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**ABSTRACT**
The objective of this paper is to build feedback control solutions for LTI systems with unknown constant input-delay. The control scheme includes a time-delay estimation algorithm and a predictor-based controller. The Lyapunov-Krasovskii theorem provides the global exponential stability of the closed-loop system. Some examples illustrate the theoretical results and the performances of the proposed method.

**KEYWORDS**
Time-delay system; delay estimation; predictor-based controller; Lyapunov-Krasovskii functional

## 1. Introduction

Time-delay system (TDS) has attracted the attention of the control community for the last decades. On the one hand, time-delay appears in many real systems, *e.g.* networked control systems (Bemporad, Heemels, & Johansson, 2010) and congestion control (Hollot, Misra, Towsley, & Gong, 2002). On the other hand, time-delay is a source of instability of the controlled system (Fridman, 2014a). In a TDS, time-delays can be found in the input, the output, the state and even several of them. This paper deals with the stabilization of LTI systems with unknown constant input-delay. The main objective of this paper is to estimate the unknown constant delay and then stabilize the system with an appropriate controller based on the delay estimation.

To stabilize LTI systems with known input-delay, predictive control is one of the most popular methods. The Smith predictor (Smith, 1959) is firstly introduced to stabilize TDS, and it can compensate arbitrarily long time-delay. The Smith predictor is extended to the state-space method by finite spectrum assignment (FSA) (Manitius & Olbrot, 1979) and reduction method (Artstein, 1982). The predictor-based controller (Karafyllis & Krstic, 2017; Cacace, Conte, Germani, & Pepe, 2016; Mazenc & Malisoff, 2016) is a new extension of the Smith predictor that stabilizes linear and nonlinear systems with constant or time-varying delays. Another advantage of the predictor-based controller is the robustness with respect to the slight delay mismatch (Bresch-Pietri & Petit, 2014). Although the delay value is not perfectly known (with small errors), the controller is able to stabilize the system. In this paper, the predictor-based controller is adopted to robustly stabilize the linear TDS with arbitrarily long input-delay.

A vast literature is avaliable on the stabilization of linear TDS with unknown constant input-delay. The adaptive backstepping approach (Krstic & Bresch-Pietri, 2009; Bekiaris-Liberis & Krstic, 2010) is the most effective way to stabilize such system. The unknown time-delay is considered as a transport partial differential equation (PDE). Then the adaptive backstepping controller is designed to stabilize the coupled system that is composed of the original system and the transport PDE. However, this approach is hard to implement due to the fact that both the controller and the delay estimation scheme contain

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integral terms. A piecewise-constant delay estimator is proposed in (Cacace, Conte, & Germani, 2017), the exponential stability of the system is ensured and there is no integral term in the state feedback controller. However, in this approach, the delay identifiability of the closed-loop system must be checked off-line with distinct delay values. A low gain feedback controller is designed in (Wei & Lin, 2019) in order to stabilize TDS. The advantage of this method is that the controller design is not based on the delay estimation, but all the open-loop poles of the system are assumed to be zero. In (Herrera & Ibeas, 2012), a delay estimator and a modified Smith predictor are combined to regulate the step response of stable, unstable and integrating TDS. However, this technique is complex since it runs large numbers of fixed-models with different delays at the same time in order to estimate the time-delay.

This paper introduces a control scheme that includes an on-line delay estimator and an approximated predictor-based controller. The global exponential stability of the closed-loop system is guaranteed by the Lyapunov-Krasovskii theorem (Krasovskii, 1963; Hale & Lunel, 1993; Fridman, 2014a). This paper is organized as follows. The problem formulations and some preliminaries are given in section 2. The main results of this paper are stated in section 3. In section 4, some simulations are given to illustrate the performances. Finally, section 5 ends the paper with some conclusions and future works.

## 2. Problem formulations

Firstly, one introduces some useful symbols and notations in the paper. The maximum and minimum eigenvalues of a matrix $P \in \mathbb{R}^{n\times n}$ read as $\bar{\lambda}(P)$ and $\underline{\lambda}(P)$, respectively. Let $\|\cdot\|$ denote the euclidean norm of a vector $x \in \mathbb{R}^n$ and the spectral norm of a matrix $P \in \mathbb{R}^{n\times m}$. The notation $x_t : [-h, 0] \to \mathbb{R}^n$ is defined as $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-h, 0]$. The space of continuous functions $\phi : [a, b] \to \mathbb{R}^n$ is denoted as $\mathbf{C}([a, b], \mathbb{R}^n)$. The definitions of the Lyapunov stability of the retared functional differential equation (RFDE) are given in (Hale & Lunel, 1993).

Consider a linear system with input-delay

$$\dot{x}(t) = Ax(t) + Bu(t-h), \qquad t \geq 0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, and $A$, $B$ have appropriate dimensions. The input-delay $h$ is supposed to be constant and unknown. Moreover, one assumes that $h$ is bounded in $[\underline{h}, \bar{h}]$, the bounds $\underline{h}$ and $\bar{h}$ are known. The initial condition of (1) is given by

$$\begin{aligned} x(0) &= x_0, \\ u(\theta) &= \phi_u(\theta) \in \mathbf{C}([-2\bar{h}, 0), \mathbb{R}^p). \end{aligned} \tag{2}$$

In (2), the initial condition of $u(t)$ is defined on $[-2\bar{h}, 0)$ instead of $[-\bar{h}, 0)$ because the delay estimator (13) given in Theorem 3.1 needs the knowledge of $u(t)$ on $[-2\bar{h}, -\bar{h})$.

**Assumption 1.** The pair $(A, B)$ is stabilizable.

Assumption 1 ensures the existence of a feedback matrix $K \in \mathbb{R}^{p\times n}$ such that $A + BK$ is Hurwitz. It yields that for all positive definite matrix $Q \in \mathbb{R}^{n\times n}$, there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that the following algebraic equation (Slotine, Li, et al., 1991, equation (3.19)) holds

$$(A + BK)^T P + P(A + BK) = -Q. \tag{3}$$

Based on the feedback matrix $K$ given after Assumption 1, one defines the controller as

$$u(t) = Kz(t), \quad t \geq 0 \tag{4}$$

where $z(t)$ is the approximated predictor relies on the knowledge of the delay estimation $\hat{h}(t)$ (the definition of $\hat{h}(t)$ will be given in section 3)

$$z(t) = e^{A\hat{h}(t)}x(t) + \int_{t-\hat{h}(t)}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s, \quad t \geq 0. \tag{5}$$

The initial condition of $z(t)$ is defined as

$$z(\theta) = \phi_z(\theta) \in \mathbf{C}([-2\bar{h}, 0), \mathbb{R}^n). \tag{6}$$

Similarly, the initial condition of $z(t)$ is defined on $[-2\bar{h}, 0)$ in order to ensure that the delay estimator (13) is computable.

**Remark 1.** To simplify the calculations, one chooses the initial condition of $z(t)$ such that

$$\|\phi_u(\theta)\| \leq \|K\|\|\phi_z(\theta)\|, \quad \theta \in [-2\bar{h}, 0). \tag{7}$$

Moreover, one requires that $u(t)$ and $z(t)$ are continuous at $t = 0$ gives that

$$\begin{aligned} \lim_{\theta \to 0^-} \phi_u(\theta) &= u(0), \\ \lim_{\theta \to 0^-} \phi_z(\theta) &= z(0). \end{aligned} \tag{8}$$

Equations (8) ensures the continuity of the controller $u(t)$ and the predictor $z(t)$ for all $t \geq -2\bar{h}$.

