S$ is called left regular semigroup if for all $a$, $b \in S$ the condition $ba^2 = a$ holds.

1.5: A semigroup $S$ is called right regular semigroup if for all $a$, $b \in S$ the condition $a^2b = a$ holds.

1.6: A semigroup $S$ is called complete regular semigroup if for all $a$, $b \in S$ the conditions $aba = a$, $a^2b = a$, and $ba^2 = a$ are satisfied.

1.7: A semigroup $S$ is called E-semigroup if all the idempotents of $S$ also form semigroup.

1.8: A semigroup $S$ is called Orthodox semigroup if $S$ is E-semigroup as well as regular semigroup.

1.9: A groupoid $S$ is called locally associative if for all $a \in S$ condition $a^2a = aa^2$ is satisfied. If each element $a \in S$ is idempotent then $S$ is locally associative groupoid.

1.10: LA-Semigroup $S$ is called LA-Group if there exists left identity in $S$ and left inverse of each element exists.

1.11: If for all $a$, $b$, $c$ and $d \in S$, $(ab)(cd) = (db)(ca)$ then $S$ is Paramedial.

1.12: A groupoid $S$ is called LDD-semigroup if for all $a$, $b$, $c$ and $d \in S$, $(ab)(cd) = (cb)(ad)$.

1.13: A groupoid $S$ is called RDD-semigroup if for all $a$, $b$, $c$ and $d \in S$, $(ab)(cd) = (ad)(cb)$.

1.14: If for all $a$, $b$, $c$ and $d \in S$, $(ab)(cd) = (db)(ca)$ then $S$ is Paramedial.

**New-Structures with Following Definitions:** If $S$ is a groupoid and for all $a$, $b$, $c$ and $d \in S$ if

2.1: Right shift double displacement law holds i.e. $(ab)(cd) = (ad)(bc)$ then $S$ is called right shift right double displacement groupoid (RSDR-groupoid).

2.2: Right shift medial law holds i.e. $(ab)(cd) = (ac)(db)$ then $S$ is called right shift medial (RSM).

2.3: Left Shift medial law holds i.e. $(ab)(cd) = (ca)(bd)$ then $S$ is called left shift medial (LSM).

2.4: Reverse right shift double displacement law holds i.e. $(ab)(cd) = (bc)(ad)$ then $S$ is called reverse right shift right double displacement (Reverse RSDR).

2.5: Reverse right shift medial law holds i.e. $(ab)(cd) = (db)(ac)$ then $S$ is called reverse right shift medial (R,RSM).

2.6: Reverse left shift medial law holds i.e. $(ab)(cd) = (bd)(ca)$ then $S$ is called reverse left medial (R,LSM).

2.7: Left shift right double displacement law holds i.e. $(ab)(cd) = (da)(cb)$ then $S$ is called left shift right double displacement groupoid (LSDRD-groupoid).

2.8: Reverse left shift double displacement law holds i.e. $(ab)(cd) = (cb)(da)$ then $S$ is called reverse left shift right double displacement (Reverse LSDR).

2.9: Reverse medial law holds i.e. $(ab)(cd) = (bd)(ac)$ then $S$ is reverse medial (R,M).

2.10: Reverse Paramedial law holds i.e. $(ab)(cd) = (ca)(db)$ then $S$ is Reverse Paramedial (R,PM).

2.11: Reverse law holds i.e. $(ab)(cd) = (dc)(ba)$ then $S$ is Reverse.

2.12: Double shift double displacement law holds i.e. $(ab)(cd) = (da)(bc)$ then $S$ is called double shift double displacement groupoid (DSDD-groupoid).
Though we have not found any example of these laws in chemistry, physics or in any branch of science yet these new assumed structures with the definitions helped us in achieving following results if a groupoid $S$ satisfies these properties:

**Open Problem Put By N. Ahmad et al:** Here we only study about LDD-semigroup where RDD-semigroup is left as open problem for Researchers.

**Answer:** Every $L_{0}$ is RDD-semigroup as well as semigroup because when we have non empty set $S$ and binary operation is defined on $S$ by such way that for all $a$ and $b$ being inputs of $S$, $ab = a$ then by the definition $(ab)c = ac = a$ and also $a(bc) = ab = a$. This also holds medial law as well RDD-law, RSM-law and RSRD-law because $ab = a$ and so we have $(ac)(bd) = ac = a$, $(ad)(cb) = ac = a$ and also $(ad)(bc) = ab = a$.

