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Abstract

The use of the Google Scholar produces about 78,000 hits on the term "Timoshenko beam." The question of priority is of great importance for this celebrated theory. For the first time in the world literature, this study is devoted to the question of priority. It is that Stephen Prokofievich Timoshenko had a co-author, Paul Ehrenfest. It so happened that the scientific work of Timoshenko dealing with the effect of rotary inertia and shear deformation does not carry the name of Ehrenfest as the co-author. In his 2002 book, Grigolyuk concluded that the theory belonged to both Timoshenko and Ehrenfest. This work confirms Grigolyuk's discovery, in his little known biographic book about Timoshenko, and provides details, including the newly discovered letter of Timoshenko to Ehrenfest, which is published here for the first time over a century after it was sent. This paper establishes that the beam theory that incorporates both the rotary inertia and shear deformation as is known presently, with shear correction factor included, should be referred to as the Timoshenko-Ehrenfest beam theory.

Keywords

Bresse- Rayleigh- Timoshenko-Ehrenfest, beam theory, history, priority

1. Introduction

At first glance, the question in the title may appear self-evident. Indeed, the title contains the term Timoshenko beam theory. Therefore, one may conclude, naturally, that it must have been S.P. Timoshenko who authored it. Indeed, in the words of Laura, Rossi, and Maurizi [1], "the publication, by Stephen Timoshenko, of his now classical theory of vibration of beams, whereby shear and rotatory inertia effects are taken into account, constitutes one of the most remarkable events in the development of the structural dynamics of the twentieth century. Together with the Timoshenko–Mindlin theory of vibrating plates it has influenced the mathematical analysis of the quasi-infinite variety of dynamics of continuous media and structural acoustics problems from bridges and machine elements to surface, underwater, and space vehicles passing through the prediction of the behavior to electronic packages, bioengineering systems, etc." These authors describe Timoshenko's contribution as "epoch making." Archibald [2, p.61] characterized S.P. Timoshenko as "the patron saint of the American engineering." Pisarenko [3], the author of the biographical book on Timoshenko, uses the concept of "Timoshenko phenomenon." Terman – Provost Emeritus of Stanford University – in his congratulatory letter [4] to Timoshenko, in conjunction with latter's 90th birth anniversary, wrote, on December 2, 1968: "I am pleased to report that the 'Timoshenko Legend' continues to flourish undiminished on the Stanford campus." Koiter [5,6], in the late 1970s, chose to disagree with the assessment of the above type. In his

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words: “What is generally known as Timoshenko beam theory is a good example of a basic principle in the history of science: a theory which bears someone’s name is most likely due to someone else [detailed discussion of the ‘principle’ can be found in section 5 – I.E.]. In fact, additional deflections due to transverse shear occur already in the fourth edition, dated 1868, of W.J.M Rankine’s [7,8] well-known book *Applied Mechanics*… The conventional formula for the shear factor k, based on energy considerations, according to the author’s definition with the value $k = (6/5)(1 + \nu)$ for a narrow rectangular cross section, also occurs already in the literature of the nineteenth century.” Simmonds [9] noted: “…shear deformation effects were first introduced by Rankine [7] and rotary inertia effects by Bresse [10]. In his often-cited paper of 1921, Timoshenko [11] without explicit reference to either Bresse or Rankine combined these effects to create what is now almost universally referred to as the Timoshenko equations.” Indeed, the rotary inertia effect was apparently discovered independently of Bresse [10] by Rayleigh [12]. There was, however, one more contributor in developing the theory that takes into account both effects. This was Paul Ehrenfest (1880–1933) who moved in 1907 to St Petersburg from Vienna, Austria. The first appearance of the beam theory, which takes into account both shear deformation and rotary inertia, was not made in Timoshenko’s often cited paper of 1921, but already in 1916, namely in his book, in Russian, *Course in Elasticity* (second volume) where S.P. Timoshenko [13] (figure 7) explicitly writes, in the footnote, that the derivation as made together with P. Ehrenfest. It is a mystery, why the subsequent publications of Timoshenko on this topic omit the name of Ehrenfest, but mention him again in the footnote of his 1922 paper [14] (figure 3), now in English, that the work in the paper was performed by Ehrenfest and him. It appears that a more justifiable name of the above theory ought to be Bresse–Rayleigh–Timoshenko–Ehrenfest beam theory or Timoshenko–Ehrenfest theory in short, because the theory as we know it today was put by these scientists in collaboration.

Grigolyuk [15, p.39] (see also [16]) writes in his book *S.P. Timoshenko: Life and Destiny*: “At that time he solved the problem of principal importance on the effect of shear stresses during the small vibrations of the beams; equation of small vibrations was obtained with shear deformation and rotary inertia taken into account. Then S.P. Timoshenko together with physicist Ehrenfest obtained the exact solution on transverse vibration of the beam of rectangular cross section. **Usually it is considered that they constitute the content of works [11,13]**. These results are presented in the paper that appeared in Zagreb in 1920 in the journal *Glasnik Hrvatskoga prirodsalovnoga društva* [17] (emphasis added – I.E.) (figure 10). Let us remark that these results first were published in 1916 in the *Course of Theory of Elasticity. Part II. Beams and Plates* (§§38–39, pp.200–213) [13].” As is clearly seen, Grigolyuk attributes the so called Timoshenko beam theory exposed in works [11,13,17] to both Timoshenko and Ehrenfest, and thus, should be called Timoshenko- Ehrenfest beam theory.

2. Topic of priority: Timoshenko and Ehrenfest

We provide herewith the details of the above cooperation. Soderberg [18, p.329] writes: “During this period he happened to meet Paul Ehrenfest, with whom he formed a lasting friendship. Ehrenfest had come from Germany to Russia in the hope that academic freedom there would offer him better opportunities. He was not successful in establishing himself in Russia, but the two met frequently; Ehrenfest would then enlarge upon the current ideas abroad in physics – relativity and quantum mechanics – new to Timoshenko at the time” (see also Boag et al. [19]).

Pisarenko [20, p.98] writes, in the free translation from Russian: “In August 1911, S.P. Timoshenko moves to St. Petersburg, in hope to find in the capital city, with concentration of majority of institutions of higher learning, some work. He was unable to find any full time job. He had to look for adjunct appointments, though these were associated with a lower level of remuneration, than a professional job. In order to support his family, he collected about 20 hours of work in various ends of the city… Still, in 1911–1912 he was able to write papers on the action of transverse impact on the beam and effect of shear deformation on transverse vibration of beams. The former work was later translated into German, whereas the latter–into English. Both works in the future served as spring points for many scientists.” Then Pisarenko [20, p.98] mentions the meeting with Paul Ehrenfest. Let us give the podium to Pisarenko [20] himself: “In Petersburg S.P. Timoshenko rented a place on the Aptekarskiy Island, just opposite of Electrotechnical Institute. His close neighbor turned out to be Paul Ehrenfest, their acquaintance coming through Professor Abram Ioffe (1880–1960) [also spelled Joffe]. German [Pisarenko

Figure 1. Paul Ehrenfest (left) and Stephen Timoshenko in St. Petersburg, published by permission of Museum Boerhaave, Leiden, The Netherlands

making a mistake here, for P. Ehrenfest was an Austrian and later a Dutch citizen – I.E.] physicist [P. Ehrenfest] gave lectures, with pleasure, on the meetings of the Russian Physical-Chemical Society, where not a small number of participants could be found. Ehrenfest was a natural lecturer, who was in constant need of audience. Once they made an acquaintance, scientists [Timoshenko and Ehrenfest] somehow agreed to meet in the botanic garden during the morning strolls… On a clean snow surface one could draw needed figures and write formulas. By this means, before the spring of 1912 Timoshenko listened to Ehrenfest's lectures on the theory of relativity and quantum theories. In summer of 1912 Timoshenko and his wife went abroad in lieu of the D.I. Zhuravsky Prize that he got." This was due to the fact that S.P. Timoshenko was able to cover the expenses associated with the trip from the price which was equal to annual salary. "In September 1912, S.P. Timoshenko returned to St. Petersburg, to continue his pedagogical activity," writes Pisarenko [20, p.103].

Hollestelle [21] writes: "During his studies, he went to Göttingen for a longer period of time, to study with the famous mathematician Felix Klein, whose ideas also had a lasting influence on Ehrenfest. Here he also got to know his future wife, Tatiana Affanassjewa – a Russian mathematics student. After Ehrenfest finished his dissertation, the couple married…"

Ehrenfest moved to St. Petersburg after his wife got a job offer there.

