Multiplicative reasoning through two-handed gestures

Sean Chorney, Canan Gunes, Nathalie Sinclair

Multiplicative reasoning through two-handed gestures

Sean Chorney, Canan Gunes and Nathalie Sinclair

Simon Fraser University, Canada; sean_chorney@sfu.ca; cgunes@sfu.ca; nathsinc@sfu.ca

This exploratory study focuses on two-handed gestures when using an innovative iPad application called TouchTimes. We suggest that the use of the two hands is relevant to multiplicative reasoning as it provides a way of expressing multiplication as a coordination of two quantities, rather than simply as repeated addition. We report on the use of TouchTimes by two third-grade girls; we first exemplify the designed two-handed gestures the girls used and then identify some emergent gestures that we see as being relevant to multiplicative thinking, particularly in terms of the relation between the multiplicand and the multiplier in a multiplication expression.

Keywords: Two-handed gestures, multiplicative thinking, touchscreen technology.

Introduction

In this paper we report on an exploratory study involving a pair of grade three students working on an iPad application called *TouchTimes* (*TT*, Jackiw & Sinclair, 2018) that was designed to offer an accessible model of multiplication that does not rely on repeated addition. *TT* uses the multi-touch affordance of the iPad to express the relationship between multiplicand, multiplier and product through the use of specifically designed two-handed gestures. It is an extension of *TouchCounts*, also designed by Jackiw and Sinclair (2014), which enables gesture-based actions for counting, adding and subtracting and has been shown to improve subitizing, the ability to differentiate between cardinality and ordinality (Sinclair & Pimm, 2015) as well as basic number sense (Sedaghatjou & Campbell, 2017; Sinclair, Chorney & Gillings, 2016). *TT* builds on this *TouchCounts* research by extending the gesture-based arithmetic. In this research study, we are interested in the way two handed gestures can support young students' multiplicative thinking.

Research on learning multiplication

Multiplication is difficult for many students, yet very important for subsequent higher-grade topics such as proportion and algebra (Siegler et al., 2012). One of the difficulties with multiplication results from students' excessive exposure to multiplication as repeated addition (Askew, 2018). It is important to develop students' understanding of multiplication in terms of relations, particularly the relationship between the multiplicand and the multiplier. Designed to develop the bi-directional functionality of multiplicand and multiplier, *TT* allows for both the multiplicand and the multiplier to be changed either during and after a multiplication computation. We hypothesise that this offers not only an empirical environment for experimentation, but also the possibility of noticing the relationship the multiplicand and the multiplier have on the product.

In the work of Clark and Kamii (1996), multiplication has two levels of abstraction. The shift from the many-to-one and an inclusion relation between multiplicand and multiplier. Multiplication is relational and functional in that both multiplier and multiplicands are abstractions that affect each other. For example, in the case of 3×4, if 3 is the multiplicand and 4 the multiplier, 3 to 1 is the abstraction of three becoming one (the unit) and the 'inclusion relation involves a transfer of units in that the multiplicand "3" counts the number of items while the multiplier "4" counts the number of triples of these' (Boulet, 1998, p. 14). In *TT* the physical use of two hands is meant to enact this dual abstraction. We are thus interested in studying the extent to which students' gestural interactions with *TT* supports their multiplicative thinking.

Brief description of *TouchTimes*

The initial screen of *TT* starts blank, except for a vertical line (Figure 1a). A user can create two kinds of mathematical objects: pips and pods. The first finger(s) down on the screen produces pips, which are coloured discs that appear under each finger. For example, if the user places three fingers on the left side (LS) of the screen, three differently coloured pips appear (Figure 1b). If the user then places four fingers on the right side (RS) while holding the three pips, four 'pods' of three pips appear (Figure 1c). Each pod is a bounded object containing a smaller version of the number, colour and configuration of the LS pips. If a finger is lifted on the LS, then there will be two pips there and each of the four pods will contain two pips. More pips can also be created by placing more fingers on the LS; this will affect the RS pods accordingly). If all the pip-making fingers are lifted, the pods disappear too (multiplication by 0), and the screen goes blank. Additional pods can be created on the RS; once created they remain on the screen even when fingers are lifted (this enables multiplication of numbers bigger than $n \times m$, where $n + m = 11$, which is the maximum number of simultaneous touches allowed). The situation can be completely reversed: if the first touch is on the RS, then pips are produced there and touches on the LS produce pods.