Every $R_{0}$ is semigroup and holds medial law but does hold RDD-law, RSRD-law but holds LDD-law, LSM-law and Reverse RSRD-law because when we have non empty set $S$ and binary operation is defined on $S$ by such way that for all $a$ and $b \in S$, $ab = b$ then by the definition $(ab)c = bc = c$ and also $a(bc) = ac = c$. This also holds medial law if we see $(ab)(cd) = bd = d$ and $(ac)(bd) = cd = d$. $R_{0}$ is LDD-semigroup because for all $a, b, c$ and $d \in S$, $(cb)(ad) = bd = d$.

$L_{0}$ (LZ) is RDD-semigroup, RSM, RSRD-groupoid and semigroup where $R_{0}$ (RZ) is LDD-semigroup, LSM, Reverse RSRD-groupoid and semigroup.

**Result-3.1 in Open-Problem:** Any groupoid $S$ that holds medial law and RSRD-Law, LDD-Law, RDD-Law and LSRD-Law is commutative.

**Proof:** Let we have set $S$ and we define binary operation on $S$ by such way that for all $a, b, c$ and $d \in S$ all following properties hold:

- $(ab)(cd) = (ac)(bd)$ (Medial law)
- $(ab)(cd) = (ad)(cb)$ (RDD-Law)
- $(ab)(cd) = (ad)(bc)$ (RSRD-Law)
- $(ab)(cd) = (cb)(ad)$ (LDD-Law)
- $(ab)(cd) = (da)(cb)$ (LSRD-Law)

From the above results this is clear that if $S$ is LDD-semigroup and RDD-semigroup i.e. $(ab)(cd) = (ad)(cb) = (cb)(ad)$ then $S$ is commutative groupoid.

**Result-3.2:** If $S$ is commutative then $S$ is locally associative because for all $a \in S$, $a^{2}a = aa^{2}$.

Also $S$ is Reverse because for all $a, b, c$ and $d \in S$ we have the following results:


We express all these results by the relation $(ab)(cd) = (dc)(ba)$. In all cases $a$ with $b$ and $c$ with $d$ comes with each other.

**Result-3.3 In Open-Problem:** If $S$ commutative and medial then $S$ is RSRD-groupoid, RDD-semigroup, LDD-semigroup and Reverse RSDD-groupoid.

**Proof:** Given that structure $S$ holds commutative law and medial law. So for all $a, b, c$ and $d$ belongs to $S$ holds following two properties: $ab = ba$ and $(ab)(cd) = (ac)(bd)$.
Questions arises that if S is not commutative but medial then on what condition S is RDD-semigroup.

**Result-3.4:** If S is medial and RSM then S is RDD-semigroup.

**Proof:** S is medial and RSM so for all a, b, c and d ∈ S the two conditions (ab)(cd) = (ac)(bd) and (ab)(cd) = (ac)(db) hold. So (ab)(cd) = (ac)(db) = (ad)(cb). First right medial and then medial law is applied and we get result.

**Result-3.5:** If S is medial and RSM then S is RSRD-groupoid.

**Proof:** This is already proved in Result-3.4 that if S is medial and RSM then S is RDD-semigroup. We are to show that S is RSRD-groupoid. So (ab)(cd) = (ac)(bd) = (ac)(db) = (ad)(cb), so (ab)(cd) = (ac)(bd) = (ad)(bc). First medial and then RDD-Law is applied and we get required result.

**Result-3.6:** If S is RSM then S is RSRD-groupoid.

**Proof:** S is RSM so for all a, b, c and d ∈ S, (ab)(cd) = (ac)(db) = (ad)(bc) which shows that S is RSRD-groupoid.

**Result-3.7:** If S is RSRD-groupoid then S is RSM.

**Proof:** S is RSRD-groupoid so for all a, b, c and d ∈ S, (ab)(cd) = (ad)(bc) = (ac)(db) which shows that S is RSM.

We left these as open problems for researchers to find the following results:

**Open Problem 1:** Prove or disprove generally that if S is non commutative but RSM and then S is medial.

**Open Problem 2:** Prove or disprove generally that if S is non commutative but RSRD-groupoid then S is medial.

**Open Problem 3:** Prove or disprove generally that if S is non commutative but RDD-semigroup then S is medial.

**Remarks:**

RSRD-law upon RSRD-law is applied and we get RSM i.e (ab)(cd) = (ad)(bc) = (ac)(db)  
RSM-law upon RSM-law is applied and we get RSRD-groupoid i.e. (ac)(db) = (ad)(bc)  
RSM-law upon RSRD-law is applied and (ab)(cd) comes again.  
RSRD-law upon RSM-law is applied and (ab)(cd) comes again.  
RDD-law upon RDD-law is applied and (ab)(cd) comes again.

Questions arises that on what condition groupoid S is medial and LDD-semigroup.

**Result-3.8:** If S is medial and LSM then S is LDD-groupoid.

**Proof:** S is medial and left shift medial then for all a, b, c and d ∈ S the two conditions always
hold, \((ab)(cd) = (ac)(bd)\) and \((ab)(cd) = (ca)(bd)\). First LSM-law and then medial law is applied.

**Result-3.9:** If \(S\) is medial and LSM then \(S\) is Reverse RSRD-groupoid.

**Proof:** We already proved in Result-3.8 that \(S\) is medial and left shift medial then \(S\) is LDD-semigroup. So for all \(a, b, c\) and \(d \in S\) the conditions \((ab)(cd) = (ac)(bd)\) and \((ab)(cd) = (ca)(bd)\) and \((ab)(cd) = (cb)(ad)\). So \((ab)(cd) = (ac)(bd) = (bc)(ad)\). First medial and then LDD-law is applied and we get result.

**Result-3.10:** If \(S\) is medial and LDD-semigroup then \(S\) is LSM and Reverse RSRD-groupoid.

**Proof:** \(S\) is medial and LDD-semigroup then for all \(a, b, c\) and \(d \in S\), \((ab)(cd) = (ac)(bd)\) and also \((ab)(cd) = (cb)(ad)\).

So \((ab)(cd) = (cb)(ad) = (ca)(bd)\) which shows that \(S\) is LSM and also \((ab)(cd) = (ca)(bd) = (bc)(ad)\). First LSM-law and then again LSM-law is applied.

**Result-3.11:** If \(S\) is LSM then \(S\) is Reverse RSRD-groupoid.

**Proof:** \(S\) is LSM then for all \(a, b, c\) and \(d \in S\), \((ab)(cd) = (ca)(bd) = (bc)(ad)\). This shows that if LSM-law is again applied then we get Reverse RSRD-law.

**Result-3.12:** If \(S\) is Reverse RSRD-groupoid then \(S\) is LSM.

**Proof:** \(S\) is Reverse RSRD-groupoid then for all \(a, b, c\) and \(d \in S\), \((ab)(cd) = (bc)(ad) = (ca)(bd)\). This shows that if we apply Reverse RSRD-law on \(S\) then \(S\) is LSM.

We left these as open problems for researchers to find the following results:

**Open Problem 4:** Prove or disprove generally that if \(S\) non commutative but LSM then \(S\) is medial.

**Open Problem 5:** Prove or disprove generally that if \(S\) non commutative but Reverse RSRD-groupoid then \(S\) is medial.

**Open Problem 6:** Prove or disprove generally that if \(S\) is non commutative but LDD-semigroup then \(S\) is medial.

**Remarks:**

LSM-law upon LSM-law is applied and we get Reverse RSRD-groupoid. Reverse RSRD-law upon Reverse RSRD-law is applied and we get LSM. LDD-law upon LDD-law is applied and \((ab)(cd)\) comes again. Reverse RSRD-law upon LSM-law is applied and \((ab)(cd)\) comes again. LSM-law upon Reverse RSRD-law is applied and \((ab)(cd)\) comes again.

**Result-3.13:** Commutative groupoids which are not medial, RDD-semigroup, LDD-semigroup, RSRD-groupoid, LSRD-groupoid, Reverse RSRD-groupoid and Reverse LSRD-groupoid.

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This is commutative but not associative because \((ab)(bb) = aa = b\) but \((ab)b = ab = a\). Also this is not medial and RDD-semigroup because \((aa)(bb) = ba = a\) but \((ab)(ab) = aa = b\). Also this is
not RSM because \((aa)(bb) = ba = a\) but \((ab)(ba) = aa = b\). We can also prove that this is not LDD-semigroup, LSM and Reverse RSRD-groupoid.