"The circle of physicists in St. Petersburg welcomed Ehrenfest enthusiastically. That, however, did not lead to a job for him." Moreover, "Once in St. Petersburg, Ehrenfest tried to make up for his shortcomings as a non-Russian as best as he could. He tried to fit into Russian society, almost symbolically, by growing a beard and long hair" (Figure 1).

It is preferable to listen to Timoshenko's [23, pp.122-123] (see also [21–24]) own testimony: "I took lodging on Aptekarkiy Island, across for the Electrical Engineering Institute. My nearest neighbor was a physicist, Ehrenfest, to whom Ioffe had introduced me. After finishing at the University of Vienna, Ehrenfest had gone to Göttingen, and there worked successfully in theoretical physics… he found it difficult to build an academic career in Germany, so [he] decided to come to Russia, to Petersburg, where – after the introduction of "Academic Freedom"– the interest in physics had markedly increased… He did not succeed in obtaining professorship, or even as a teacher. Yet he loved to teach and explain things, and we arranged to meet mornings in the Botanical Garden, when usually there were no visitors, and where on the clean surface of the snow one could make the necessary drawings with a stick. There he delivered to me a number of lectures on matters then new, such as relativity and quantum theory. In the spring of 1912 the snow in the Botanical Garden melted, and the lectures stopped. Ehrenfest had obtained a professorship and left Russia forever."

Timoshenko [23] does not mention in his book *As I Remember* (see also its editions in Russian [24,25,27] and in German [26]) their scientific cooperation during this period. He neither submits a paper on combined effect of rotary inertia and shear deformation to any Russian or non-Russian journals. Only 4 years later, Timoshenko [13, p.206] (figure 4) mentions his cooperation with Ehrenfest, in the book *Theory of Elasticity*.

Let us return to Timoshenko's book [13] (see also its reissue [28]). In the beginning of his discussion of the combined effect of rotary inertia and shear deformation: "We jointly with Prof. P. Ehrenfest obtained also the exact solution of the problem of layer of rectangular cross-section." The important word here appears to be "also." Thus, Timoshenko informs, that not only he and Ehrenfest obtained the equations for what is now usually referred as Timoshenko beam theory, but they also, obtained some other problem's solution! The pertinent question arises why they did not publish these results immediately? Why did Timoshenko include results obtained with someone else, namely Ehrenfest, in his own book? Did he ask the permission from Ehrenfest? Or did neither Ehrenfest nor Timoshenko attach any importance to what they did together, and hence Timoshenko did not feel obliged to get permission? Is it possible to assume that they thought that the incorporation of rotary inertia and shear deformation effects was unimportant? The answer to the latter question must be an unequivocal no. Timoshenko must have known that Bresse [10] and Rayleigh [12] already took into account rotary inertia. Perhaps Timoshenko did not know French and could have not read the book by Bresse [10]? It should be noted that Timoshenko in 1913 published a paper in French [29]; thus he either knew French or asked someone to translate his manuscript into French. Perhaps Timoshenko did not publish any papers in 1912 and 1913, so he likewise refrained from publication of his results with Ehrenfest? Again, we must answer this inquiry with no. In 1912, Timoshenko published two papers, and in 1913 he published six papers! Many years later, when Timoshenko's [13] book was republished in the Russian language [28] the scientific editor of the re-publication Professor Eduard Ivanovich Grigolyuk contacted Timoshenko and asked him about his comment made in 1916, that "we together with Prof. P. Ehrenfest obtained also the exact solution for the problem for rectangular cross section" (figure 3).

What does this expression mean? There are several possibilities. It may mean, and this is possible when considering the different professions of the two persons, that Timoshenko presented the approximate beam theory to Ehrenfest, who helped to find an exact frequency relation for calibration. On the other hand, one should remember that Ehrenfest's Ph.D. dissertation was in classical mechanics; he only later changed his scientific interests. Such interpretation seems to be dismissed by Grigolyuk (see [28, p.338]) who writes: "This result, obtained by S.P. Timoshenko with P. Ehrenfest, was not published jointly. In his letter to the editor of current edition, dated 2 July 1970, S.P. Timoshenko comments that 'this work was conducted in 1913, when Ehrenfest lived in St. Petersburg. Later on I did not work with him.'" Here following clarifying comment is needed. It may appear impossible that the joint work of Timoshenko and Ehrenfest could have been done in 1913, for Ehrenfest left Russia in 1912. Ehrenfest arrived to Leiden in October 1912, delivering customary inaugural lecture on December 2, 1912. Thus, Timoshenko [28, p. 338] might appear as misremembering the year when he and Ehrenfest obtained the equations describing rotary inertia and shear effect, as well as the elasticity solution for rectangular cross-section beams. Indeed, the year he mentions to Grigolyuk may be inferred as incorrect also from the autobiographical book of Timoshenko [23–27] *As I Remember* (with a Russian edition appearing in Paris, earlier, in 1963 [24]; see also [25–27]) where he mentions that "in the spring of 1912 the snow in the Botanical Garden [of St. Petersburg – I.E.] melted, and the lectures [of Ehrenfest, delivered to Timoshenko in the Botanic Garden – I.E.] stopped. Ehrenfest had obtained a professorship and left Russia forever." Here too Timoshenko makes a mistake. Although Ehrenfest left Russia in 1912, he did return to Russia several times, and was elected to the Academy of Sciences of the former USSR, as its Corresponding Member, in 1924, three years prior to Timoshenko being elected to an analogous position.

In his 1968 book *As I Remember* [23], Timoshenko does not mention that the effect shear deformation and rotary inertia were studied by him in cooperation with P. Ehrenfest. As stressed above, this cooperation is mentioned in three places by him, namely in his 1916 book [13] (figure 4) on theory of elasticity in Russian, in its 1972 republication [28], again in Russian, and in his 1922 paper [14] (figure 3), in English. However, he did not mention Ehrenfest in either his 1920 paper [17] or in his 1921 paper [11]. The mentioning of Ehrenfest in the 1922 paper was probably influenced with the fact that he met Ehrenfest in Germany, Jena in 1921; by this time Ehrenfest "had become a famous professor and now held the chair of physics" [23, p.223]. Perhaps Timoshenko refrained from mentioning Ehrenfest due to the fact that he did not mention the details of scientific work in the autobiographical [23] rather than scientific book? The answer to this inquiry is negative. On page 119 of his autobiography [23], Timoshenko writes: “In the winter of 1911–1912 I wrote a paper on the effect of transverse impact on a beam and investigated the influence of shear stresses on the transverse vibrations of beams. Both these articles were translated, one into German, the other into English, and subsequently served as a point of departure for the research of many authors.” So, in his autobiography he does mention his scientific works; at the same time, he omits mentioning the cooperation he had with Ehrenfest on this topic.

Timoshenko could have combined writing about his scientific work on page 119 with mentioning Ehrenfest on page 123, but somehow, inexplicably, he did not do so. Perhaps by year 1963 when Timoshenko wrote the book [24], in Russian, he didn’t remember the above cooperation? Indeed, the English version [23] of his book is called *As I Remember*, implying perhaps that not everything was remembered half a century later. In their foreword [23], James. M. Gere and Donovan H. Young write, in March 1967: “We have had the full support of Professor Timoshenko during the translation, and he has carefully checked this edition.” This “carefully checked” edition still leaves many questions unanswered.

Timoshenko does not explain why work that was done with Ehrenfest in winter of 1911–1912 waited until 1916 book [13] publication with only a footnote about Ehrenfest. Why did Timoshenko wait till 1916 to publish results? Why did neither of the three Timoshenko’s papers in English [11,14,17] bear Ehrenfest’s name alongside that of his? Perhaps perplexed by this, Grigolyuk [28] inquired about this as the editor of the second edition of Timoshenko’s [28] (figure 5) book on elasticity. Timoshenko’s response [28] (Figure 5) was terse, noting, “This work was conducted in 1913, when Ehrenfest lived in St. Petersburg. Later on I did not work with him.” (Figures 5 and 9).