Figure 1: (a) Initial screen of *TT*; (b) Creating 3 pips; (c) Creating 4 pods

As soon as there are both pips and pods on the screen, a multiplication expression appears on the top of the screen. For the above situation, the mathematical expression is $3 \times 4 = 12$ (Figure 1c). Once created, pods can be dragged anywhere on the screen and can also be trashed. Pips can also be moved, as long as the fingers are not lifted. In the reversed situation, the multiplication expression is $12 = 4 \times 3$. The two hands have different functions, and in this way, multiplication is experienced relationally, rather than in terms of repeated addition.

Theoretical framing

Research into iPads is still relatively new and there have been various frameworks proposed to study how the iPad interrelates with mathematical thinking. Embodied theories have been common since the iPad offers unique kinaesthetic experiences with number, particularly in terms of the gestures required to create and manipulate mathematical objects. Bairral and Arzarello (2015) use the idea of embodiment to suggest that the action of the body in any type of activity acts as a grounding for cognition. However, we align with researchers based in a nondualist paradigm (challenging the separation of mind and body), and see embodied action not as a reflection of thought, nor as a grounding of thought, but rather as an aspect of mathematical thinking (de Freitas & Sinclair, 2014).

We employ an embodied framework that sees both the tool (the iPad and application) and the children as a cohesive unit, as an assemblage (de Freitas & Sinclair, 2014). By seeing *TT* not as a separate entity from the students but as 'entwined' (Nemirovsky et al. 2013; Arzarello et al., 2002) we can theorize a formation of bodily-based mathematical activity. This form of embodiment shifts from the tool-student duality, suggesting instead that the mathematical thinking *is* action-with-tools.

Given the design of *TT*, we are particularly interested in how gestures are indicative of students' mathematical thinking. We follow Sinclair and Pimm (2014) in seeing gesture on the iPad as epistemic *and* communicative. Bairral and Arzarello (2015) claim that there are "six basic finger actions for input on a touchscreen: tap, double tap, long tap (hold), drag, flick, and multi-touch (rotate)" (p. 2). For *TT*, we suggest that the last type of gesture, multi-touch, can be expanded and elaborated, particularly in relation to the use of two-handed gestures, on which little research has been conducted in mathematics education. An exception is Duijzer et al. (2017), who study the coordination of action and perception in terms of proportional reasoning in an environment where children move their two hands in the same direction, but at different rates. Bairral and Arzarello (2015) address two kinds of two-handed gestures: (1) when one hand is holding and the other is dragging; (2) when both hands are doing the same gesture (as in when two fingers are dragging away or towards each other to expand or reduce an image). In *TT* each hand performs different movements and different functionalities. One hand acts as the multiplier; the other as the multiplicand. Our research questions therefore are: What are the different types of two-handed gestures used by novice, young learners? How do the two coordinating hands express multiplicative thinking?

Methods

In this study, we examined two students' first contact with *TT*. We worked with two grade three students in an elementary school in a downtown part of British Columbia. We will call them Jessica and Ava in this paper. This pair of students was chosen by the classroom teacher who thought that they would work well together and to provide Jessica, who often ignores instructions, with a different kind of mathematical experience. We video recorded their interaction with the application. The session lasted for 27 minutes. For the initial seven minutes, the students interacted with the tablet without being given a specific task by the interviewer (third author). After the seventh minute, the interviewer occasionally proposed a mathematical task. We transcribed the video, paying attention to when the girls made two-handed gestures, as well as to their awareness of multiplication. For example, if the girls moved their fingers in such a way that the multiplicand reduced by one, we identified their awareness of changes based on what they said. We distinguished situations in which each girl used a hand, which we call a two-person bi-handed gesture, from situations in which one girl used two hands, which we call a solo bi-handed gesture. We see the former as being highly relevant to exploring the social aspect of joint coordination (see Sebanz & Knoblich, 2009).

Analysis

We present our analysis in two parts. First, we use an *a priori* analysis of the "designed" gestures, that is, the gestures that *TT* was designed to allow, and describe which ones were observed in the video. We then present two of the "emergent" gestures found, that is, gestures that were not explicitly designed, but that were identified as being relative to the girls' thinking about multiplication.