**An example of structure S which is commutative and holds all properties from 2.1 to 2.12 but is not semigroup, neither LA-Semigroup nor RA-Semigroup.**

**Exp:** On Real Numbers let we define binary operation such that \(ab = (a+b)/2\). So for all a, b, c and d \(\in S\), \((ab)(cd) = ((a+b)/2)((c+d)/2) = (a+b+c+d)/4\). If we apply any rule explained in the 2.1 to 2.12, we always get \((a+b+c+d)/4\). Also each element in R is idempotent because \(aa = (a+a)/2 = a\). On Real Numbers if we define binary operation such that \(ab = (a+b)/3\) or \(ab = (a+b)/k\) where \(k \neq 0\) and 1 then R w.r.t binary operation is commutative and holds all properties from 2.1 to 2.12. This is not semigroup, neither LA-Semigroup nor RA-Semigroup because \((ab)c = ((a+b)/2)c = (a+b+2c)/4\) which is not equal to \((2a+b+c)/4\) in general if we take \(a = 1\), \(b = 2\) and \(c = 3\). Also this is neither LA-Semigroup nor RA-Semigroup.

**Result-3.14:** If \(S\) is right shift medial and left shift medial then \(S\) is commutative.

**Proof:** \(S\) is right medial and left medial then for all a, b, c and d \(\in S\) the two conditions \((ab)(cd) = (ac)(db)\) and \((ca)(bd)\) holds.

Let \(a = cd\) and \(b = ef\) then \(ab = (cd)(ef)\)

\[
= (ce)(fd) \quad \text{RSM}
\]
\[
= (cf)(de) \quad \text{RSM}
\]
\[
= (dc)(fe) \quad \text{LSM}
\]
\[
= (df)(ec) \quad \text{RSM}
\]
\[
= (ed)(fc) \quad \text{LSM}
\]
\[
= (ef)(cd) \quad \text{RSM}
\]
\[
= \quad \text{ba}
\]

This shows \(S\) is commutative and \(S\) also holds medial law, RDD-law, LDD-law and all properties from 2.1 to 2.12.

**Theorem-3.1:** If \(S\) is RDD-semigroup with left identity then \(S\) is commutative semigroup.

**Proof:** \(S\) is RDD-semigroup then for all a, b, c and d \(\in S\), \((ab)(cd) = (ad)(cb)\) and \(S\) also contains left identity then for all a \(\in S\), there exists e such that \(ea = a\). So \(ab = (ea)(eb) = (eb)(ea) = ba\), which shows \(ab = ba\). To show this is associative we have \((ab)e = (ab)(ce) = (ae)(cb) = a(cb) = a(bc)\). [\(S\) is commutative and so \(ae = ea\) and \(cb = bc\)]

**Corollary Related to Theorem-3.1:** If \(S\) is RDD-semigroup with left identity then \(S\) is LA-Semigroup as well as RA-Semigroup.

**Proof:** Straightforward by using Theorem-1.

**Theorem-3.2:** If \(S\) is RDD-semigroup with right identity then \(S\) is semigroup if \((ab)c = (ac)b\).

**Proof:** \(S\) is RDD-semigroup then for all a, b, c and d \(\in S\), \((ab)(cd) = (ad)(cb)\) and \(S\) also contains right identity then for all a \(\in S\), there exists e such that ae = a. Simple way is to apply conditions i.e. \((ab)c = (ac)b = (ac)(be) = (ae)(bc) = a(bc)\) which shows that for all a, b and c \(\in S\), \((ab)c = a(bc)\).

**Theorem-3.3:** If \(S\) is RSRD-groupoid with left identity then \(S\) is semigroup if \(a(bc) = b(ca)\).

**Proof:** \(S\) is RSRD-groupoid then for all a, b, c and d \(\in S\), \((ab)(cd) = (ad)(bc)\) and \(S\) also
contains left identity then for all \( a \in S \), there exists \( e \) such that \( ea = a \). Simple way is to apply conditions i.e. \( a(bc) = b(ca) = (eb)(ca) = (ea)(bc) = a(bc) \) which shows that for all \( a, b \) and \( c \in S \), \( (ab)c = a(bc) \).