It appears surprising is that the book *Collected Papers by S.P. Timoshenko* [30] reproduces his paper *On the Transverse Vibrations of Bars of Uniform Cross-Sections* [14], but in contrast with the original publication, does not mention Ehrenfest anymore (figure 6). The persistent question arises as to whether this took place, inadvertently, via the editors, E.L. Eriksen, J.N. Goodier, J.P. Den Hartog, L.S. Jacobsen, R.E. Peterson, and D.H. Young. Or perhaps a typist omitted the name of Ehrenfest by mistake? Omitting Ehrenfest’s name apparently contradicts the editors’ statement on page V, which states: “the papers themselves… are printed as they were originally published, any corrections or additions appearing in the form of footnotes.” We must then assume that S.P. Timoshenko made these corrections and additions, though the editors do not explicitly mention this. This fact appears in contradiction with statements made by Timoshenko [30] (see also [31–33]) about priority in publications mentioning Ehrenfest. One cannot refrain from asking if Tomoshenko himself deleted or asked secretary or editor(s) to delete mentioning Ehrenfest.

Of course, no clear answer can be given to questions of this type on the basis of the presented facts, as perplexing these facts may be. But these facts cannot be avoided. Also, when judging about publications that were written the first decades of the twentieth century, one should keep in mind that the manners of publishing and citing were possibly different in that period, the latter moreover having been affected by the terrible events of World War I.

In the chapter 22, titled “We Move to Petersburg,” Timoshenko [23, pp.119–123] informs us that in August 1911, after being fired from the Dean’s job at Kiev Polytechnic Institute, he went to St. Petersburg to look for an adjunct position, since he “could not get a regular position in the schools (universities – I. E.).” On page 119 of his autobiography Timoshenko [23] writes

> “Although little time was left for science, for which anyway I lacked the necessary peace of mind, I did manage to get something done. In the winter of 1911–12 I wrote a paper on the effect of transverse impact on a beam and investigated the influence of shear stresses on the transverse vibrations of beams. Both these articles were translated, one into German, the other in to English, and subsequently served as a point of departure for the research of many authors.”

The results of the study on the impact on the beam were published in his papers [36, 37], in German. Indeed, the careful reading of the above passage of Timoshenko book reveals, that about the work on “effect of transverse impact on a beam” he uses the statement “I wrote a paper”. However, about his work on “the influence of shear stresses on the transverse vibrations of beams” he uses the verb “investigated.” This causes questions to arise: “In the winter of 1911–1912 Timoshenko investigated the influence of shear stresses on the transversal vibrations of beams. Did he write a paper on this topic? If yes, when?” The first inquiry is answered positively from Timoshenko’s own statement that “both those articles were translated, one into German, the other into English, and subsequently served as a point of departure for the research of many authors.”

Timoshenko fails to inform that he refrained to publish the material of "the influence of shear stresses on transverse vibrations of beams" in a journal article form more or less immediately after completing the investigation. He waited a staggering five years, until the work was reported in his book *Course in Theory of Elasticity* [13] (figure 4) which was republished [28] in 1972 – the year of Timoshenko's death – by the Ukrainian Publishing House "Naukova Dumka." Later, in 1920, the material was published in the Croatian journal *Glasnik Hrvatskoga Prirodaslovnoga Društva (Herald of the Croatian Nature Association)* [17], in English (Figure 10). It should be stressed that this very article is not mentioned in the section "Articles in English, French, and German" in the list of publications given by Timoshenko's [23, pp.421 –427] autobiography that appeared in Russian. A year later, in 1921, the same article was published in the British Journal *Philosophical Magazine* [11], without mentioning that it was already published in Russian [13] and in English [14]. Is there any difference between Timoshenko's articles dated 1920 [14] and 1921 [11]? Essentially none in the content, except that the 1921 paper [11] does not reference work by Rayleigh [12] who took into account the effect of rotary inertia on beam vibrations; additionally, the papers [11,14] use different shear correction factors.

Timoshenko [38] attributes the term $(Ir/g)(\partial^3 w/\partial x \partial t^2)dx$ in the differential equation for the lateral vibration of the prismatic bar to Lord Rayleigh [12, paragraph 18b]. Leissa [39, p.314] writes: "In 1877 Lord Rayleigh showed how the addition of 'rotatory' (in the language of his day) inertia effects, to those of classical transitional inertia affected the flexural vibration frequencies of beams."

Timoshenko [38], followed by numerous investigators, also cites Rayleigh [12] in his second exposition of the effects of rotary inertia and shear deformation [17]. He fails to mention either Bresse [10] or Rayleigh [12] in his subsequent paper [11]. The latter paper is essentially identical to his first publication [13] on this topic except different shear corrections factors, as mentioned above.

3. Overlooking contribution of Bresse

The third edition of the book *Vibration Problems in Engineering*, written by Timoshenko in 1955 (p.329), in collaboration with D.H. Young, again, mentions Lord Rayleigh's [12] contribution to the accounting of rotary inertia. Authors refrain from mentioning Bresse's [10] work although the book, by Timoshenko and Young published in 1955, was published after Timoshenko's 1953 [30, p.151] book on the history of strength of materials. This is where Timoshenko specifically mentioned that Bresse [7] was the first investigator who introduced the rotary inertia!

The fourth edition of the book *Vibration Problems in Engineering*, now co-authored by three authors namely by Timoshenko, Young, and Weaver in 1974, also contains reference to Rayleigh [12]. The latest edition (fifth), co-authored by Weaver, Timoshenko, and Young, in 1990 (p.435) appeared after Timoshenko's and Young's death, in 1990. It does not mention Bresse [10] or Rayleigh [12] but only Timoshenko's [11] paper, with the note that the experimental verification of the shear effect was conducted. Strangely, neither of these editions contain reference to the first author who took into account the rotary inertia effect, namely, Bresse [10].

It ought to be emphasized that Rayleigh [12] did not refer to the work by Bresse [10]. The partial explanation could be attributed to the circumstances in which his seminal book *The Theory of Sound*, came into being. Crandall [40] writes: "In 1871 at the age of 28 he married Evelyn Balfour. Six months later, he fell seriously ill with rheumatic fever and nearly died. While on recovery, he was advised not to spend the first winter after his illness in England and so the young couple spent the three winter months on a houseboat traveling on the Nile River in Egypt. In the spring, they toured in Greece and Italy, returning to London in May 1873. A month later, John's father, the second Baron Rayleigh, died. Thus at the age of 30, John William Strutt assumed the title and the responsibilities of Lordship."

To a large extent the discipline of vibrations as we know it today was born during those 3 months on the River Nile. According to Rayleigh's son's account, "It was a monotonous life but it suited them. My father began his well-known treatise on the Theory of Sound, and worked at it all the morning in the cabin. He was able to write much of the earlier part of the book without access to a large library…In the mornings it was difficult to persuade him to land, even to see the most enchanting temple…" The treatise was eventually completed 4 years later and published in two volumes: Vol. 1, on vibrations, in 1877, and Vol. 2 on acoustics, in 1878. Major additions were made in the second editions, in 1894 for Vol. 1 and in 1896 for Vol. 2."

It is quite possible that Rayleigh [12] did not know the work by Bresse [10]. Alternatively, he may have thought that the extension of the Bernoulli–Euler beam theory to include rotary inertia was a minor issue. Naturally, during his tour on the Nile he could not have searched for references.

Several authors refer to the so-called Timoshenko beams as Bresse–Timoshenko beams. The encyclopedia of vibration [41, p.239] uses the term "Bresse–Timoshenko theory," noting: "Credit for including the rotatory inertia effect is given to Rayleigh in the 1870s, while Timoshenko added the shear flexibility effect in 1921."

Bert and Gordaninejad [42,43] note that the effect of "rotatory inertia dates back to the work of Bresse [10], which predates…Timoshenko's [11] shear deformable beam theory." They also note: "A beam model taking into consideration the shear deformation effect is usually known as the Timoshenko beam model." Then, they add: "A beam model taking into consideration the shear deformation effect is usually known as the Timoshenko beam model." Hagedorn and DasGupta [44, p.144]. write about "some…aspects of the Timoshenko (or Bresse) beam." Laurenson and Yuceoglu [45] state: "Bresse [10] in 1859 was the first to include both the rotatory inertia and shear flexibility. However, today this theory is referred to as the Timoshenko beam theory." Pascariu [46] stresses: "Bresse is the first who is attributed with corrections of rotatory inertia and shear…" Likewise, Beyer [47] states: "It is of some historical interest that the rotatory inertia correction, usually attributed to Rayleigh (*Theory of Sound*, [12, pp.744–746]), is given by Bresse [10] in his *Cours de Mechanique Applique* (1859, p.126), which was largely unknown when Timoshenko proposed his model in 1921." Jahsman [48] wrote: "Bresse was the first to develop the beam equations corrected for rotatory inertia and shear…" Naturally such a beam model that takes into account rotary inertia ought to be referred to as a Bresse beam, or as Bresse–Rayleigh beam. Ironically, there is only a single work that refers, in the title, to Bresse–Rayleigh beam. This is the paper by Jackson and Oyadiji [49]. Other authors, when discussing the rotary inertia alone, mention Rayleigh's name. The Google Scholar website found, in March 2017, 390,000 hits on the term Rayleigh's beam; this number is not precise since numerous results contained the concept of *Timoshenko beam*.