A priori analysis of designed gestures

Each gesture identified in Table 1 is done with a single hand. The two-handed gesture comes into existence when any two of these gestures (including identical gestures) are done on different sides of the screen and with different hands (though it is technically possible to use one hand to touch both sides of the screen). The number of two-handed gestures then grows to include all the combinations of any two of the gestures in Table 1.

Single tap (T): Tapping the screen with one or many fingers.
Hold (H): Pressing a screen object continuously without lifting the finger/s.
Drag (D): Moving an object with finger/s on the screen continuously without lifting
Hold and tap (H&T): While some fingers are holding, other finger/s are tapping
Hold and drag (H&D): While some fingers are holding, other finger/s are dragging
Hold and lift (H&L): While some fingers are holding, other finger/s are lifting

Table 1: Designed gestures in *TT*

In Figure 2, we depict three of the more frequent gestures: T, T; T, H; H&T, H. Below, we provide a brief description of each as it occurred in the video.

Figure 2: (a) T, T; (b) H, T; (c) H&L, H

From the students' perspective, in Figure 2a the middle finger of the first hand taps the RS of the screen and the index finger of the second hand taps the LS of the screen. This two-handed gesture was mostly observed at the beginning of the video as the girls were first exploring *TT*. They stopped using this gesture after a while, either because they realized that this did not create more than one object at a time, or that a holding gesture was necessary to create a pod. In Figure 2b, the first hand holds five fingers on the LS all-at-once, the second hand taps the index finger on the RS. This gesture produces one pod, which is a unit of five pips. In Figure 2c, there are three fingers of one hand on the LS, while lifting the pinky and thumb, while the index finger of the second hand was held on the RS. This gesture was very common and changed the number of pips within each of the pods.

A posteriori analysis of the emergent gestures

We will present two episodes, each of which involved the use of an emergent gesture, which we will refer to as a *three-handed gesture* and a *dancing gesture*, respectively.

At the beginning of the first episode chosen, Jessica holds five fingers of her right hand on the RS and Ava holds two pods on the LS (H, H). There were 15 pods on the screen showing $75 = 15 \times 5$. Jessica then used her left hand to make another pip, thereby making six pips in total (Figure 3a), changing the equation to $15 \times 6 = 90$, and changing all the pods to include six pips.

Jessica:	What the
Ava	Wooow (*drags index finger, producing 7 pips, as in Figure 3b*)
Jessica	di di di di didi di di di di oh wait I can change the shape

When she put her sixth finger down, Jessica brought her second hand into play so that the girls were making a *three-handed gesture* of hold pips, drag pip and hold pods (Figure 3a). When she said "What the" she seemed to be surprised at the effect of the sixth pip, which changed the configuration of each of the 15 pods, as well as the multiplication statement. Ava then let go of the pods she was holding with her left hand and used her index finger on the RS to create another pip, which she dragged on the screen, saying "wooow". There were now seven pips in each pod and a new equation $105 = 15 \times 7$. This new three-handed gesture now included a hold, a drag and a drag (Figure 3b). As Jessica dragged her sixth finger, the shape of each pod changed to mimic the configuration of her six fingers, which she seemed to realise when she says "I can change the shape".

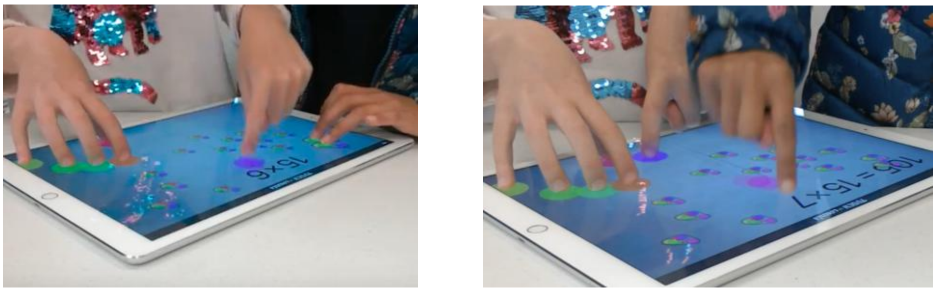

Figure 3: (a) Adding a 6th pip; (b) Adding a 7th pip

In this episode the girls make two different three-handed gestures. In the first instance, Ava had been busy making pods and it was only when Jessica brought her third hand into play that she seemed to notice that the additional pip created an important change on the screen, prompting her to say "What the". Then when Ava sees what Jessica is doing, she changes from a holding pods gesture to a dragging pips gesture. In the first three-handed gesture, Ava does not need to be holding the pods, so the gesture might feel extraneous to the activity. However, it is perhaps because of the fact that she is holding pods that she noticed the interesting effect produced by Jessica's added pip (the pods would have changed size and configuration right under her fingertips, so to speak). In the second case, Ava and Jessica are both doing drag-pip gestures, and so replicating the same gesture that could have been done by just one hand. However, it is in wanting to mimic Jessica that Ava starts dragging a pip, and so we see how the third hand enables both girls to do the same thing at the same time, together producing a change in the shape of the pods that is more than what each of them could have produced on their own.