**Note:** If in Open Problem-2 this is proved that RSRD-groupoid \( S \) is medial then \( S \) is also RSM and RDD-semigroup, so \( S \) becomes commutative semigroup and LA-semigroup as well as RA-Semigroup.

**Result-3.15:** If \( S \) is RDD-semigroup with right identity and for all \( a, b \) and \( c \in S \), \( (ab)c = (ac)b \) but if \( S \) is not commutative then \( S \) is neither LA-Semigroup nor RA-Semigroup.

**Important Note and Results which we used in Next Theorems:**
(i) A commutative LA-Semigroup is commutative semigroup and RA-Semigroup.
(ii) A commutative RA-Semigroup is commutative semigroup and LA-semigroup.
(iii) If \( S \) is semigroup and LA-Semigroup then \( S \) is also RA-semigroup and commutative.
(iv) If \( S \) is semigroup and RA-Semigroup then \( S \) is also LA-semigroup and commutative.
(v) \( S \) is paramedial if \( S \) is medial one of these following conditions holds:
(a) \( S \) is commutative. (b)\( S \) is unipotent and also left (right) cancellative.
(c) \( S \) is LA-semigroup with left identity. (d) \( S \) is left (right) modular groupoid.
Further we prove that on what conditions groupoids are commutative.

**Theorem-3.4:** If \( S \) is Reverse medial (R\(_{v}\)M) then \( S \) is commutative.
**Proof:** \( S \) is R\(_{v}\)M then for all \( a, b, c \) and \( d \in S \), \( (ab)(cd) = (bd)(ac) \) and if we again apply reverse medial law then we get \( (dc)(ba) \) which shows that \( S \) holds Reverse-law, so \( S \) is commutative.

**Theorem-3.5:** If \( S \) is Reverse Paramedial (R\(_{v}\)PM) then \( S \) is commutative.
**Proof:** \( S \) is R\(_{v}\)PM then for all \( a, b, c \) and \( d \in S \), \( (ab)(cd) = (ca)(db) \) and if we again apply reverse paramedial law we get \( (dc)(ba) \), so \( S \) is Reverse and commutative.

**Result-3.16:** If \( S \) is Reverse RSM then \( S \) is Reverse LSRD-groupoid because \( S \) is Reverse RSM then for all \( a, b, c \) and \( d \in S \), \( (ab)(cd) = (db)(ac) \) and if we again apply Reverse RSM-law then we get \( (cb)(da) \) which shows that \( S \) is Reverse LSRD-groupoid.

**Result-3.17:** If \( S \) is Reverse LSM then \( S \) is LSRD-groupoid because if \( S \) is Reverse LSM then for all \( a, b, c \) and \( d \in S \), \( (ab)(cd) = (bd)(ca) \) and if we again apply Reverse LSM-law then we get \( (da)(cb) \) which shows that \( S \) is LSRD-groupoid.

**Result-3.18:** If \( S \) is DSDD-groupoid then \( S \) is commutative.
**Proof:** \( S \) is DSDD-groupoid then for all \( a, b, c \) and \( d \in S \), \((ab)(cd) = (da)(bc)\) and if we again apply DSDD-law then we get \( (cd)(ab) \) which shows that \( S \) is commutative. So if \( S \) is DSDD-groupoid then \( S \) is commutative.

**Result-3.19:** If \( S \) is DSDD-groupoid then \( S \) holds medial-law, RDD-law, LDD-law and all properties defined in 2.1 to 2.11.

**Theorem-3.6:** If \( S \) is LA-Semigroup and DSDD-groupoid then \( S \) is commutative semigroup.
**Proof:** Straightforward by using Important Note (i) and Result-3.19.
Theorem-3.7: If $S$ is RA-Semigroup and DSDD-groupoid then $S$ is commutative semigroup.
Proof: Straightforward by using Important Note (ii) and Result-3.19.

Result-3.20: If $S$ is Reverse then this is not necessary that $S$ is DSDD-groupoid e.g. if we have the following table then we can prove this in effortless way:

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This is locally associative and commutative but this is not DSDD-groupoid. $(aa)(bb) = ba = a$ but $(ab)(ab) = aa = b$. This is commutative and Reverse but this is not DSDD-groupoid and also this is not medial, LDD-semigroup, RDD-semigroup.