In addition to the pioneering work by Bresse [10], Timoshenko [11,17] did not cite additional work in his two papers, published in the West, on the effect of rotary inertia. This work is done by Searle [50], and published by a universally accessible journal *Philosophical Magazine*. This is the very journal in which a year after his first publication [11] in English, his second publication [17] in English appeared, of the material first exposed in his book [13]. It should be stressed that Timoshenko's [11] second publication was a close copy of the first, 1920 publication in the Yugoslav, namely Croatian, journal [17] (figure 10). The only difference between two papers consists in the fact that Timoshenko [17] cites Lord Rayleigh's contribution, whereas the later paper [11] does not mention Rayleigh's work; likewise, he adopts different shear correction factors in these two papers.

The Google Scholar website produces over 78,000 hits on the term *Timoshenko beam*. Hence it is not surprising that the work by Laura, Rossi, and Maurizi [1] designates this method as "epoch making" in their review some 70 years after the theory was enunciated. These authors, as many others in the West, did not know that the method made a maiden appearance in 1916 (figure 4), and not in 1921, as it is uniformly thought.

These authors cannot be blamed, as it were, owing to the fact that the paper by Timoshenko [17] is not well known. In Timoshenko's autobiography [23, p.422], the 1920 paper [17] is not mentioned in the list of Timoshenko's papers. Timoshenko might have not been proud of the double publication he committed. Note that ref. [17] is not mentioned in the list of his publication dedicated to Timoshenko's 65th Birth Anniversary (figure 8). Nevertheless, we cannot blame Timoshenko either; it was a turbulent time in his life: he desperately wanted publicity in the West, in order to find a professorial job. As someone remarked, "without publicity there is no prosperity."

While at Zagreb Polytechnic, Timoshenko, by all probability, was looking elsewhere; thus, he probably considered the publication of his works, in then-famous journal *Philosophical Magazine*, as a gateway to a better position. The present writer is not aware of the publication etiquette at that time. Nowadays, multiple publications of the same paper, are not welcomed, to state mildly, and often are retracted.

The differential equation that takes into account both shear and rotary inertia effect reads:

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\left(1+\frac{E}{k'G}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\rho^2 I}{k'G}\frac{\partial^4 w}{\partial t^4}=0 \tag{1}$$

This equation is usually referred to as the Timoshenko equation. Timoshenko had predecessors, namely Bresse [10] and Rayleigh [12], as far as the rotary inertia is concerned; and in relation to the shear deformation effect, albeit in statics, Koiter [5,6] pointed at others' contributions. It appears that one ought to refrain from using the uniformly known term *Timoshenko beam theory*, and utilize more informative terms. We will return to this topic later. It should be remarked that Mindlin [59,60], Bert [61], and Elishakoff and Lubliner [62] refer to these sets as Bresse–Timoshenko equations. They felt, as Mindlin and Deresiewitz [51] did, that the historical justice demanded that at least the name of Bresse should have being reinstated when one discusses the effect of rotary inertia and shear deformation. Neither of these authors knew that the modern theory that S.P. Timoshenko advanced was developed with Paul Ehrenfest.

It appears therefore fully justified to call it *Bresse–Rayleigh–Timoshenko–Ehrenfest theory* or BRTE theory if one wants to include all participants in the development of this theory. A shorter name would be *Timoshenko–Ehrenfest beam theory*. Indeed, Bresse [10] and Rayleigh [12] derived the rotary inertia effect but the contribution of the shear deformation is often more important than that due to rotary inertia. Now, Bresse [10] derived, according to Artobolevsky, Bobrovnitsky, and Genkin [52] (see also Challamel [55,56]) the effect of shear deformation but he did not utilize the shear coefficient. Thus the modern theory of beams, as it stands now, taking into account both rotary inertia and shear deformation incorporating shear correction factor, is due to two scientists: S.P. Timoshenko and P. Ehrenfest, in a personal jaw-droping testimony of Timoshenko – one of the two co-authors. This appears to be relatively straight forward case of establishing priority. This is why hereinafter we will adopt the name *Timoshenko–Ehrenfest beam theory*.

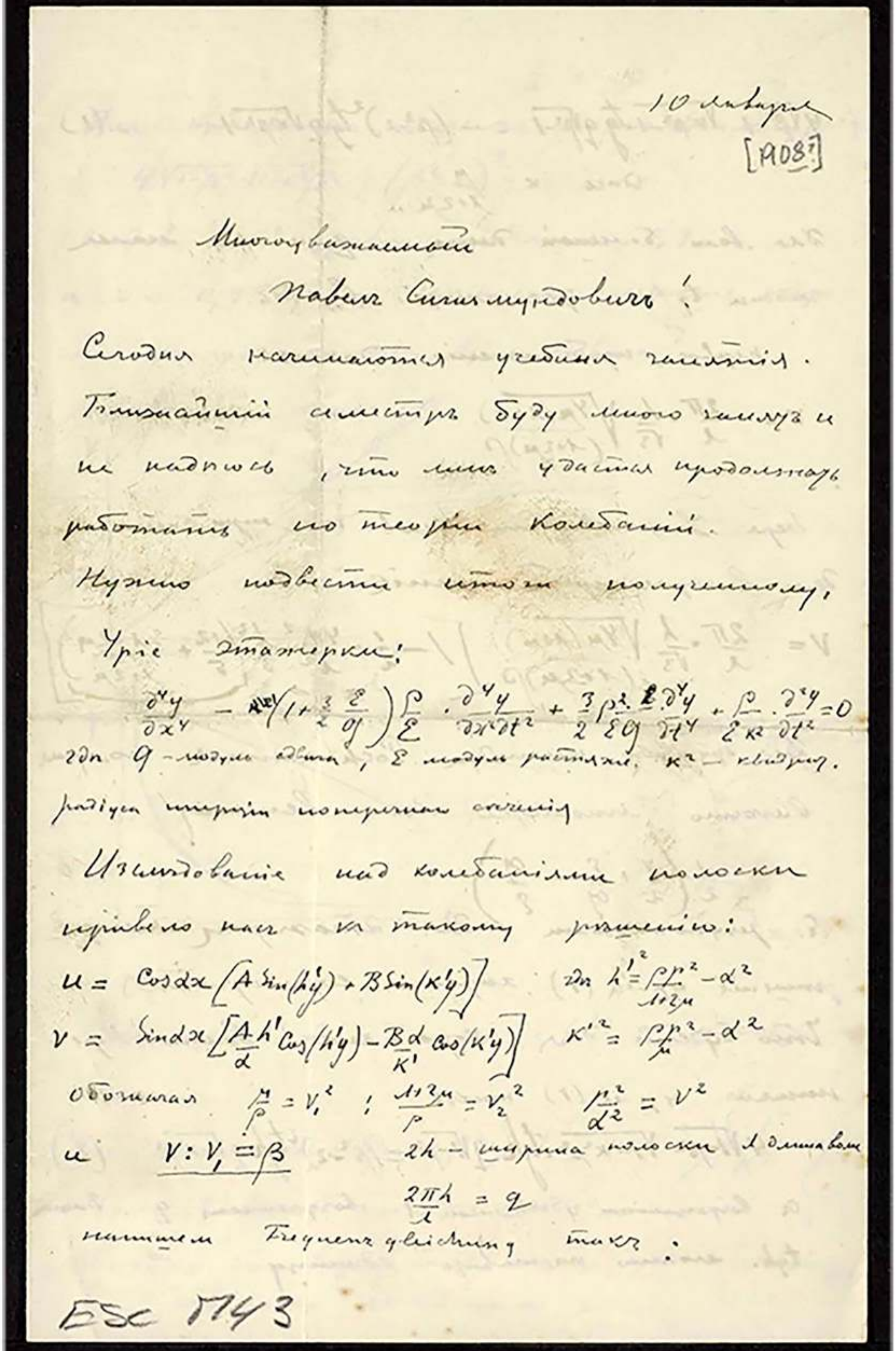
In his book on history of strength of materials, Timoshenko [53, p.418] provides his own take on what became to be known as Timoshenko beam theory: "Lateral vibrations of shafts and beams core also of great practical importance. The simplest cases of vibration of prismatic bars were their solutions included in books on acoustics. In applying these solutions to practical beams in which the cross-sectional dimensions are not very small in comparison to the length, or to cases where higher modes of vibration are of importance, it has become necessary to derive, more complete differential equation, in which the effect of shearing forces on deflections is considered."