In the second episode, Ava was holding five fingers on the left screen (which she held for the complete episode; see Figures 4a, 4b) and Jessica had just finished creating pods and dragging them around the screen. There were 31 pods on the screen and the multiplication sentence at this point was $5 \times 31 = 155$. Jessica touched one finger on the LS under Ava's hand (H, T) and then lifted it up again (H, L). Once more she tapped in the same spot, down and up. The multiplication sentence changed from $5 \times 31 = 105$ to $6 \times 31 = 186$, back and forth. She repeated this a third time and said, "They're dancing again" and both girls giggled. She continued to tap this way for a total of eight repetitions. Jessica, still maintaining one finger on the screen, then adds a second finger, causing each pod to increase to seven pips (Figure 4b). She then began to touch up and down repeatedly with two fingers; the multiplication sentence changing from $5 \times 31 = 105$ to $7 \times 31 = 217$, back and forth (Figure 4a, 4b). This time, however, Jessica added a secondary rhythm. She sang 'di di didi di di' while tapping; on each 'di' she tapped once' on the 'didi' she tapped quickly, and in between phrases, she did not tap. She continued to tap with two fingers like this through four phases of 'di di didi di di'.

Figure 4: (a) Jessica about to tap; (b) Jessica tapping two

We see this 'dancing gesture' as an emergent one that actually combines a repetitive set of (H, T) gestures. It becomes a single gesture for the girls that enables them to make the pods "dance". Aside from the evident visual appeal of this dancing gesture, it also embodies the girls' functional awareness of the relation between the pips and the pods, showing as it does how the number of pips affects the shape and colour of each of the pods. With it, the girls can change the multiplicand very quickly, seeing the effect on both the visual dynamics of the screen, as well as on the multiplication sentence.

Discussion

Our first research question asked, what were the different types of two-handed gestures used by novice, young learners? In our *a priori* table we identified the designed, that is, the basic gestures supported in *TT*, which were the gestures that we expected the children to make. The most frequently-used gesture was the (H, T) in which one girl would hold pips (usually Ava), and the other would create pods one finger at a time. The girls rather quickly began to use hold gestures when they realised that lifting their fingers off the pips would clear the screen. They can often be seen holding both pips and pods, especially at the beginning. Dragging was used both to move pips on the screen, as discussed in the first episode above, and also to drag pods to the trash. The use of the gestures in Table 1 provided a starting point for their developing fluency with the tool.

We also identified emergent gestures that seemed to play more important roles in the girls' understanding of how the pips and pods related to each other. In the first episode, we describe two different three-handed gestures made by the girls that helped both to draw their attention to the relation between the number and position of pips on the screen, and to the shape of the pods. The second three-handed gesture seemed to also have an important social component in allowing both girls to do the same gesture at the same time. As with these *three-handed gestures*, the emergent *dancing gesture* combined two or more of the gestures from Table 1. The dancing gesture can thus be seen as a bundling of several, sequentially produced (H&T, H) and (H&L, H) gestures. Over the course of the experiment, this gesture occurred multiple times with different numbers of pips and pods. The girls' continued return to this gesture, as well as the identification of its presence by such utterances as, 'they're dancing again', highlight their awareness of the multiplicative relation between multiplicand and multiplier.

Our second research question asked: How do the two coordinating hands express multiplicative thinking? It was in the use of the two emergent gestures that the girls seemed to express and notice the relation between pips and pods. In making the number of pips change, the girls noticed that the number of pips determined the shape and size of the pod, thus observing the effect of the size of the unit on the pods. The fact that the relation between the pips and the pods took some time to notice and control suggests that this is not a simple relation to understand, both in terms of physically coordinating two hands (or more) across two people, and in terms of tracking the bi-directional relation between multiplicand and multiplier. This embodied approach to multiplication seemed to initiate gestures through which the girls could explore/express this multiplicative relation.