**Assumption 2.** The delayed input value $u(t-h)$ is available for measurements.

In this paper, one uses the knowledge of $u(t-h)$ to estimate the unknown input-delay. The delayed input value $u(t-h)$ can be stored and sent to the controller in practice. The same assumption is given in (Léchappé et al., 2016; Diop, Kolmanovsky, Moraal, & Van Nieuwstadt, 2001) in order to solve the delay estimation problem. For the stabilization problems, the adaptive backstepping approaches (Krstic & Bresch-Pietri, 2009, p.4500), (Bekiaris-Liberis & Krstic, 2010, p.278) also claim that the infinite-dimensional state

$$u(x,t) = U(t + D(x-1)) \tag{9}$$

with the unknown input-delay $D$, is available for measurement for all $x \in [0,1]$. Thus, Assumption 2 is a standard assumption for this problem.
Assumption 2 can be verified in a class of remote control systems (RCS) (Yang, 2011, Chapter 3.4). Consider a RCS with full-state measurement:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t-h_i) \\ y(t) = x(t-h_o) \end{cases} \tag{10}$$

where $h_i$ and $h_o$ are the unknown constant input and output delays arisen from the remote data transmission. Firstly, one assumes that the plant sends the measured state $x(t)$ and the delayed input $u(t-h_i)$ back to the controller in the same communication channel (the data transmission of the delayed control signal is shown by the dashed arrows in Figure 1), then the controller receives $y(t)$ and $u(t-h_i-h_o)$ at the same time. In other words, at the controller side, the delayed signal $u(t-h_i-h_o)$ is available for measurements. Secondly, since $h_o$ is constant, the dynamics of $y(t)$ reads as

$$\dot{y}(t) = \dot{x}(t-h_o) = Ax(t-h_o) + Bu(t-h_i-h_o). \tag{11}$$

Remind that, if one defines $h = h_i + h_o$, system (11) is equivalent to an input-delay system

$$\dot{y}(t) = Ay(t) + Bu(t-h) \tag{12}$$

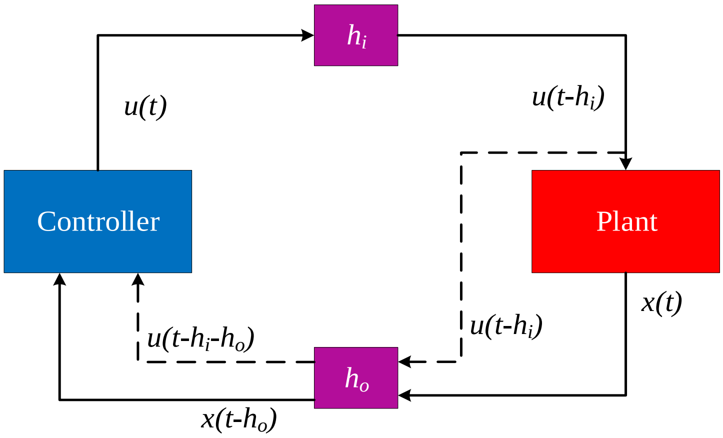


**Figure 1.** Control architecture of RCS (10) with full state measurement.

where $u(t-h)$ is available for measurements. This example shows that the input and output delays of the remote control system (10) can be lumped together, and this system can be equally considered as an input-delay system (12) in which Assumption 2 is satisfied. This method can also be used for the output feedback of a RCS (Deng, Léchappé, Moulay, & Plestan, 2019).
Moreover, the authors of (Bresch-Pietri & Krstic, 2010, Section IV) stated that the adaptive backstepping approach is able to be used for the pitch/ flight path rate control of an aircraft. It also means that Assumption 2 can be used to handle such real control problems.

## 3. Main results

The main results of this paper are stated as follows.

**Theorem 3.1.** *Consider the system* (1) *and the approximated predictor-based controller* (4)-(5) *with initial conditions* (2) *and* (6)*. Assume that Assumptions 1-2 are fulfilled. Define the delay estimator as follows:*

$$\dot{\hat{h}}(t) = \frac{\|u(t-h) - u(t-\hat{h}(t))\|}{\max_{s\in[t-2\bar{h},t]} \|z(s)\| + \epsilon}, \quad t \geq 0 \tag{13}$$

*where $\epsilon$ is a sufficiently small positive constant and the initial condition $\hat{h}(0) = h_0$ satisfies that $h_0 \leq h$. If the initial condition of the delay estimator $h_0$ is sufficiently close to $h$, then there exist $M_1 > 0$ and $\eta > 0$ such that $z(t)$ is globally uniformly exponentially stable (GUES) with decay rate $\eta$:*

$$\|z(t)\| \leq M_1 \max_{s\in[-\bar{h},0]} \|z(s)\| e^{-\eta t}, \quad t \geq 0. \tag{14}$$

*Moreover, there also exists $M_2 > 0$ such that the state $x(t)$ globally converges to zero with decay rate $\eta$ such that:*

$$\|x(t)\| \leq M_2 \max_{s\in[-\bar{h},0]} \|z(s)\| e^{-\eta t}, \quad t \geq 0. \tag{15}$$

■

***Proof.*** The proof of Theorem 3.1 is divided into two parts. The first part of the proof deals with the delay estimator and the second part of the proof provides the Lyapunov stability analysis.

***Part 1.*** The estimation scheme (13) provides that $\dot{\hat{h}}(t) \geq 0$ for all $t \geq 0$. Then, one proves the inequality

$$\hat{h}(t) \leq h, \quad t \geq 0. \tag{16}$$

by contradiction. Assume that there exists an instant $t_0 > 0$ such that $\hat{h}(t_0) > h$, then there must exist another instant $t_1 \in [0, t_0)$ such that

$$\hat{h}(t_1) = h, \quad \dot{\hat{h}}(t_1) > 0 \tag{17}$$

thanks to the fact $h_0 \leq h$, the continuity of $\hat{h}(t)$, the monotonicity of $\hat{h}(t)$ and the intermediate value theorem (Beals, 2004, Theorem 8.9). However, $\dot{\hat{h}}(t_1) > 0$ cannot be true while $\hat{h}(t_1) = h$ holds. As a consequence of $\hat{h}(t_1) = h$, one has

$$\dot{\hat{h}}(t_1) = \frac{\|u(t_1 - h) - u(t_1 - \hat{h}(t_1))\|}{\max_{s\in[t_1-2\bar{h},t_1]} \|z(s)\| + \epsilon} = \frac{\|u(t_1 - h) - u(t_1 - h)\|}{\max_{s\in[t_1-2\bar{h},t_1]} \|z(s)\| + \epsilon} = 0. \tag{18}$$

There is a contradiction between (17) and (18), then inequality (16) is proven. Since $\hat{h}(t)$ is increasing and upper bounded by $h$, define $D = h - h_0$, it implies that

$$h - \hat{h}(t) \leq D, \quad t \geq 0. \tag{19}$$

Inequality (19) will be used in the Lyapunov stability analysis.
***Part 2.*** Differentiating the predictor (5) and using the control law (4) leads to

$$\begin{aligned}\dot{z}(t) =&(A + BK)z(t) + e^{A\hat{h}(t)}B[u(t - h) - u(t - \hat{h}(t))] + \dot{\hat{h}}(t)Az(t)\\ &- \dot{\hat{h}}(t)A\int_{t-\hat{h}(t)}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s + \dot{\hat{h}}(t)e^{A\hat{h}(t)}Bu(t - \hat{h}(t)), \quad t \geq 0.\end{aligned} \tag{20}$$

Combining (7) and (4), one concludes that

$$\|u(t)\| \leq \|K\|\|z(t)\|, \quad t \geq -2\bar{h}. \tag{21}$$

Define the three positive-definite terms as follows:

$$\begin{aligned}V_1(z) &= z(t)^T P z(t),\\ V_2(u_t, t) &= \int_{t-\hat{h}(t)}^{t} (2\bar{h} + s - t)\|u(s)\|^2\mathrm{d}s,\\ V_3(\dot{u}_t, t) &= \int_{t-h}^{t} (h + s - t)\|\dot{u}(s)\|^2\mathrm{d}s,\end{aligned} \tag{22}$$

for $t \geq 0$. Consider the Lyapunov-Krasovskii functional

$$V(z, u_t, \dot{u}_t, t) = V_1(z) + \alpha V_2(u_t, t) + \beta V_3(\dot{u}_t, t) \tag{23}$$

with $\alpha, \beta > 0$. Lemma B.1 (Appdendix B) and the Lyapunov-Krasovskii theorem provides that

$$V(t) \leq V(\bar{h})e^{-2\eta(t-\bar{h})}, \quad t \geq \bar{h} \tag{24}$$

for well-tuned $\alpha$, $\beta$, $\eta$ and sufficiently small $D$. Finally, Lemma A.1 (Appdendix A), Lemma C.1 (Appdendix C) and inequality (24) result in the exponential stability (14) and the exponential convergence (15). □