4. Did Ehrenfest refuse his name to appear as that of the co-author?

In S.P. Timoshenko's paper "On Transverse Vibration of Bars of Uniform Cross-Section" (Philosophical Magazine and Journal of Science, Series 6, 1922, Vol. 43, pp.125–131), on page 127 it is remarked that the work was conducted by P. Ehrenfest and by Timoshenko jointly but solution (i.e. its numerical implementation of the frequency equation, belonged to S.P. Timoshenko. Ehrenfest's name, however, does not appear as one of the authors. A nagging thought persists: Did Ehrenfest refuse to be the co-author of the paper on the effect of shear deformation and rotary inertia with Timoshenko? This possibility cannot be totally discarded, since Ehrenfest's approach to science is best illustrated by what he wrote to Robert Oppenheimer (1904-1967) in the summer of 1928, after Oppenheimer invited himself for an extended stay in Leiden, "If you intend to mount heavy mathematical artillery again during your coming year in Europe, I would ask you not only to not come to Leiden, but if possible not even to Holland, and just because I am really so fond of you and want to keep it that way. But if, on the contrary, you want to spend at least your few months patiently, comfortably, and joyfully in discussions that keep coming back to the same few points, chatting about a few basic questions with me and our young people – and **without thinking much about publishing (!!!)** (emphasis by present writer – I.E.) – why then I welcome you with open arms!" (see [74,75]).

As stressed above, in his 1968 book *As I Remember* Timoshenko [20] does not mention that the effect shear deformation and rotary inertia were studied by him in cooperation with P. Ehrenfest. As emphasized above this cooperation is mentioned in three places by him, namely in his 1916 book on theory of elasticity in Russian [10], in its 1972 republication [55], again in the Russian version of that book on the theory of elasticity, and in his 1922 paper [11]. However, he did not mention Ehrenfest in neither his 1920 paper [14] nor in his 1921 paper [8] or in three editions of the vibration text book.

Fortunately, Paul Ehrenfest found his fame in the Netherlands. He tragically committed suicide in 1933. After some years, the grateful Dutch created the Paul Ehrenfest Archive at the Museum Boerhaave in Leiden. Moreover, Bruce Wheaton [76] published the Catalogue listing all available documents in the Ehrenfest Archive. Two pages in the Catalogue mention a letter and postcards received



10 января
[1908?]

Многоуважаемый
Павел Сигизмундович!

Сегодня начинаются учебные занятия. Ближайший семестр буду много занят и не надеюсь, что мне удастся продолжать работать по теории колебаний. Нужно подвести итоги полученному.

Урiе этажерки!

$$\frac{\partial^4 y}{\partial x^4} - \left(1+\frac{3}{2}\frac{E}{G}\right)\frac{\rho}{E}\cdot\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{3}{2}\frac{\rho^2}{EG}\frac{\partial^4 y}{\partial t^4} + \frac{\rho}{E k^2}\cdot\frac{\partial^2 y}{\partial t^2} = 0$$

где G - модуль сдвига, E модуль растяжения, k^2 - квадрат радиуса инерции поперечного сечения

Исследование над колебаниями полоски привело нас к такому решению:

$u = \cos\alpha x\left[A\sin(h'y) + B\sin(k'y)\right]$ где $h'^2 = \frac{\rho p^2}{\lambda+2\mu} - \alpha^2$

$v = \sin\alpha x\left[\frac{A h'}{\alpha}\cos(h'y) - \frac{B\alpha}{k'}\cos(k'y)\right]$ $k'^2 = \frac{\rho p^2}{\mu} - \alpha^2$

Обозначая $\frac{\mu}{\rho} = V_1^2$; $\frac{\lambda+2\mu}{\rho} = V_2^2$ $\frac{p^2}{\alpha^2} = V^2$

и $V : V_1 = \beta$ $2h$ - ширина полоски λ длина волны

$\frac{2\pi h}{\lambda} = q$

напишем Frequenz gleichung так:

ESC M43

Figure 2. Copy of Timoshenko's letter to Ehrenfest, apparently dated January 10, 1913, including the equation taking into account shear deformation and rotary inertia (Courtesy of Prof. Dirk van Delft Director of Museum Boerhaave, Leiden, The Netherlands). Full letter is reproduced in ref. [64]

$$
\left.
\begin{aligned}
m &= \alpha\sqrt{1-\left(\frac{V}{V_1}\right)^2}, \\
n &= \alpha\sqrt{1-\left(\frac{V}{V_2}\right)^2}, \\
V &= \frac{p}{\alpha}, \text{ denotes the velocity of waves of transverse vibration,} \\
V_1 &= \sqrt{\frac{\lambda+2\mu}{\rho}}, \text{ denotes the velocity of waves of dilatation,} \\
\text{and}\quad V_2 &= \sqrt{\frac{\mu}{\rho}}, \text{ denotes the velocity of waves of distortion.}
\end{aligned}
\right\} \quad (4)
$$

It is easily verified that equations (3) are satisfied by the expressions

$$
\left.
\begin{aligned}
u &= \cos \alpha x\,(M \sinh my + N \sinh ny)\cos pt, \\
v &= \sin \alpha x\left(M\frac{m}{\alpha}\cosh my + N\frac{\alpha}{n}\cosh ny\right)\cos pt,
\end{aligned}
\right\}
$$

and the conditions that the boundaries ($y=\pm c$) are free from traction give us

$$
\left.
\begin{aligned}
&[(\lambda+2\mu)m^2-\lambda\alpha^2]\,M \sinh mc + 2\mu\alpha^2 N \sinh nc = 0, \\
\text{and}\quad &2mn\,M \cosh mc + (\alpha^2+n^2)\,N \cosh nc = 0.
\end{aligned}
\right\}
$$

Hence, by eliminating the ratio M/N, we obtain the "frequency equation" in the form *

$$4\mu\alpha^2 mn \tanh nc = (\alpha^2+n^2)[(\lambda+2\mu)m^2-\lambda\alpha^2]\tanh mc;$$

whence, denoting the length of the waves by l, and putting $V/V_1=f$, $V/V_2=h$, we have

$$4\sqrt{(1-f^2)(1-h^2)}\tanh\left(\frac{2\pi c}{l}\sqrt{1-h^2}\right)$$
$$=(2-h^2)^2\tanh\left(\frac{2\pi c}{l}\sqrt{1-f^2}\right). \quad (5)$$

If V be given, the corresponding value of the ratio $l/2c$

* This "frequency equation" was found by Prof. P. Ehrenfest and myself in collaboration. The solution given in this paper is my own.

Figure 3. "Copy of the page in ref. [14] Timoshenko, 1992. The footnote states "This "frequency equation" was found by Prof. P. Ehrenfest and myself in collaboration. The solution given in this paper is my own".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С. П. Тимошенко.

КУРСЪ

ТЕОРІИ УПРУГОСТИ.

Часть II.

СТЕРЖНИ и ПЛАСТИНКИ.

ПЕТРОГРАДЪ.

1916.