Monoids which are not LA-Semigroups and RA-Semigroups:
(1) Take set $S = \{1, 2\}$ and let $F$ be set of all functions from $S$ to $S$ then w.r.t binary operation of composition of mapping $F$ is monoid but $F$ is not LA-Semigroup, RA-Semigroup and even $F$ is not medial.
(2) $LO_n$ and $RO_n$ are semigroups but they are not LA-semigroups, RA-semigroups.
(3) Every non commutative monoid and semigroup is neither LA-semigroup nor RA-Semigroup.

Groups which are not LA-Semigroups and RA-Semigroups:
(1) Group of bijective mappings from set $S$ to $S$ i.e. $S_3, S_4$ and $S_n$.
(2) $GL(n, R)$ General linear group of invertible matrices w.r.t multiplication.
(3) Every non commutative group is neither LA-Semigroup nor RA-Semigroup.

Result-3.21: An example of groupoid which contains identity and contains inverse of each element but does not hold associative law, left invertive law and right invertive law.

Exp: Let we have set $S = \{1, 2, 3, 4, 5\}$ and . be binary operation defined on $S$ such that we have the following table:

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This is locally associative groupoid which contains identity element and inverse of each element exists. $2.4 = 4.2 = 1$ where $3.5 = 5.3 = 1$ and from table this is clear that 1 is identity. But this structure does not hold associative law because $(2.2).4 = 2.4 = 1$ and $2.(2.4) = 2.1 = 2$. Also this does not hold left invertive law and right invertive law. This is not LDD-semigroup, RDD-semigroup and this is not even medial if we see $(2.4).(3.3) = 1.3 = 3$ but $(2.3).(4.3) = 2.4 = 1$. Also this is not RDD-semigroup and LDD-semigroup.

Theorem-3.8: If $S$ is LA-Semigroup and Reverse RSM then $S$ is commutative semigroup.
Proof: Using Theorem-2 if $S$ is Reverse RSM then $S$ is commutative and by using important
Note (i) if $S$ is commutative LA-Semigroup then $S$ is commutative semigroup and RA-Semigroup.

**Theorem-3.9:** If $S$ is LA-Semigroup and reverse paramedial then $S$ is commutative semigroup. **Proof:** Using Theorem-3 if $S$ is reverse paramedial then $S$ is commutative and if $S$ is commutative LA-Semigroup then $S$ is commutative semigroup and RA-semigroup.

**Corollaries Related To Theorems 3.8 and 3.9:** If $S$ is RA-semigroup and holds any one law from reverse paramedial law, RSM-law or LSM-law then $S$ is commutative.

**Result-3.22:** Not every semigroup and monoid is medial.
(1) Set of all square matrices of order $m \times m$ w.r.t multiplication.
(2) Set of all functions from set $S$ to $S$ where $S = \{1, 2, 3\}$ and $S = \{1, 2, 3, 4\}$ and so on with binary operation composition of mapping.

**Theorem-3.10:** If $S$ is RDD-semigroup, Reverse RSRD-groupoid then $S$ is commutative.
**Proof:** If $S$ is RDD-semigroup and Reverse RSRD-groupoid so for all $a, b, c$ and $d \in S$ $(ab)(cd) = (ad)(bc)$ and $(ab)(cd) = (bc)(ad)$. We are to show that $S$ is commutative.

$$(ab)(cd) = (bc)(ad) \quad \text{Reverse RSRD-Law}$$
$$= (bd)(ac) \quad \text{RDD-Law} \quad (RvM)$$
$$= (da)(bc) \quad \text{Reverse RSRD-Law}$$
$$= (ab)(dc) \quad \text{Reverse RSRD-Law}$$
$$= (ac)(db) \quad \text{RDD-law}$$
$$= (cd)(ab) \quad \text{Reverse RSRD-Law}$$

So this is proved generally that $(ab)(cd) = (cd)(ab)$. We can also use Theorem 3.4 that if $S$ Reverse Medial then $S$ is commutative.

**Corollary in Theorem-3.10:** If $S$ is RDD-semigroup, Reverse RSRD-groupoid and semigroup then $S$ is commutative semigroup so $S$ is obvious case of LA-Semigroup and RA-Semigroup.