Theorem 3.1 provides a control strategy to the stabilization of LTI systems with unknown constant input-delay. The main contributions are:

- The controller design is simple, no parameter needs to be tuned for the controller. The parameter $\epsilon$ does not influence the stability conditions (B24)-(B29), it is only added to avoid the singularity of (13). In practice, one can just set $\epsilon$ sufficiently small (*e.g.* $\epsilon < 10^{-3}$). The constraint on the initial condition $h_0$ is conservative in the proof, but the system can be stabilized in simulations although the distance between $h$ and $h_0$ is much further than the theoretical one. This argument will be discussed in Remark 2 and will be illustrated by the simulations given in section 4.
- Although the closed-loop system has nonlinear behavior, it does not have finite escape time by virtue of (A9), Lemma A.1 (Appendix A).
- Assumption 2 ensures that one can measure the delayed value $u(t-h)$, then the delay estimation scheme will not be influenced by the model uncertainties and the external disturbances since $u(t)$ is stored and $u(t-h)$ is measured.

Moreover, several remarks are given to discuss the main results.

**Remark 2.** Theorem 3.1 requires that the initial condition $h_0$ should be sufficiently close to the unknown time-delay $h$. But the difference between $h$ and $h_0$ can be much larger in applications.
Indeed, this condition on $h_0$ is only used in the proof of Lemma B.1 to ensure that the error term $h-\hat{h}(t)$ is sufficiently small **for all $t \geq \bar{h}$. Consider the case when $h_0$ is not close to $h$, but $\hat{h}(t)$ is sufficiently close to $h$ for all $t \geq \bar{h}$**, the right-hand sides of (B5) and (B6) are still sufficiently small and Lemma B.1 still holds. Thus, the exponential stability of the closed-loop system is guaranteed iin this case. Moreover, the proof of Lemma B.1 is also conservative.
Some simulations with different $h_0$ will be given in subsection 4.1 to test if the system can be stabilized with less restrictive initial conditions.

**Remark 3.** Consider the definition of the exponential stability given in (Fridman, 2014a, p.136, Definition 4.1), one makes sure that $z(t)$ is GUES although its trajectory passes through a transient on $t \in [0, \bar{h}]$. However, one cannot say that $x(t)$ is also GUES in the sense of (Fridman, 2014a, p.136, Definition 4.1). Thus, one claims in Theorem 3.1 that $x(t)$ globally converges to zero.

**Remark 4.** One can compare the main results of this article and the one of (Deng et al., 2019), in which the delay estimator is defined as

$$\dot{\hat{h}}(t) = \min\{\|u(t-h) - u(t-\hat{h}(t))\|, \bar{\delta}\}, \quad t \geq 0. \tag{25}$$

The differences between the proposed method and the one of (Deng et al., 2019) are stated in the sequel:

- The global exponential stability (14) and global exponential convergence (15) are ensured by Theorem 3.1 by using the Lyapunov-Krasovskii theorem, rather than the global asymptotic stability and global convergence provided in (Deng et al., 2019) with Lyapunov-Razumikhin theorem[1].
- This work requires a smaller amount of tuning parameters than (Deng et al., 2019). Only the initial condition $h_0$ need to be tuned in this work; but in (Deng et al., 2019), $\bar{\delta}$ is also required to be sufficiently small in order to obtain the stability.
- Delay estimator (13) makes the convergence $\lim_{t\to\infty} \hat{h}(t) = h$ easier to achieve in practice. The comparison between the two delay estimators will be given in Section 4.2.

[1]The Lyapunov-Razumikhin theorem given in (Hale & Lunel, 1993, Theorem 4.2, p.152) ensures the uniform asymptotic stability of the TDS. To obtain the exponential stability by using the Lyapunov-Razumikhin theorem, the Halanay's inequality must be additionally checked (Fridman, 2014b, p.273-274).

## 4. Simulations

In this section, a system inspired by the system studied in (Zhou, 2014a) is considered. The system is described as

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t-h), \quad t \geq 0. \tag{26}$$

In subsection 4.1, a simulation illustrates the discussions given in Remark 2 such that the constraint on the initial condition $h_0$ is less restrictive than Theorem 3.1. In subsection 4.2, one compares the proposed delay estimator (13) and the existing method (25) mentioned in Remark 4. In subsection 4.3, one shows that the delay estimator (13) is not influenced by the model uncertainties.

**Remark 5.** As stated in (Van Assche, Dambrine, Lafay, & Richard, 1999), some numerical problems may arise during when calculating the integral term of (5) (because the integration must be replaced by finite sums). Several methods can handle this problem:

- in (Zhou, 2014b; Zhou, Liu, & Mazenc, 2017), an artificial finite-dimensional system that predicts the future state is used to replace the infinite-dimensional predictor (5);
- in (Bresch-Pietri, Chauvin, & Petit, 2012, p.1548) a trapezoidal discretization method and a periodic reset are used to overcome the numerical issues.

In this paper, one uses the technique proposed in (Bresch-Pietri et al., 2012, p.1548) to calculate the prediction (5) in order to avoid the numerical problems.

### *4.1. Simulation results with different initial conditions $h_0$*

Consider the system (26) with a large input-delay $h = 4.5$s, define $\underline{h} = 0$s and $\bar{h} = 5$s. The initial conditions of the system are given by $x(0) = \begin{bmatrix} 0.5 & 0.6 & -0.6 & 0.5 \end{bmatrix}^T$, and $u(\theta) = -3.1452$ for all $\theta \in [-2\bar{h}, 0)$. The feedback matrix $K = \begin{bmatrix} 6 & -8 & 13 & -12 \end{bmatrix}$ leads to the eigenvalues $\{-1; -1; -3; -5\}$ for $A + BK$. The initial condition of $z(t)$ is set to $z(\theta) = \begin{bmatrix} 0.2962 & 0.0502 & -0.6335 & -0.3096 \end{bmatrix}^T$ for all $\theta \in [-2\bar{h}, 0)$ that guarantees (7), and the parameter $\epsilon$ is set to $10^{-4}$. In this simulation, one tests the proposed method with two initial conditions $h_0 = 2$s and $h_0 = 4$s.

The simulation results are presented in Figure 2, the system (26) is stabilized in both cases. The simulation results illustrate the discussions given in Remark 2 such that the estimation error $h - \hat{h}(t)$ is only required to be sufficiently small for all $t \geq \bar{h}$, but not necessary for all $t \geq 0$. In Figure 2, since the unknown time-delay is almost estimated for all $t \geq \bar{h}$ in both cases, then Lemmas A.1, B.1 and C.1 hold, and they guarantee the exponential stability of the closed-loop system.

The simulation results given in this subsection illustrate the discussions of Remark 2, and it shows that the choice of $h_0$ is much more flexible in practice.

### *4.2. Comparison between two delay estimators*

Consider that the system (26) is subject to an unknown input-delay $h = 0.5$s that is upper bounded by $\bar{h} = 0.8$s. The initial condition of the delay estimator is set to $h_0 = 0.2$s. The other parameters are the same as the ones defined in subsection 4.1, and the parameter $\bar{\delta}$ in (25) is set to 0.1.