Figure 4. Title page of S.P. Timoshenko's book [13] *A course in theory of elasticity. Part II. Beams and plates*, Petrograd, 1916, in the Russian language.

from S.P. Timoshenko. When the present author applied to the Archive with the request to receive copies of the correspondence, the Archive was in the process of digitization, and hence the correspondence was not available. However, the present author obtained later on the of the additional material, specifically the letter and postcards received by S.P. Timoshenko while Ehrenfest was still in St. Petersburg. These correspondences were, by all probability, considered important by Ehrenfest since he brought these with him to Leiden. Only one letter [58] (figure 2) is associated with the beam theory that takes into account the rotary inertia and shear deformation. The top right corner contains the date "January 10;" the handwriting, apparently by Bruce Wheaton, below the date indicates the year as 1908. But this letter couldn't have been written in 1908 for then Timoshenko and Ehrenfest didn't work together. Moreover, Timoshenko was in Kiev whereas Ehrenfest was in St. Petersburg. The very next page, however, which is not reproduced here, reveals that the letter was written in 1913. Indeed, Timoshenko informs Ehrenfest: "After you left [St. Petersburg I.E] I did not have a chance of undertaking walks." Ehrenfest left St. Petersburg in 1912; thus the letter must have been written in 1913. Indeed, Timoshenko mentioned the year 1913 in his response to Grigolyuk.

Just before the equation, Timoshenko writes [58]: "Let us summarize the obtained results." Since Timoshenko does not say what the equation signifies, that shows that they both knew what they were discussing, clearly indicating that this equation was known to both of them, not only to Timoshenko – the composer of the letter.

Over half a century later, Timoshenko [21] composed his autobiography titled *Remembrances*, which appeared in Russian, in Paris. In 1968 the English version appeared [20] titled *As I Remember*, translated from the Russian edition by Robert Addis. In their foreword [20], James M. Gere and Donovan H. Young write, in March 1967: "It is only natural that the School of Engineering at Stanford University, where Timoshenko taught for so many years, has arranged for the English translation and publication of this book." To digress, let's note that the Timoshenko's book translation was a highly unusual gesture of good will and respect on the behalf of Stanford University. In Chapter 22 titled "We Move to Petersburg," Timoshenko [20, p.119] writes: "In the winter 1911–12 I wrote a paper on the effect of a transverse impact on a beam and investigated the influence of shear deformation on the transverse vibrations of the beams. Both these articles were translated, one into German, the other into English, and subsequently served as a point of departure for the research of many authors". Later on, in the same chapter, Timoshenko [20, pp.122–123] tells us the story of his meeting with Paul Ehrenfest, and

Выделим из стержня двумя бесконечно близкими сечениями элемент длиной δx и составим для него уравнения движения. Положение выделенного элемента определится перемещением y его центра тяжести и углом поворота φ относительно нейтральной линии [1].

Так как прогиб стержня обусловлен не только относительным поворотом поперечных сечений, но также и сдвигом одного из них относительно другого, то угол наклона касательной к изогнутой оси стержня не будет равняться углу поворота сечения. Разность между этими углами будет, очевидно, равняться относительному сдвигу, который обозначим через β.

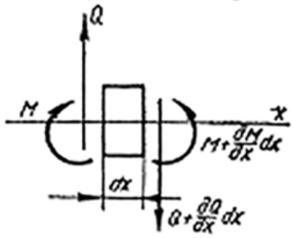

Рис. 76.

Таким образом, получаем зависимость

$$\frac{\partial y}{\partial x} = \varphi + \beta. \qquad \text{(f)}$$

Кроме того, для изгибающего момента и для перерезывающей силы будем иметь выражения

$$M = -EJ\frac{\partial \varphi}{\partial x}; \quad Q = k'\beta FG = k'\left(\frac{\partial y}{\partial x} - \varphi\right)FG. \qquad \text{(h)}$$

где k' — коэффициент, зависящий от формы поперечного сечения.

Принимая во внимание приложенные к выделенному элементу силы (рис. 76), напишем уравнение движения, соответствующее вращению элемента, в таком виде:

$$-\frac{\partial M}{\partial x}\delta x + Q\delta x = \frac{\gamma J}{g}\frac{\partial^2 \varphi}{\partial t^2}\delta x,$$

или, вставляя вместо M и Q их значения (h),

$$EJ\frac{\partial^2 \varphi}{\partial x^2} + k'\left(\frac{\partial y}{\partial x} - \varphi\right)FG - \frac{\gamma J}{g}\frac{\partial^2 \varphi}{\partial t^2} = 0. \qquad \text{(k)}$$

Уравнение движения, соответствующее перемещению элемента в направлении оси y, запишем так:

$$\frac{\partial Q}{\partial x}\delta x = \frac{\gamma F}{g}\frac{\partial^2 y}{\partial t^2}\delta x$$

или

$$\frac{\gamma F}{g}\frac{\partial^2 y}{\partial t^2} - k'\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \varphi}{\partial x}\right)FG = 0. \qquad \text{(l)}$$

Исключая из (k) и (l) величину φ, приходим к такому уравнению для поперечных колебаний стержня:

$$EJ\frac{\partial^4 y}{\partial x^4} + \frac{\gamma F}{g}\frac{\partial^2 y}{\partial t^2} - \left(\frac{\gamma J}{g} + \frac{EJ\gamma}{gk'G}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\gamma J}{g}\frac{\gamma}{gk'G}\frac{\partial^4 y}{\partial t^4} = 0. \qquad (178)$$

Сравнивая (177) и (178), устанавливаем те члены, которыми определяется поправка на касательные напряжения. Ниже на примере покажем, как сказывается эта поправка на частоте различных типов собственных колебаний.

[1] При составлении уравнений рассматриваем элемент как абсолютно твердое тело и пренебрегаем искривлением поперечных сечений, поэтому поправка лишь приближенно оценивает влияние касательных сил. Нами совместно с проф. П. Эренфестом получено также точное решение задачи для полосы прямоугольного поперечного сечения. [Речь идет о поперечных колебаниях стержня прямоугольного поперечного сечения на основе уравнений теории упругости. Этот результат, полученный С. П. Тимошенко и П. Эренфестом, совместно не публиковался. В письме к отв. редактору настоящего издания от 2 июля 1970 г. С. П. Тимошенко замечает, что «эта работа была сделана в 1913 г., когда Эренфест жил в Петербурге. Позже я с ним не работал». В статье С. П. Тимошенко «On the transverse vibration of bars of uniform cross-section». Philosophical Magazine and Journal of Science, Series, 6. 1922, Bd 43, SS. 125—131 на стр. 127 указывается: «Это частотное уравнение было найдено мною совместно с проф. П. Эренфестом. Само же решение, изложенное в статье, принадлежит мне»]. Это решение, как и упомянутое на стр. 153 решение Л. Похгаммера для круглого цилиндра, показывает, что приводимое здесь приближенное решение с достаточной точностью учитывает влияние касательных сил.

Figure 5. Copy of page 338 from S.P. Timoshenko's [28] book *A course in theory of elasticity*, second edition, 1972. The footnote mentioning Ehrenfest, was included along with the comment of the book's editor Prof. E.I. Grigolyuk, S.P. Timoshenko responded that the work was conducted when P. Ehrenfest lived on St, Petersburg, "later, i didn't work with him," wrote S.P. Timoshenko".

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Taking the x-axis in the direction of the central line, and choosing for Δ an even and for $\bar{\omega}$ an uneven function of y, we may write

$$(3)\qquad \begin{cases} \Delta = A \sin \alpha x \sinh my \cos pt, \\ 2\bar{\omega} = B \cos \alpha x \cosh ny \cos pt, \end{cases}$$

where A and B are undetermined coefficients,

$$(4)\qquad \begin{cases} m = \alpha\sqrt{1-(V/V_1)^2}, \\ n = \alpha\sqrt{1-(V/V_2)^2}, \\ V = p/\alpha, \text{ denotes the velocity of waves of transverse vibration,} \\ V_1 = \sqrt{(\lambda+2\mu)/\varrho}, \text{ denotes the velocity of waves of dilatation, and} \\ V_2 = \sqrt{\mu/\varrho}, \text{ denotes the velocity of waves of distortion.} \end{cases}$$

It is easily verified that equations (3) are satisfied by the expressions

$$u = \cos \alpha x \,(M \sinh my + N \sinh ny) \cos pt,$$
$$v = \sin \alpha x \left(M \frac{m}{\alpha} \cosh my + N \frac{\alpha}{n} \cosh ny\right) \cos pt,$$

and the conditions that the boundaries ($y = \pm c$) are free from traction give us

$$[(\lambda + 2\mu)m^2 - \lambda\alpha^2]M \sinh mc + 2\mu\alpha^2 N \sinh nc = 0,$$

and
$$2mnM \cosh mc + (\alpha^2 + n^2)N \cosh nc = 0.$$

Hence, by eliminating the ratio M/N, we obtain the "frequency equation" in the form

$$4\mu\alpha^2 mn \tanh nc = (\alpha^2 + n^2)[(\lambda + 2\mu)m^2 - \lambda\alpha^2] \tanh mc;$$

whence, denoting the length of the waves by l, and putting $V/V_1 = f$, $V/V_2 = h$, we have

$$(5)\qquad 4\sqrt{(1-f^2)(1-h^2)} \tanh\left(\frac{2\pi c}{l}\sqrt{1-h^2}\right) = (2-h^2)^2 \tanh\left(\frac{2\pi c}{l}\sqrt{1-f^2}\right).$$

If V be given, the corresponding value of the ratio $l/2c$ can be calculated from this equation. Some results are given in the table below [1]).

h	0,5	0,7	0,9	0,9165	0,9192
$l/2c$	4,8	2,5	0,75	0,47	0,30

[1]) In these calculations, σ has been taken to be 0,25.