**Theorem-3.11:** If $S$ is LDD-semigroup, RSRD-groupoid then $S$ is commutative.
**Proof:** If $S$ is RDD-semigroup and Reverse RSRD-groupoid so for all $a, b, c$ and $d \in S$ $(ab)(cd) = (cb)(ad)$ and $(ab)(cd) = (ad)(bc)$. We are to show that $S$ is commutative.

$$(ab)(cd) = (ad)(bc) \quad \text{RSRD-Law}$$
$$= (bd)(ac) \quad \text{LDD-Law} \quad (RvM)$$
$$= (bc)(da) \quad \text{RSRD-Law}$$
$$= (ba)(cd) \quad \text{RSRD-Law}$$
$$= (ca)(bd) \quad \text{LDD-Law}$$
$$= (cd)(ab) \quad \text{RSRD-Law}$$

We can use Theorem 3.4 that if $S$ is Reverse Medial $(RvM)$ then $S$ is commutative.

**Corollary in Theorem-3.11:** If $S$ is LDD-semigroup, RSRD-groupoid and semigroup then $S$ is commutative semigroup so being commutative semigroup $S$ is LA-Semigroup and RA-Semigroup.
Examples of RDD-semigroups which do not hold not commutative law, Associative law, left invertive law and right invertive law:
(1) On positive real numbers if \( ab = \ln(a) \).
(2) On positive real numbers if \( ab = e^a \).
(3) On Rational numbers and on Real numbers if \( ab = a^2 \).
(4) On a set containing square matrices of order \( m \times m \), \( AB = A^t \). These hold RDD-law, medial law, RSM-law and RSRD-law.

Examples of LDD-semigroups which do not hold not commutative law, Associative law, left invertive law and right invertive law:
(1) On positive real numbers if \( ab = \ln(b) \).
(2) On positive real numbers if \( ab = e^b \).
(3) On Rational numbers and on Real numbers if \( ab = b^2 \).
(4) On a set containing square matrices of order \( m \times m \), \( AB = A^t \). These hold LDD-law, medial law, LSM-law and Reverse RSRD-law.

Open Problem 7: We left this to researchers as an open problem to construct a non commutative groupoid \( T \) which is only medial and does not hold RDD-law, LDD-law, Paramedial law, associative law, left invertive law, right invertive law and does not hold any property from 2.1 to 2.8.

[Obviously 2.9 to 2.12 are cases of commutative groupoid, so they will not be discussed]

Result-3.23: If \( S \) is LSRD-groupoid then \( S \) is Reverse LSM.
Proof: \( S \) is LSRD-groupoid so for all \( a, b, c \) and \( d \in S \), \( (ab)(cd) = (da)(cb) \) and if we again apply LSRD-law then we get \( (bd)(ca) \) which is reverse LSM. So if \( S \) is LSRD-groupoid then \( S \) is Reverse LSM. So \( (ab)(cd) = (da)(cb) = (ab)(cd) \). Reverse LSM law is applied on LSRD-law and we get \( (ab)(cd) \).

Result-3.24: If \( S \) is Reverse LSM then \( S \) is LSRD-groupoid.
Proof: \( S \) is Reverse LSM so for all \( a, b, c \) and \( d \in S \), \( (ab)(cd) = (da)(cb) \) and if we again apply Reverse LSM-law then we get \( (da)(cb) \) which shows that \( S \) is LSRD-groupoid.

Result-3.25: If \( S \) is medial and LSRD-groupoid then \( S \) is commutative.
Proof: \( S \) is medial and LSRD-groupoid so for all \( a, b, c \) and \( d \in S \), \( (ab)(cd) = (ac)(bd) \) and \( (ab)(cd) = (da)(cb) \) and we also proved in Result-3.23 that if \( S \) is LSRD-groupoid then \( S \) is Reverse LSM and \( (ab)(cd) = (bd)(ca) \). So we use these three results by the following steps: \( (ab)(cd) = (ac)(bd) = (da)(bc) \) which shows that \( S \) is DSDD-groupoid and by using Result-3.18 that if \( S \) is DSDD-groupoid then \( S \) is commutative.

Result-3.26: If \( S \) is medial and Reverse LSM then \( S \) is commutative.
Proof: Starightforward by using Results 3.23 to 3.25.