In Figure 3, the two delay estimators (13) and (25) are compared. The system is stabilized in both cases, but Figure 3 (Bottom) shows that the convergence $\lim\limits_{t\to\infty} \hat{h}(t) = h$ is only ensured by the proposed delay estimator (13). Consider the case when $\hat{h}(t)$ is sufficiently close to $h$ and the system is almost stabilized, in this case the error term $\|u(t-h) - u(t-\hat{h}(t))\|$ is approximately zero. Hence, (13) makes the convergence $\lim\limits_{t\to\infty} \hat{h}(t) = h$ easier to achieve since the denominator $\max_{s\in[t-2\bar{h},t]} \|z(s)\| + \epsilon$ is much

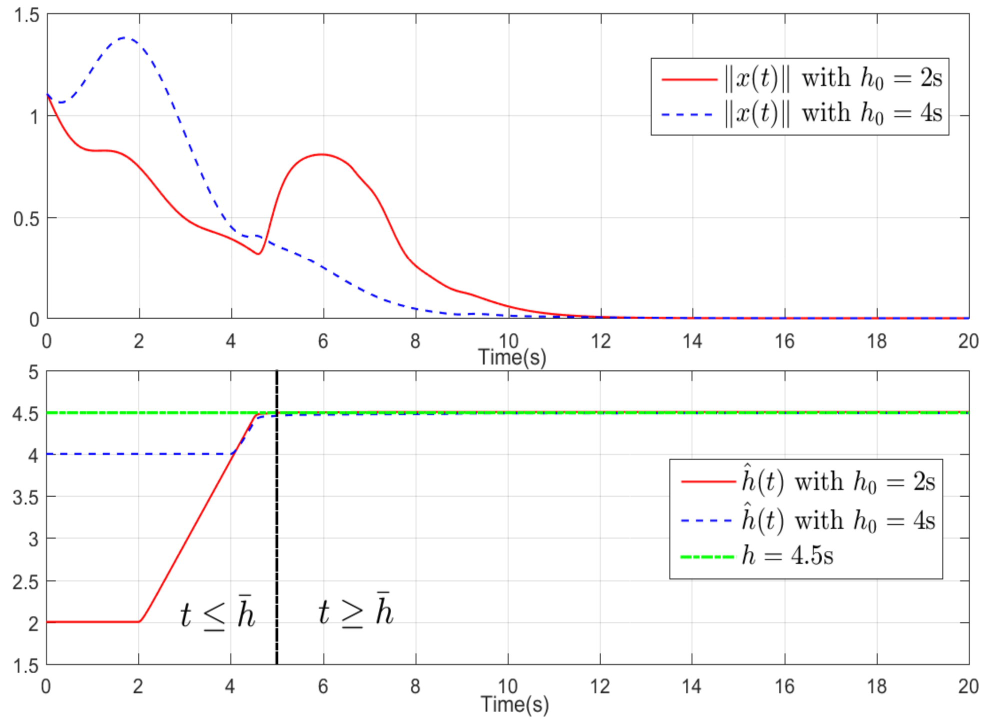


**Figure 2.** Evolution of $\|x(t)\|$ (Top) and $\hat{h}(t)$ (Bottom) versus time for system (26) with two cases: $h_0 = 2$s and $h_0 = 4$s.

smaller than 1. In other words, the denominator of (13) speeds up the delay estimation when the system is almost stabilized.
Furthermore, Figure 3 (Top) shows that delay estimator (25) makes $\|x(t)\|$ have an oscillation behavior, but (13) does not. Then one concludes that delay estimator (13) can also improve the performances of the system.

### *4.3. Simulation with model uncertainties*

In this subsection, one shows that the proposed delay estimator (13) is robust with respect to the model uncertainties. Considers the system (26) with model uncertainties:

$$\dot{x}(t) = \left( \underbrace{\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 & 0.2 & 0 & 0.1 \\ 0.1 & 0 & 0.1 & 0 \\ -0.1 & 0.1 & 0 & 0 \\ 0.1 & 0 & -0.1 & 0 \end{bmatrix}}_{\Delta A} \right) x(t) + \left( \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{B} + \underbrace{\begin{bmatrix} 0.1 \\ -0.2 \\ 0.1 \\ -0.1 \end{bmatrix}}_{\Delta B} \right) u(t-h), \quad t \geq 0 \tag{27}$$

where $\Delta A$ and $\Delta B$ are unknown. All of the parameters are the same as the ones given in subsection 4.2, and the control laws (5)-(4)-(13) are calculated by using $A$ and $B$.
The simulation results are presented in Figure 4, the uncertain system (27) is stabilized although there exist model uncertainties $\Delta A$ and $\Delta B$. Moreover, as stated in Section 3, the delay estimator (13) can deal with uncertain systems since $u(t)$ is stored and $u(t-h)$ is measured. This simulation highlights the robustness of the proposed method.

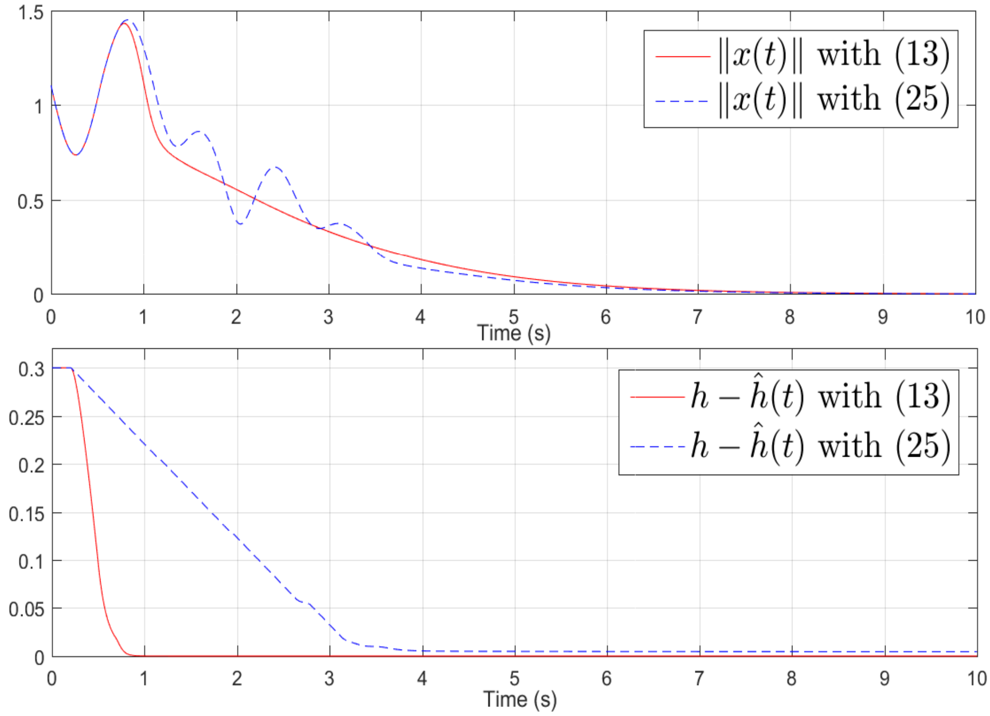


**Figure 3.** Evolution of $\|x(t)\|$ (Top) and $h - \hat{h}(t)$ (Bottom) versus time for system (26) ($h = 0.5$s) with delay estimators (13) and (25).

## 5. Conclusions

This paper deals with the stabilization of LTI systems with unknown constant input-delay. The control scheme is comprised of a delay estimator and an approximated predictor-based controller. The global uniform exponential stability of the closed-loop system is ensured by using the Lyapunov-Krasovskii theorem. Several simulations are given to illustrate the performance of the proposed method, they also show that the constraint on the initial condition of the delay estimator is not restrictive in practice. The stabilization of the LTI system with time-varying input-delay will be investigated for future works.