Figure 6. Copy of the page from S.P. Timoshenko's article *On the Transverse Vibrations of Bars of Uniform Cross-Section* [14] included in *The Collected Papers of Stephen P. Timoshenko* [30]. The footnote on Paul Ehrenfest appearing in his original paper [14] (figure 3) is missing. It is unknown who deleted the footnote appearing on the original paper. The footnote and manner of writing differ from original, meaning that the paper was re-typed, with the footnote about P. Ehrenfest was deleted.

Если выдѣлить двумя безконечно близкими поперечными сѣченіями элементъ стержня длиной δx, то силы инерціи, соотвѣтствующія вращенію этого элемента, дадутъ намъ моментъ равный

$$-\frac{J\gamma}{g}\frac{\partial^3 y}{\partial x\,\partial t^2}\delta x \quad \text{(e)}$$

При составленіи дифференціальнаго ур-ія для изогнутой оси стержня нужно принять во вниманіе не только непрерывно распредѣленную нагрузку, соотвѣтствующую силамъ инерціи поступательнаго движенія, но и непрерывно распредѣленные моменты, опредѣляемые ф-ой (e). Въ такомъ случаѣ измѣненіе изгибающаго момента вдоль оси стержня представится такъ

$$\frac{\partial M}{\partial x} = Q - \frac{J\gamma}{g}\frac{\partial^3 y}{\partial x\,\partial t^2}$$

Здѣсь Q поперечная сила, получающаяся отъ силъ инерціи поступательнаго движенія элементовъ стержня.

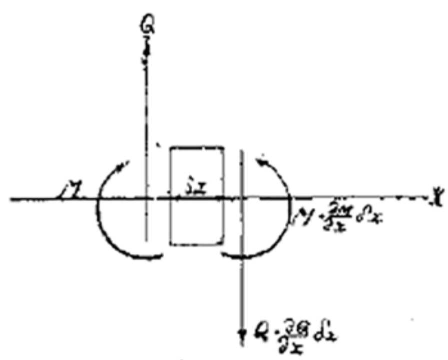
Вставляя найденное значеніе производной $\frac{\partial M}{\partial x}$ въ дифференціальное ур-іе

$$EJ\frac{\partial^4 y}{\partial x^4} = -\frac{\partial^2 M}{\partial x^2},$$

получаемъ

$$EJ\frac{\partial^4 y}{\partial x^4} = -\frac{\gamma F}{g}\frac{\partial^2 y}{\partial t^2} + \frac{J\gamma}{g}\frac{\partial^4 y}{\partial x^2\,\partial t^2} \quad (177)$$

Второй членъ въ правой части ур-ія представляетъ собой поправку, обусловленную поворотомъ поперечныхъ сѣченій при колебаніяхъ.



Черт. 76.

Болѣе точное ур-іе мы получимъ, если примемъ во вниманіе не только вліяніе силъ инерціи, соотвѣтствующихъ повороту сѣченій, но также и вліяніе касательныхъ напряженій на прогибъ стержня [1]). Выдѣлимъ изъ стержня двумя безконечно близкими сѣченіями элементъ длиной δx и составимъ для него ур-ія движенія. Положеніе выдѣленнаго элемента опредѣлится перемѣщеніемъ y его центра тяжести и угломъ поворота φ относительно нейтральной линіи [2]). Такъ какъ прогибъ стержня обусловленъ не только относительнымъ поворотомъ поперечныхъ сѣченій но также и сдвигомъ ихъ одного относительно другого, то уголъ наклона каса-

[1]) Вліяніе касательныхъ напряженій на періодъ поперечныхъ колебаній стержня обыкновенно не принимается въ расчетъ, а между тѣмъ соотвѣтствующая ему поправка по своей величинѣ больше той, которая получается отъ поворачиванія сѣченій.

[2]) При составленіи ур-ій мы разсматриваемъ элементъ, какъ абсолютно твердое тѣло и пренебрегаемъ искривленіемъ поперечныхъ сѣченій, поэтому наша поправка лишь приближенно оцѣниваетъ вліяніе касательныхъ силъ. Нами совмѣстно съ проф. П. Эренфестомъ получено также точное рѣшеніе задачи для полосы прямоугольнаго поперечнаго сѣченія. Это рѣшеніе, какъ и цитированное нами выше (стр. 184) рѣшеніе Pochhammer'а для круглаго цилиндра, показываютъ, что приводимое здѣсь приближенное рѣшеніе съ достаточной точностью учитываетъ вліяніе касательныхъ силъ.

Figure 7. The copy of page 206 of Timoshenko's elasticity book [13]. After exposing the new beam theory, S.P. Timoshenko mentions, in the footnote 2: "We jointly with Prof. P. Ehrenfest, obtained also an exact solution for the layer of rectangular cross-section."

6 JOHN M. LESSELLS

2. Über die erzwungenen Schwingungen von prismatischen Stäben. *Zeitschrift für Mathematik und Physik.* Volume 59, page 163 (1911).
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4. Sur la stabilité des systèmes élastiques. *Annales des Ponts et Chaussées.* 9th Series, Volume 13, page 496; Volume 14, page 73; Volume 17, page 372 (1913). Reprint of all three parts: Paris, A. Dumas (1913).
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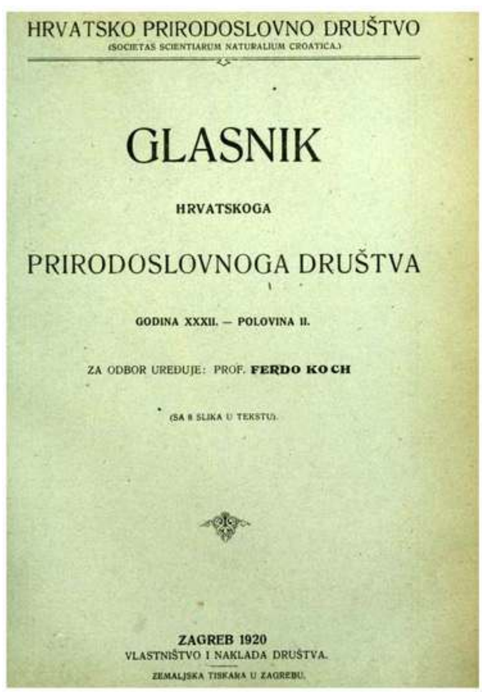
6. Über die Stabilität versteifter Platten. *Der Eisenbau,* Volume 12, page 147 (1921).
7. Étude de l'action des charges roulantes sur les rails. *Le Génie Civil,* Volume 79, page 555 (1921).
8. On the Correction for Shear in the Differential Equation for Transverse Vibration of Prismatic Bars. *Philosophical Magazine,* Volume 41, page 744 (1921).
9. A Membrane Analogy to Flexure. *Proceedings London Mathematical Society,* Series 2, Volume 20, page 398 (1921).
10. On the Torsion of a Prism, One of the Cross-Sections of Which Remains Plane. *Proceedings London Mathematical Society,* Series 2, Volume 20, page 389 (1921).
11. On the Transverse Vibration of Bars of Uniform Cross-Section. *Philosophical Magazine,* Volume 43, page 125 (1922).
12. Beams with Loads Irregularly Distributed. *Engineering,* Volume 113, page 196 (1922).
13. On the Forced Vibration of Bridges. *Philosophical Magazine,* Volume 43, page 1018 (1922).
14. Über die Biegung der allseitig unterstützten rechteckigen Platte unter Wirkung einer Einzellast. *Der Bauingenieur,* Volume 3, page 51 (1922).
15. On the Distribution of Stresses in a Circular Ring Compressed by Two Forces Acting along a Diameter. *Philosophical Magazine,* Volume 44, page 1014 (1922).
16. Calcul des arcs élastiques. *Le Ciment, Paris* (1922).
17. Berechnung der Schubspannungen im gebogenen Balken. *Zeitschrift für angewandte Mathematik und Mechanik,* Volume 2, page 160 (1922).
18. Elasticity of Pipe Bends. *Transactions of the American Society of Mechanical Engineers,* Volume 44, page 585 (1922).
19. Torsion of Crankshafts. *Transactions of the American Society of Mechanical Engineers,* Volume 45, page 96 (1923).
20. Kippsicherheit des gekrümmten Stabes mit kreisförmiger Mittellinie. *Zeitschrift für angewandte Mathematik und Mechanik,* Volume 3, page 358 (1923).
21. Bending Stresses in Curved Tubes of Rectangular Cross-Section. *Transactions of the American Society of Mechanical Engineers,* Volume 45, page 135 (1923).