Conclusion

In this exploratory study, which looks at how gestures were used in the *TT* app, we identified 6x6 different "designed" gestures and exemplified three of the most commonly-used ones. We then described two additional emergent gestures that were used several times by the girls and seemed important to their growing fluency with the app and their awareness of the multiplicative relation between the pips and pods (multiplicand and multiplier). We propose that the gesture-based multiplicative actions of the girls offered them novel ways to explore and express multiplicative relationships. In the repeated addition approach, the relation privileges a focus on the multiplier (3+3+3+3+…). In *TT*, however, both operands can be manipulated with immediate feedback, both in terms of the visual presentation of the pods, and symbolically through the equation. If repeated addition tends to be algorithmic, linear, sequential and unidirectional, *TT* seems to offer a more interactive, bi-directional and simultaneous instantiation of multiplication.

This research provides insight into the link between action and mathematical reasoning, and has implications for supporting students' multiplicative thinking. In future work, a closer examination of how learners coordinate the gestural expression with symbolic expressions (the equations shown on the screen) should be examined more closely. Also, the collaborative interactions of learners working together, as expressed through the three-handed gestures described here, deserves further attention.

References

Alibali, M., & Nathan, M. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *The Journal of the Learning Sciences*, *21*(2), 247-286.

Arzarello, F., Bairral, M., Dané, C., & Iijima Yasuyuki. (2013). Ways of manipulation touchscreen in one geometrical dynamic software. *International Conference on Technology in Mathematics Teaching* (ICTMT11), pp. 59-61. Bari, Italy.

Arzarello, F., Olivero, F., and Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM 34*(3), 66-72.

Askew, M. (2018). Multiplicative reasoning: Teaching primary pupils in ways that focus on functional relationships. *The Curriculum Journal*, *29*(3), 406–423.

Bairral, M., & Arzarello, F. (2015). The use of hands and manipulation touchscreen in high school geometry classes. In *CERME 9-Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2460-2466).

Boulet, G. (1998). On the essence of multiplication. *For the Learning of Mathematics*, *18*(3), 12-19.

Clark, F. B. & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5, *Journal for Research in Mathematics Education*, *27*(1), 41-51.

de Freitas, E. & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom.* New York, NY: Cambridge University Press.

Duijzer, C. A. C. G., Shayan, S., Bakker, A., Van der Schaaf, M. F., & Abrahamson, D. (2017). Touchscreen tablets: Coordinating action and perception for mathematical cognition. *Frontiers in Psychology*, *8*(144). doi:10.3389/fpsyg.2017.00144

Jackiw, N., & Sinclair, N. (2018). *TouchTimes* (iPad App). Burnaby, BC: Tangible Mathematics Group.

Jackiw, N., & Sinclair, N. (2014). *TouchCounts* (iPad App). Burnaby, BC: Tangible Mathematics Group. [https://itunes.apple.com/ca/app/touchcounts/id897302197?mt=8]

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the Classroom Floor Becomes the Complex Plane: Addition and Multiplication as Ways of Bodily Navigation. *Journal of the Learning Sciences*, *21*(2), 287-323.

Sebanz, N., & Knoblich, G. (2009). Prediction in joint action: what, when, and where. *Topics in Cognitive Science*, *1*(2), 353-367.

Sedaghatjou, M. & Campbell, S. (2017). Exploring cardinality in the era of touchscreen-based technology. *International Journal of Mathematical Education in Science and Technology 48*(8), 1225–1239.

Sinclair, N., Chorney, S. & Gillings, S. (2016). Rhythm in number: Exploring the affective, social and mathematical dimensions of using *TouchCounts*. *Mathematics Education Research Journal*, *28*(1), 31-51.

Sinclair, N., & Pimm, D. (2015). Whatever be their number: Counting on the visible, the audible and the tangible. In M. Meletiou-Mavrotheris, K. Mavrou & E. Paparistodemou (Eds), *Integrating touch-enabled and mobile devices into contemporary mathematics education* (pp. 50–80). Hershey, PA: IBI Global.

Sinclair, N., & Pimm, D. (2014). Number's subtle touch: expanding finger gnosis in the era of multi-touch technologies. In P. Liljedahl, C. Nicol, S. Oesterle & D. Allan (Eds), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education* (Vol. 5, p. 209-216). Vancouver, Canada: PME.