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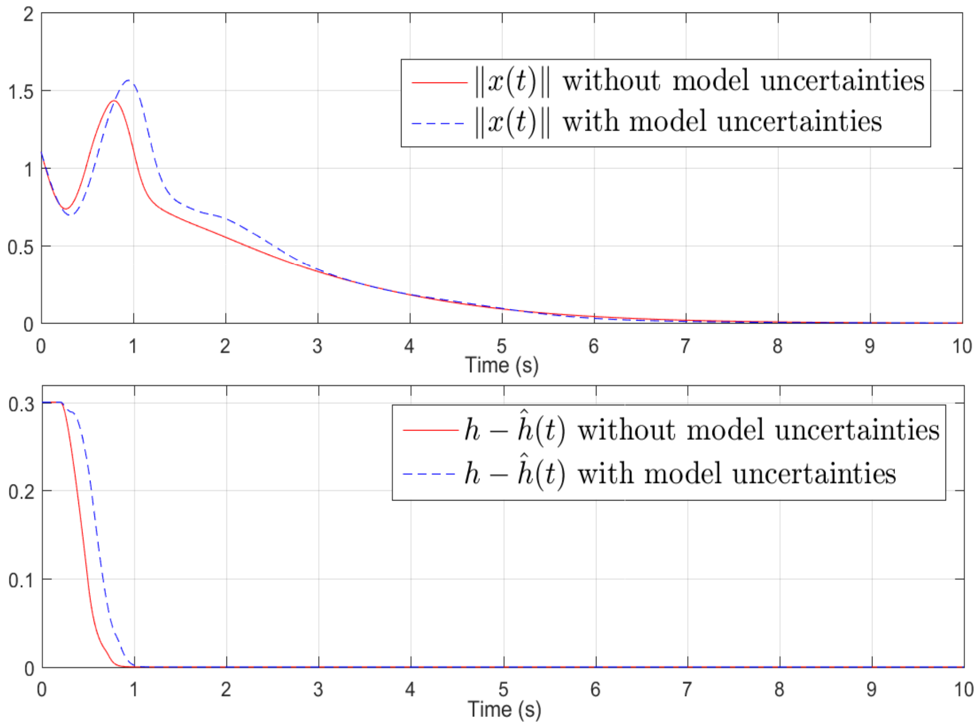


**Figure 4.** Evolution of $\|x(t)\|$ (Top) and $h - \hat{h}(t)$ (Bottom) versus time for the standard system (26) and the uncertain system (27) with control strategies (5)-(4)-(13).


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## Appendix A. Behaviors of system (20) during the transient on $t \in [0, \bar{h}]$

Lemma A.1 provides the bounds of $\|z(t)\|$ and $\|\dot{z}(t)\|$ during the transient on $t \in [0, \bar{h}]$.

**Lemma A.1.** *Define*

$$\begin{aligned} k &= \max_{s \in [-\bar{h}, 0]} \|z(s)\|, \\ b &= \|A + BK\| + 2\|K\|\|A\| + 2e^{\|A\|\bar{h}}\|B\|\|K\|(1 + \|K\| + \|K\|\|A\|\bar{h}). \end{aligned} \tag{A1}$$

*Then the trajectory of the $z$-system satisfies that*

$$\max_{s \in [0, \bar{h}]} \|z(s)\| \leq k e^{b\bar{h}}, \quad \max_{s \in [0, \bar{h}]} \|\dot{z}(s)\| \leq b k e^{b\bar{h}}. \tag{A2}$$

■

***Proof.*** Taking account of (21), (16) and the triangle inequality, (13) is upper bounded as follows

$$\dot{\hat{h}}(t) \leq \frac{\|u(t-h)\| + \|u(t - \hat{h}(t))\|}{\max_{s \in [t-2\bar{h}, t]} \|z(s)\| + \epsilon} \leq 2\|K\| \frac{\max_{s \in [t-2\bar{h}, t]} \|z(s)\|}{\max_{s \in [t-2\bar{h}, t]} \|z(t)\| + \epsilon} \leq 2\|K\|, \quad t \geq 0. \tag{A3}$$

Consider a scalar function $N(t)$ defined as follows

$$N(t) = \begin{cases} z^T(t) z(t) = \|z(t)\|^2, & t \geq 0 \\ \phi_z^T(\theta) \phi_z(\theta) = \|\phi_z(\theta)\|^2, & -2\bar{h} \leq \theta < 0 \end{cases}. \tag{A4}$$

Remind that one has (8), then $N(t)$ is continuous for all $t \geq -2\bar{h}$. Differentiating $N(t)$ along the trajectories of (20) leads to

$$\begin{aligned}\dot{N}(t) =& z^T(t)\left[(A+BK)^T+(A+BK)+\dot{\hat{h}}(t)(A^T+A)\right]z(t)+z(t)^T e^{A\hat{h}(t)}B[u(t-h)-u(t-\hat{h}(t))]\\ &+[u(t-h)-u(t-\hat{h}(t))]^T B^T e^{A^T\hat{h}(t)}z(t)+\dot{\hat{h}}(t)z^T(t)e^{A\hat{h}(t)}Bu(t-\hat{h}(t))\\ &+\dot{\hat{h}}(t)\left[e^{A\hat{h}(t)}Bu(t-\hat{h}(t))\right]^T z(t)-\dot{\hat{h}}(t)z^T(t)A\int_{t-\hat{h}(t)}^{t}e^{A(t-s)}Bu(s)\mathrm{d}s\\ &-\dot{\hat{h}}(t)\left[A\int_{t-\hat{h}(t)}^{t}e^{A(t-s)}Bu(s)\mathrm{d}s\right]^T z(t), \quad t\geq 0.\end{aligned} \tag{A5}$$

Taking the norm of the right-hand side of (A5) and using (21), (A3), it follows that

$$\begin{aligned}\dot{N}(t) \leq\ & 2\|A+BK\|\|z(t)\|^2+4e^{\|A\|\bar{h}}\|B\|\|K\|\|z(t)\|\max_{s\in[t-\bar{h},t]}\|z(s)\|+4\|K\|\|A\|\|z(t)\|^2\\ &+4\|K\|\|A\|e^{\|A\|\bar{h}}\|B\|\|K\|\bar{h}\|z(t)\|\max_{s\in[t-\bar{h},t]}\|z(s)\|+4\|K\|e^{\|A\|\bar{h}}\|B\|\|K\|\|z(t)\|\max_{s\in[t-\bar{h},t]}\|z(s)\|, \quad t\geq 0.\end{aligned} \tag{A6}$$

Due to the fact that $\|z(t)\| \leq \max_{s\in[t-\bar{h},t]}\|z(s)\|$, (A6) is upper bounded as

$$\dot{N}(t) \leq 2b\left(\max_{s\in[t-\bar{h},t]}\|z(s)\|\right)^2 = 2b\max_{s\in[t-\bar{h},t]}N(s), \quad t\geq 0 \tag{A7}$$

where $b$ is defined in (A1). The analysis given in (Bresch-Pietri, Mazenc, & Petit, 2018, Appendix B, p.232-233) proves that the trajectory of $N(t)$ satisfies the following inequality:

$$N(t) \leq \max_{s\in[-\bar{h},0]}N(s)e^{2bt}, \quad t\geq 0. \tag{A8}$$

Since $\max_{s\in[-\bar{h},0]}N(s)=k^2$, one proves that[2]

$$\|z(t)\| \leq ke^{bt}, \quad t\geq 0 \tag{A9}$$

where $k$ and $b$ are defined in (A1). Given that $\|z(\theta)\|\leq k$ for all $\theta\in[-\bar{h},0]$, then one has

$$\max_{s\in[t-\bar{h},t]}\|z(s)\| \leq ke^{bt}, \quad t\geq 0. \tag{A10}$$

Next, one takes the norm of (20) and one repeats the calculations given in (A6) and (A7), it follows that

$$\|\dot{z}(t)\| \leq b\max_{s\in[t-\bar{h},t]}\|z(s)\|, \quad t\geq 0. \tag{A11}$$

Combining (A10) and (A11) leads to

$$\|\dot{z}(t)\| \leq bke^{bt}, \quad t\geq 0. \tag{A12}$$

Inequalities (A9) and (A12) result in (A2), and this ends the proof. □

## Appendix B. Lyapunov Stability of $z(t)$

This section gives the Lyapunov stability of $z(t)$ for all $t\geq\bar{h}$.