Figure 8. J.M. Lessels's account of Timoshenko's papers in his article titled S. Timoshenko, in the book Contributions in the Mechanics of Solids: Dedicated to Stephen Timoshenko by His Friends on the Occasion of the Sixtieth Birthday Anniversary, pp. 1-8, New York: The Macmillan Company, 1938, does not list Timoshenko's 1920 paper [17].

[1] При составлении уравнений рассматриваем элемент как абсолютно твердое тело и пренебрегаем искривлением поперечных сечений, поэтому поправка лишь приближенно оценивает влияние касательных сил. Нами совместно с проф. П. Эренфестом получено также точное решение задачи для полосы прямоугольного поперечного сечения. [Речь идет о поперечных колебаниях стержня прямоугольного поперечного сечения на основе уравнений теории упругости. Этот результат, полученный С. П. Тимошенко и П. Эренфестом, совместно не публиковался. В письме к отв. редактору настоящего издания от 2 июля 1970 г. С. П. Тимошенко замечает, что «эта работа была сделана в 1913 г., когда Эренфест жил в Петербурге. Позже я с ним не работал». В статье С. П. Тимошенко «On the transverse vibration of bars of uniform cross-section». Philosophical Magazine and Journal of Science, Series, 6, 1922, Bd 43, SS. 125—131 на стр. 127 указывается: «Это частотное уравнение было найдено мною совместно с проф. П. Эренфестом. Само же решение, изложенное в статье, принадлежит мне»]. Это решение, как и упомянутое на стр. 153 решение Л. Похгаммера для круглого цилиндра, показывает, что приводимое здесь приближенное решение с достаточной точностью учитывает влияние касательных сил.

Figure 9. An extract from page 338 of the second edition of Timoshenko's [58] book *A course in theory of elasticity* published in 1972 with E.I. Grigolyuk serving as editor. In square parentheses Grigolyuk comments: "The discussion concerns the transverse vibrations of the beam with rectangular cross section based on theory of elasticity. This result, obtained by S.P. Timoshenko with P. Ehrenfest, was not published [by them] jointly. In the letter to the editor of the current edition, S.P. Timoshenko notes that this work was performed in 1913, when Ehrenfest lived in Petersburg. Later I did not work with him."


HRVATSKO PRIRODOSLOVNO DRUŠTVO
(SOCIETAS SCIENTIARUM NATURALIUM CROATICA.)

GLASNIK
HRVATSKOGA
PRIRODOSLOVNOGA DRUŠTVA

GODINA XXXII. — POLOVINA II.

ZA ODBOR UREDUJE: PROF. FERDO KOCH

(SA 8 SLIKA U TEKSTU.)

ZAGREB 1920
VLASTNIŠTVO I NAKLADA DRUŠTVA.
ZEMALJSKA TISKARA U ZAGREBU.

Figure 10. The title page of the journal *Glasnik Hrvatskoga Prirodoslovnoga Drustva where* S.P. Timoshenko and Ehrenfest's work on effect of rotary inertia and shear deformation appeared first in English.

reproduces the picture taken in winter 1912 (figure 1). In light of the fact that in his book on the theory of elasticity (Timoshenko, 1916, p. 206) (tittle page in figure 4) mentioned that the theory was developed with Ehrenfest, Timoshenko could have combined writing about his scientific work on page 119 of autobiography with mentioning the cooperation with Ehrenfest. However, he delayed mentioning Ehrenfest till later pages, namely 122 and 123, without mentioning their cooperation and the beautiful theory that was a result of this joint work!

5. General comments on priority and conclusion

One correspondent wrote to this writer: "…The questions of priority were discussed by many authors who noted that it is difficult to expect a fair judgement in this field." For example, discussing history of mathematics, Borodich [81] wrote: "There is even a joke that the fundamental law of the history of mathematics says: "Any proposition named after someone traces its origin back to another one" (translation from German:"Ein Satz, der einen Namen tragt, stammt von einem Anderen") [82] (note the similarity with W.T. Koiter's statement above). It is interesting to note that Sir Michael Berry referred to a similar statement as the "theorem of Arnold." Arnold's [83] statement reads: "Integration had already been encountered with Archimedes, and differentiation with Pascal and Fermat; the connection between these operations was known to Barrow. What did Newton do in analysis? What was his main mathematical discovery? Newton invented Taylor series, the main instrument of analysis" might be brought as an illustration of the "law."

Another correspondent wrote "The Gibbs phenomenon was discovered and described in detail 50 years before Gibbs by Wilderham…. One of the physicists, with annoyance said, if you did something worthwhile, predecessors immediately show up! This phenomenon was studied in detail by van Dyke [84] as applied to boundary layer theory. He noted that some of the ideas of this theory were used by many scientists long before Prandtl. However, these works can be compared with grains that have not sprouted, and Prandtl's work is rooted from which the new theories gave grown up."

To reiterate in this particular case Timoshenko himself writes in 1916 that the theory was developed by him together with Ehrenfest. Moreover, in his 1922 paper [14] Timoshenko writes that the theory was developed by Ehrenfest (noting his co-author as the first one!) and him, and that solution, i.e. enumeration of results, was conducted by Timoshenko.

Additionally, based upon the response by Timoshenko, Grigolyuk [15,16] – who was a fellow Ukrainian and a fan of Timoshenko's – unequivocally concluded in the biographical book on Timoshenko that the theory belonged to Timoshenko and Ehrenfest (see the quote of Grigolyuk [15,16] in the end of section 1).

The foreword to Arnold's [85] book mentions so-called "Arnold's law (implied by statements in his many letters disputing priority…): Discoveries are rarely attributed to the correct person." In words of Sir Michael Berry [86] "Arnold's law applies to itself. I have heard it referred to by other names." Indeed, it appears perhaps earlier as the Stigler's [87] law of eponymy, stating "No scientific discovery is named after its original discoverer," which probably is an exaggerated maxim.

In this particular case, the judgment on priority is much simpler than in numerous other cases in the history of ideas: Timoshenko was one of the co-authors of the now celebrated beam theory; the other was Ehrenfest. This work confirms the conclusion made by Grigolyuk [15,16] on this matter: This study shows that theory of beams that incorporates shear deformation and rotary inertia, ought to be referred to as the Timoshenko–Ehrenfest beam theory.

The extensive evidence provided in this article, and especially Timoshenko's own revealing testimonies in [10,11,55] and his 1913 letter to Ehrenfest, show that adopting the name Timoshenko-Ehrenfest beam theory will serve both the justice and truth.

In this respect, it appears apropos to quote Psalm 85:12: "Truth will sprout from the Earth." In the words of A.J. Heschel (1907–1972) – theologian and philosopher –"truth lies buried, stifled in the grave, yet it remains alive. Truth wants to emerge, but man does not permit its appearance. Man's structure stands like a mausoleum on the grave of truth, preventing it from raising its head." Indeed, the search conducted by the present writer might strongly remind the archeological digging and the construction of the entire picture by sealing pieces of the puzzle together. Once I learned that Timoshenko and Ehrenfest developed the beam theory together, this idea consumed me, in light of human responsibility for searching for truth and bringing it to the community of readers who are advised to make their own conclusions.

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