Theorem-3.12: If \( S \) is LA-Semigroup and LSRD-groupoid then \( S \) is commutative semigroup.
Proof: \( S \) is LA-Semigroup then \( S \) is medial and if \( S \) is also LSRD-groupoid then by using Result-3.25, \( S \) is commutative and so \( S \) is commutative semigroup.

Theorem-3.13: If \( S \) is LA-Semigroup and Reverse LSM then \( S \) is commutative semigroup.
Proof: Straightforward by using Result-3.23 to 3.25.
Corollary Related To Theorem-3.12: If S is RA-Semigroup and LSRD-groupoid then S is commutative semigroup.

Corollary Related To Theorem-3.13: If S is RA-Semigroup and Reverse LSM then S is commutative semigroup.

Result-3.27: If S is Reverse RSM then S is Reverse LSRD-groupoid.
Proof: S is Reverse RSM then for all a, b, c and d ∈ S, (ab)(cd) = (db)(ac) and if we again apply Reverse RSM-law then we get (cb)(da) which shows that S is Reverse LSRD-groupoid. So S is Reverse RSM and Reverse LSRD-groupoid. (ab)(cd) = (db)(ac) = (ab)(cd), which shows that when Reverse RSM law is applied on Reverse LSRD-law then we get the original position of elements and vice versa.

Result-3.28: If S is Reverse LSRD-groupoid then S is Reverse RSM.
Proof: Straightforward.

Result-3.29: If S is medial and Reverse RSM then S is commutative.
Proof: S is Reverse RSM then S is also Reverse LSRD-groupoid. So for all a, b, c and d ∈ S, (ab)(cd) = (db)(ac) and (ab)(cd) = (cb)(da). S is also medial so (ab)(cd) = (ac)(bd). So by using these properties we get (ab)(cd) = (db)(ac) = (da)(bc) which shows that S is DSDD-groupoid and if S is DSDD-groupoid then S is commutative.

Result-3.30: If S is medial and Reverse LSRD-groupoid then S is commutative.
Proof: Straightforward by using Result-26 and Result-27.

Theorem 3.14: If S is LA-Semigroup and Reverse RSM then S is commutative semigroup as well as RA-Semigroup.
Proof: S is LA-Semigroup so S is medial and if S is medial and Reverse RSM then this is proved in Result-3.29 that S is commutative. So S is commutative LA-Semigroup which is obviously commutative semigroup and RA-Semigroup.

Theorem 3.15: If S is LA-Semigroup and Reverse LSRD-groupoid then S is commutative semigroup as well as RA-Semigroup.
Proof: Straightforward.

Corollary Related To Theorem-3.14: If S is RA-Semigroup and Reverse RSM then S is commutative semigroup.
Corollary Related To Theorem-3.15: If S is RA-Semigroup and Reverse LSRD-groupoid then S is commutative semigroup.

Conclusions:
(1) If S is commutative medial then S is RDD-semigroup but if S is non commutative medial then S is RDD-semigroup if S is RSM.
(2) If S is commutative medial then S is LDD-semigroup but if S is non commutative medial then S is LDD-semigroup if S is LSM.
(3) If S is DSDD-groupoid then S is medial, RDD-semigroup, LDD-semigroup and holds all properties defined in 2.1 to 2.11.
(4) If S is RvM then S is medial, RDD-semigroup, LDD-semigroup and holds all properties
defined in 2.1 to 2.8 and 2.10 to 2.12.

(5) If $S$ is $R_vPM$ then $S$ is medial, RDD-semigroup, LDD-semigroup and holds all properties defined in 2.1 to 2.9 and 2.11 to 2.12.

(6) If $S$ is semigroup and holds any law from Reverse-law, $R_vM$-law, $R_vPM$-law or DSDD-law then $S$ is commutative semigroup, LA-Semigroup and RA-Semigroup.

(7) DSDD-law, $R_vM$-law and $R_vPM$-law are strong commutative laws than Reverse-law.

**Next Work:**

(NW-1) Patterns to construct finite medials which become LDD-semigroup, RDD-semigroup and contain cyclic groups, orthodox semigroups, LA-Semigroups and RA-Semigroups.

(NW-2) Patterns to construct finite commutative groupoids containing cyclic groups and commutative orthodox semigroups.

**References:**