[2] A similar result is also given in (Selivanov & Fridman, 2016, Appendix A, (A.5)-(A.6)) with the use of the Gronwall-Bellman Lemma.

**Lemma B.1.** *Consider the Lyapunov-Krasovskii functional* (23)*, if the parameter $D$ described in* (19) *is sufficiently small, then there exists $\eta > 0$ such that*

$$V(t) \leq V(\bar{h})e^{-2\eta(t-\bar{h})}, \quad t \geq \bar{h}. \tag{B1}$$

■

***Proof.*** Firstly, for all $r \in [\hat{h}(t), h]$, one has

$$\begin{aligned} \dot{z}(t-r) =& (A+BK)z(t-r) + e^{A\hat{h}(t-r)}B[u(t-r-h) - u(t-r-\hat{h}(t-r))] \\ &+ \dot{\hat{h}}(t-r)Az(t-r) + \dot{\hat{h}}(t-r)e^{A\hat{h}(t-r)}Bu(t-r-\hat{h}(t-r)) \\ &- \dot{\hat{h}}(t-r)A\int_{t-r-\hat{h}(t-r)}^{t-r} e^{A(t-r-s)}Bu(s)\mathrm{d}s, \quad t \geq \bar{h}. \end{aligned} \tag{B2}$$

Taking the norm of (B2) leads to

$$\|\dot{z}(t-r)\| \leq b \max_{s\in[t-2\bar{h},t]} \|z(s)\|, \quad t \geq \bar{h} \tag{B3}$$

for all $r \in [\hat{h}(t), h]$ (the parameter $b$ is defined in (A1)). Consider the term $u(t-h) - u(t-\hat{h}(t))$ and the mean value theorem for vector-valued functions (Rudin et al., 1964, Theorem 5.19), there exists $r_1 \in [\hat{h}(t), h]$ such that

$$\|u(t-h) - u(t-\hat{h}(t))\| \leq (h - \hat{h}(t))\|K\|\|\dot{z}(t-r_1)\|, \quad t \geq \bar{h} \tag{B4}$$

which implies that

$$\|u(t-h) - u(t-\hat{h}(t))\| \leq (h-\hat{h}(t))\|K\|b \max_{s\in[t-2\bar{h},t]} \|z(s)\|, \qquad t \geq \bar{h}. \tag{B5}$$

Substituting (B5) into (13) yields that

$$\dot{\hat{h}}(t) \leq (h-\hat{h}(t))\|K\|b \frac{\max_{s\in[t-2\bar{h},t]}\|z(s)\|}{\max_{s\in[t-2\bar{h},t]}\|z(s)\| + \epsilon} \leq D\|K\|b, \quad t \geq \bar{h}. \tag{B6}$$

To simplify the writing, one defines

$$\hat{\delta} \triangleq D\|K\|b \tag{B7}$$

and one assumes that $D$ is sufficiently small such that $\hat{\delta} < 1$. Inequality (B6) provides that, if $h - \hat{h}(t)$ is sufficiently small, then $\dot{\hat{h}}(t)$ is also sufficiently small. This property will be used in the Lyapunov analysis. The next step is to consider the Lyapunov-Krasovskii functional (23). For all $t \geq \bar{h}$, differentiating $V_1(z)$ along the trajectory of (20) and using Lyapunov-equation (3) leads to:

$$\begin{aligned} \dot{V}_1(z(t)) =& -z(t)^TQz(t) + z(t)^TPe^{A\hat{h}(t)}B[u(t-h) - u(t-\hat{h}(t))] + [u(t-h) - u(t-\hat{h}(t))]^TB^Te^{A^T\hat{h}(t)}Pz(t) \\ &+ \dot{\hat{h}}(t)z^T\left[A^TP + PA\right]z(t) + \dot{\hat{h}}(t)z(t)^TPe^{A\hat{h}(t)}Bu(t-\hat{h}(t)) + \dot{\hat{h}}(t)z(t)^TPe^{A\hat{h}(t)}Bu(t-\hat{h}(t)) \\ &+ \dot{\hat{h}}(t)\left[e^{A\hat{h}(t)}Bu(t-\hat{h}(t))\right]^TPz(t) - \dot{\hat{h}}(t)z(t)^TPA\int_{t-\hat{h}(t)}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s \\ &- \dot{\hat{h}}(t)\left[A\int_{t-\hat{h}(t)}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s\right]^TPz(t). \end{aligned} \tag{B8}$$

Taking the norm of (B8), applying (B6), (B7) and the triangle inequality for integrals (Rudin, 2006, Theorem 1.33) one gets

$$\begin{aligned}\dot{V}_1(z(t)) \leq &-\underline{\lambda}(Q)\|z(t)\|^2 + 2\hat{\delta}\|PA\|\|z(t)\|^2 + 2\hat{\delta}e^{\|A\|\bar{h}}\|B\|\|P\|\|z(t)\|\|u(t-\hat{h}(t))\| \\ &+ 2\hat{\delta}\|PA\|e^{\|A\|\bar{h}}\|B\|\|z(t)\|\|v(t)\| + 2e^{\|A\|\bar{h}}\|B\|\|P\|\|z(t)\|\|w(t)\|\end{aligned} \tag{B9}$$

where $\|v(t)\| = \int_{t-\hat{h}(t)}^{t}\|u(s)\|\mathrm{d}s$ and $\|w(t)\| = \int_{t-h}^{t-\hat{h}(t)}\|\dot{u}(s)\|\mathrm{d}s$. Define $c_1 = 2\|PA\|$, $c_2 = 2e^{\|A\|\bar{h}}\|B\|\|P\|$, $c_3 = 2\|PA\|e^{\|A\|\bar{h}}\|B\|$, then (B9) is equivalent to

$$\dot{V}_1(z) \leq -(\underline{\lambda}(Q) - \hat{\delta}c_1)\|z(t)\|^2 + \hat{\delta}c_2\|z(t)\|\|u(t-\hat{h}(t))\| + \hat{\delta}c_3\|z(t)\|\|v(t)\| + c_2\|z(t)\|\|w(t)\|. \tag{B10}$$

Then, by completing the squares, (B10) implies that

$$\begin{aligned}\dot{V}_1(z) \leq &-(\underline{\lambda}(Q) - \hat{\delta}c_1)\|z(t)\|^2 + \frac{\hat{\delta}c_2}{2}(\|z(t)\|^2 + \|u(t-\hat{h}(t))\|^2) \\ &+ \frac{\hat{\delta}c_3}{2}(\|z(t)\|^2 + \|v(t)\|^2) + \left(\frac{Dc_2^2}{\beta}\|z(t)\|^2 + \frac{\beta}{4D}\|w(t)\|^2\right).\end{aligned} \tag{B11}$$

For all $t \geq \bar{h}$, differentiating $V_2(u_t, t)$ gives that

$$\dot{V}_2(u_t, t) \leq 2\bar{h}\|K\|^2\|z(t)\|^2 - (1 - \dot{\hat{h}}(t))(2\bar{h} - \hat{h}(t))\|u(t-\hat{h}(t))\|^2 - \int_{t-\hat{h}(t)}^{t}\|u(s)\|^2\mathrm{d}s. \tag{B12}$$

Remind that $1 - \dot{\hat{h}}(t) \geq 1 - \hat{\delta} \geq 0$ and $2\bar{h} - \hat{h}(t) \geq \bar{h} \geq 0$, one has

$$(1 - \dot{\hat{h}}(t))(2\bar{h} - \hat{h}(t)) \geq (1 - \hat{\delta})\bar{h} \geq 0. \tag{B13}$$

Combining (B12) and (B13) leads to

$$\dot{V}_2(u_t, t) \leq 2\bar{h}\|K\|^2\|z(t)\|^2 - (1 - \hat{\delta})\bar{h}\|u(t-\hat{h}(t))\|^2 - \int_{t-\hat{h}(t)}^{t}\|u(s)\|^2\mathrm{d}s. \tag{B14}$$

Using the Jensen's inequality (Gu, Chen, & Kharitonov, 2003, Proposition B.8) and the fact that $0 \leq \hat{h}(t) \leq \bar{h}$, the integral term of (B14) is upper bounded as follows

$$-\int_{t-\hat{h}(t)}^{t}\|u(s)\|^2\mathrm{d}s \leq -\frac{1}{\hat{h}(t)}\left[\int_{t-\hat{h}(t)}^{t}\|u(s)\|\mathrm{d}s\right]^2 \leq -\frac{1}{\bar{h}}\left[\int_{t-\hat{h}(t)}^{t}\|u(s)\|\mathrm{d}s\right]^2. \tag{B15}$$

Using (B15), inequality (B14) is developed to

$$\dot{V}_2(u_t, t) \leq 2\bar{h}\|K\|^2\|z(t)\|^2 - (1 - \hat{\delta})\bar{h}\|u(t-\hat{h}(t))\|^2 - \frac{1}{2\bar{h}}\|v(t)\|^2 - \frac{1}{2}\int_{t-\hat{h}(t)}^{t}\|u(s)\|^2\mathrm{d}s. \tag{B16}$$

For all $t \geq \bar{h}$, differentiating $V_3(\dot{u}_t, t)$ leads to

$$\dot{V}_3(\dot{u}_t, t) \leq \bar{h}\|K\|^2\|\dot{z}(t)\|^2 - \int_{t-h}^{t}\|\dot{u}(s)\|^2\mathrm{d}s. \tag{B17}$$

Remind that one has $0 \leq h - \hat{h}(t) \leq D$, then one also has $-\frac{1}{h-\hat{h}(t)} \leq -\frac{1}{D} \leq 0$. So inequality (B17) becomes to

$$\begin{aligned} \dot{V}_3(\dot{u}_t, t) &\leq \bar{h}\|K\|^2\|\dot{z}(t)\|^2 - \frac{1}{2}\int_{t-h}^{t-\hat{h}(t)} \|\dot{u}(s)\|^2 \mathrm{d}s - \frac{1}{2}\int_{t-h}^{t} \|\dot{u}(s)\|^2 \mathrm{d}s \\ &\leq \bar{h}\|K\|^2\|\dot{z}(t)\|^2 - \frac{1}{2D}\|w(t)\|^2 - \frac{1}{2}\int_{t-h}^{t} \|\dot{u}(s)\|^2 \mathrm{d}s. \end{aligned} \tag{B18}$$

For $t \geq \bar{h}$, taking the norm of (20) and taking account of (B6), one has

$$\begin{aligned} \|\dot{z}(t)\| \leq &\|A + BK\|\|z(t)\| + \hat{\delta}\|A\|\|z(t)\| + e^{\|A\|\bar{h}}\|B\|\|w(t)\| \\ &+ \hat{\delta}\|A\|e^{\|A\|\bar{h}}\|B\|\|v(t)\| + \hat{\delta}e^{\|A\|\bar{h}}\|B\|\|u(t-\hat{h}(t))\|. \end{aligned} \tag{B19}$$

Squaring (B19) and using the power means inequality given in (Bullen, 2013, p.203, Theorem 1) yields that

$$\|\dot{z}(t)\|^2 \leq (c_4 + c_5\hat{\delta}^2)\|z(t)\|^2 + c_6\|w(t)\|^2 + c_7\hat{\delta}^2\|v(t)\|^2 + c_6\hat{\delta}^2\|u(t-\hat{h}(t))\|^2 \tag{B20}$$

where $c_4 = 5\|A + BK\|^2$, $c_5 = 5\|A\|^2$, $c_6 = 5e^{2\|A\|\bar{h}}\|B\|^2$ and $c_7 = 5\|A\|^2e^{2\|A\|\bar{h}}\|B\|^2$. Thus, (B18) is upper bounded by

$$\begin{aligned} \dot{V}_3(\dot{u}_t, t) \leq &\bar{h}\|K\|^2\left[(c_4 + c_5\hat{\delta}^2)\|z(t)\|^2 + c_6\|w(t)\|^2 + c_7\hat{\delta}^2\|v(t)\|^2 + c_6\hat{\delta}^2\|u(t-\hat{h}(t))\|^2\right] \\ &- \frac{1}{2D}\|w(t)\|^2 - \frac{1}{2}\int_{t-h}^{t} \|\dot{u}(s)\|^2 \mathrm{d}s. \end{aligned} \tag{B21}$$

Futhermore, the Lyapunov-Krasovskii functional (23) is upper bounded by

$$V(z, u_t, \dot{u}_t, t) \leq \bar{\lambda}(P)\|z(t)\|^2 + 2\alpha\bar{h}\int_{t-\hat{h}(t)}^{t} \|u(s)\|^2 \mathrm{d}s + \beta\bar{h}\int_{t-h}^{t} \|\dot{u}(s)\|^2 \mathrm{d}s. \tag{B22}$$

For $t \geq \bar{h}$, taking account of (B11), (B16), (B21) and (B22) gives that

$$\begin{aligned} \dot{V} + 2\eta V \leq &-\left[\underline{\lambda}(Q) - \hat{\delta}c_1 - \hat{\delta}(c_2 + c_3)/2 - Dc_2^2/\beta - 2\alpha\bar{h}\|K\|^2 - \beta\bar{h}\|K\|^2(c_4 + c_5\hat{\delta}^2) - 2\eta\bar{\lambda}(P)\right]\|z(t)\|^2 \\ &- \left[\alpha\bar{h} - \alpha\bar{h}\hat{\delta} - \hat{\delta}c_2/2 - \beta\bar{h}\|K\|^2c_6\hat{\delta}^2\right]\|u(t-\hat{h}(t))\|^2 - \left[\alpha/2 - 4\alpha\bar{h}\eta\right]\int_{t-\hat{h}(t)}^{t} \|u(s)\|^2 \mathrm{d}s \\ &- \left[\beta/(2D) - \beta/(4D) - \beta\bar{h}\|K\|^2c_6\right]\|w(t)\|^2 - \left[\alpha/(2\bar{h}) - \hat{\delta}c_3/2 - \beta\bar{h}\|K\|^2c_7\hat{\delta}^2\right]\|v(t)\|^2 \\ &- \left[\beta/2 - 2\beta\bar{h}\eta\right]\int_{t-h}^{t} \|\dot{u}(s)\|^2 \mathrm{d}s. \end{aligned} \tag{B23}$$

Thus, one obtains the stability conditions as follows

$$\begin{cases} \underline{\lambda}(Q) - \hat{\delta}c_1 - \hat{\delta}(c_2 + c_3)/2 - Dc_2^2/\beta - 2\alpha\bar{h}\|K\|^2 - \beta\bar{h}\|K\|^2(c_4 + c_5\hat{\delta}^2) - 2\eta\bar{\lambda}(P) \geq 0 & \text{(B24)} \\ \alpha\bar{h} - \alpha\bar{h}\hat{\delta} - \hat{\delta}c_2/2 - \beta\bar{h}\|K\|^2c_6\hat{\delta}^2 \geq 0 & \text{(B25)} \\ \alpha/(2\bar{h}) - \hat{\delta}c_3/2 - \beta\bar{h}\|K\|^2c_7\hat{\delta}^2 \geq 0 & \text{(B26)} \\ \beta/(2D) - \beta/(4D) - \beta\bar{h}\|K\|^2c_6 \geq 0 & \text{(B27)} \\ \alpha/2 - 4\alpha\bar{h}\eta \geq 0 & \text{(B28)} \\ \beta/2 - 2\beta\bar{h}\eta \geq 0 & \text{(B29)} \end{cases}$$

Firstly, one chooses $\alpha$, $\beta$ and $\eta$ sufficiently small in order to verify (B28), (B29) and the following inequality

$$\underline{\lambda}(Q)/2 - 2\alpha\bar{h}\|K\|^2 - \beta\bar{h}\|K\|^2 c_4 - 2\eta\bar{\lambda}(P) \geq 0. \tag{B30}$$

Secondly, it is possible to find sufficiently small $\hat{\delta}$ and $D$ satisfying (B25), (B26), (B27) and the following inequality:

$$\underline{\lambda}(Q)/2 - \hat{\delta}c_1 - \hat{\delta}(c_2 + c_3)/2 - Dc_2^2/\beta - \beta\bar{h}\|K\|^2 c_5\hat{\delta}^2 \geq 0. \tag{B31}$$

Thus, there exist $\hat{\delta}^* > 0$ and $D^* > 0$ such that for all $\hat{\delta} \leq \hat{\delta}^*$ and $D \leq D^*$, the stability conditions (B24)-(B29) are all guaranteed. Reminding the definition of $\hat{\delta}$ given in (B7), one concludes that if $h - h_0 = D$ is sufficiently small, such that

$$D \leq \min\left\{D^*, \frac{\hat{\delta}^*}{\|K\|b}\right\} \tag{B32}$$

then the Lyapunov-Krasovskii theorem (Fridman, 2014a, Theorem 3.1, Definition 4.1) ensures that the closed-loop $z$-system is uniformly exponentially stable for all $t \geq \bar{h}$. Moreover, one has

$$V(z, u_t, \dot{u}_t, t) \geq w(\|z(t)\|) \tag{B33}$$

with $w(s) = \underline{\lambda}(P)s^2$. Since one has $\lim\limits_{s\to+\infty} w(s) = +\infty$, it implies that the closed-loop $z$-system is GUES for all $t \geq \bar{h}$ such that

$$\dot{V}(z, u_t, \dot{u}_t, t) \leq -2\eta V(z, u_t, \dot{u}_t, t), \quad t \geq \bar{h} \tag{B34}$$

Finally, (B1) is proven by using (B34). □

## Appendix C. The exponential stability (14) and the exponential convergence (15)

Lemma C.1 gives the convergences (14) and (15) by using the results of Lemma A.1 and Lemma B.1.

**Lemma C.1.** *If the Lyapunov-Krasovskii functional given by* (22) *and* (23) *satisfies* (B1)*, then there exists* $M_1, M_2 > 0$ *such that* (14) *and* (15) *hold.* ■

***Proof.*** To obtain the stability of $z(t)$ for all $t \geq 0$, the boundedness of $V(\bar{h})$ is necessary, one firstly bounds the term $V(\bar{h})$ as follows

$$\begin{aligned} V(\bar{h}) &\leq \bar{\lambda}(P)\|z(\bar{h})\|^2 + 2\alpha\bar{h}\|K\|^2 \int_0^{\bar{h}} \|z(s)\|^2 \mathrm{d}s + \beta\bar{h}\|K\|^2 \int_0^{\bar{h}} \|\dot{z}(s)\|^2 \mathrm{d}s \\ &\leq \bar{\lambda}(P)\|z(\bar{h})\|^2 + 2\alpha\bar{h}^2\|K\|^2 \max_{s\in[0,\bar{h}]} \|z(s)\|^2 + \beta\bar{h}^2\|K\|^2 \max_{s\in[0,\bar{h}]} \|\dot{z}(s)\|^2. \end{aligned} \tag{C1}$$

Taking account of (C1), (A9) and (A2), the upper bound of $V(\bar{h})$ satisfies that

$$V(\bar{h}) \leq \left[\bar{\lambda}(P)e^{2b\bar{h}} + 2\alpha\bar{h}^2\|K\|^2 e^{2b\bar{h}} + \beta\bar{h}^2\|K\|^2 b^2 e^{2b\bar{h}}\right] k^2. \tag{C2}$$

Define $M_0^2 = \bar{\lambda}(P)e^{2b\bar{h}} + 2\alpha\bar{h}^2\|K\|^2 e^{2b\bar{h}} + \beta\bar{h}^2\|K\|^2 b^2 e^{2b\bar{h}}$, inequalities (B33), (C2) and (B1) yield that

$$\|z(t)\| \leq \frac{M_0 e^{\eta\bar{h}}}{\sqrt{\underline{\lambda}(P)}} k e^{-\eta t}, \quad t \geq \bar{h}. \tag{C3}$$

Moreover, during the transient on $0 \le t \le \bar{h}$, $e^{\eta(\bar{h}-t)}$ is larger than 1. From (A10), $z(t)$ is upper bounded as follows

$$\|z(t)\| \le ke^{b\bar{h}} \le ke^{b\bar{h}}e^{\eta(\bar{h}-t)} \le ke^{(b+\eta)\bar{h}}e^{-\eta t}, \quad 0 \le t \le \bar{h}. \tag{C4}$$

Define

$$M_1 = \frac{M_0 e^{\eta\bar{h}}}{\sqrt{\underline{\lambda}(P)}} \tag{C5}$$

which is larger than $e^{(b+\eta)\bar{h}}$ since

$$M_1 \ge \frac{\sqrt{\bar{\lambda}(P)e^{2b\bar{h}}}}{\sqrt{\underline{\lambda}(P)}}e^{\eta\bar{h}} \ge e^{(b+\eta)\bar{h}} \ge 1. \tag{C6}$$

Therefore, (C3) and (C4) result in the exponential stability (14). The next part of the proof is to demonstrate (15). Rearranging (5), one gets

$$x(t) = e^{-A\hat{h}(t)}z(t) - \int_{t-\hat{h}(t)}^{t} e^{A[t-\hat{h}(t)-s]}Bu(s)\mathrm{d}s, \quad t \ge 0. \tag{C7}$$

Taking the norm of (C7) and using (21), the upper bound of $\|x(t)\|$ reads as follows

$$\|x(t)\| \le e^{\|A\|\bar{h}}\|z(t)\| + e^{\|A\|\bar{h}}\|B\|\|K\|\bar{h} \max_{s\in[t-\bar{h},t]} \|z(s)\|, \quad t \ge 0. \tag{C8}$$

Consider (14) and (C6), then $\max_{s\in[t-\bar{h},t]} \|z(s)\|$ satisfies

$$\max_{s\in[t-\bar{h},t]} \|z(s)\| \le \begin{cases} M_1 k, & 0 \le t \le \bar{h} \\ M_1 k e^{-\eta(t-\bar{h})}, & t \ge \bar{h} \end{cases}. \tag{C9}$$

Note that $e^{\eta(\bar{h}-t)} \ge 1$ for all $0 \le t \le \bar{h}$, then (C9) provides that

$$\max_{s\in[t-\bar{h},t]} \|z(s)\| \le M_1 e^{\eta\bar{h}} k e^{-\eta t}, \quad t \ge 0. \tag{C10}$$

Substituting (14) and (C10) into (C8), the upper bound of $x(t)$ is developed to

$$\begin{aligned} \|x(t)\| &\le e^{\|A\|\bar{h}} M_1 k e^{-\eta t} + e^{\|A\|\bar{h}}\|B\|\|K\|\bar{h} M_1 e^{\eta\bar{h}} k e^{-\eta t} \\ &\le \left[e^{\|A\|\bar{h}} + e^{\|A\|\bar{h}}\|B\|\|K\|e^{\eta\bar{h}}\bar{h}\right] M_1 k e^{-\eta t}, \quad t \ge 0. \end{aligned} \tag{C11}$$

Thus, the exponential convergence (15) is proven with $M_2 = \left[e^{\|A\|\bar{h}} + e^{\|A\|\bar{h}}\|B\|\|K\|e^{\eta\bar{h}}\bar{h}\right] M_1$. □