The impact of a lower order term in a Dirichlet problem with a singular nonlinearity

Lucio Boccardo and Gisella Croce*

Abstract. In this paper we study the existence and regularity of solutions to the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + u|u|^{r-1} = \frac{f(x)}{u^\theta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

proving that the lower order term $u|u|^{r-1}$ has some regularizing effects on the solutions.

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1. Introduction

In this paper we study the existence and regularity of weak solutions to the following nonlinear problem

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + u|u|^{r-1} = \frac{f(x)}{u^\theta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1)$$

Here Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, f is a positive (that is $f(x) \geq 0$ and not zero a.e.) function in $L^1(\Omega)$ and $0 < \theta < 1$. Moreover $a : \Omega \rightarrow \mathbb{R}$ is Lipschitz

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continuous and we assume that there exist $0 < \alpha < \beta$ such that $\alpha \leq a(x) \leq \beta$ a.e. in Ω .

Due to the singularity in the right hand side, a solution of this problem is a function u in $W_0^{1,1}(\Omega)$ such that for all $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that $u \geq c_\omega > 0$ in ω , and satisfying

$$\int_{\Omega} a(x) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + \int_{\Omega} |u|^{r-1} u \varphi = \int_{\Omega} \frac{f \varphi}{u^\gamma} \quad \forall \varphi \in C_0^1(\Omega). \quad (2)$$

This paper is motivated by the results of [6], where the authors studied the existence, regularity and uniqueness of weak solutions to the singular semilinear problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = \frac{f(x)}{u^\theta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

M was supposed to be an elliptic bounded matrix, $0 \leq f \in L^m(\Omega)$, $m \geq 1$, or f a Radon measure and $\theta > 0$.

In [11] a nonlinear version of the above problem was studied, considering an operator as the p -laplacian instead of $-\operatorname{div}(M(x)\nabla u)$ (see also Remark 4.2). In [14] the author added a lower order term growing as $|u|^{r-1}u$, and studied existence, regularity and uniqueness of solutions to problem (1). Although the right hand side is singular at u , the lower order term in the left hand side has a regularizing effect. In the right hand side, f is assumed to belong to $L^m(\Omega)$ with $m > 1$ in the case $\theta < 1$. In this paper we will analyse the limit case $f \in L^1(\Omega)$ showing that the lower order term $|u|^{r-1}u$ has still a regularizing effect on the solutions.

We observe that the regularizing effect of the lower order term $|u|^{r-1}u$ is well known for similar problems. It was pointed out in [7, 5, 8] for elliptic problems. Later it was showed for elliptic problems with degenerate coercivity in [9, 2, 3].

In this paper we will show the following result.

Theorem 1.1. *Let $f \in L^1(\Omega)$ be a positive function ($f(x) \geq 0$ and not zero a.e.). Assume that $r > \frac{(1-\theta)(p-1)N}{N-p}$, $p \geq 1 + \frac{1-\theta}{r}$, $0 < \theta < 1$. Then there exists a function $u \in W_0^{1,q}(\Omega)$, with $q = \frac{p}{1+\frac{1-\theta}{r}}$ which is a solution to problem (1) in the sense of (2). Moreover $u^{r+\theta}$ belongs to $L^1(\Omega)$.*

In order to prove this result, we will work by approximation, “truncating” the singular term $\frac{1}{u^\theta}$ so that it becomes not singular at the origin. We will get some a priori estimates on the solutions u_n of the approximating problems, which will allow us to pass to the limit and find a solution to problem (1). In order to obtain a solution in the sense described above, we will apply a suitable form of the Strong Maximum Principle.

We observe that the regularity of the above theorem implies that q can be equal to 1, that is, u can belong to the non-reflexive space $W_0^{1,1}(\Omega)$. This will demand the proof of the equi-integrability of $|\nabla u_n|^q$.

We will then analyse the regularity obtained and point out the regularizing effects of the term $|u|^{r-1}u$.

2. Approximating problems

In this section we prove the existence of a solution to the following approximating problems:

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u_n|^{p-1}\nabla u_n) + u_n|u_n|^{r-1} = \frac{f_n}{(u_n + \frac{1}{n})^\theta} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (3)$$

Due to the nature of the approximation, the sequence u_n will be increasing with n , so that the (strict) positivity of the limit will be derived from the (strict) positivity of any of the u_n (which in turn will follow by the standard maximum principle for elliptic equations).

Let f be a nonnegative measurable function (not identically zero), let $n \in \mathbb{N}$, let $f_n(x) = \min\{f(x), n\}$.

Lemma 2.1. *Problem (3) has a nonnegative solution u_n in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let n in $\mathbb{N}$ be fixed, let v be a function in $L^2(\Omega)$, and define $w = S(v)$ to be the unique solution of

$$\begin{cases} -\operatorname{div}(a(x)|\nabla w|^{p-2}\nabla w) + |w|^{r-1}w = \frac{f_n}{(|v| + \frac{1}{n})^\gamma} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (4)$$

The existence of a solution $w \in W_0^{1,p}(\Omega)$ follows from the classical results of [13].

Taking w as test function, we have, using the ellipticity of a ,

$$\alpha \int_{\Omega} |\nabla w|^p \leq \int_{\Omega} a(x)|\nabla w|^{p-2}\nabla w \cdot \nabla w = \int_{\Omega} \frac{f_n w}{(|v| + \frac{1}{n})^\gamma} \leq n^{\gamma+1} \int_{\Omega} |w|.$$

By the Sobolev inequality on the left hand side and the Hölder inequality on the right hand side one has

$$\left[\int_{\Omega} |w|^{p^*} \right]^{p/p^*} \leq C n^{\gamma+1} \left(\int_{\Omega} |w|^{p^*} \right)^{\frac{1}{p^*}},$$

for some constant C independent on v . This implies

$$\|w\|_{L^{p^*}(\Omega)} \leq C n^{\gamma+1},$$

so that the ball of $L^{p^*}(\Omega)$ of radius $C n^{\gamma+1}$ is invariant for S . It is easy to prove, using the Sobolev embedding, that S is both continuous and compact on $L^{p^*}(\Omega)$, so that by Schauder's fixed point theorem there exists u_n in $W_0^{1,p}(\Omega)$ such that $u_n = S(u_n)$, i.e., u_n solves

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u_n|^{p-2}\nabla u_n) + |u_n|^{r-1}u_n = \frac{f_n}{(|u_n| + \frac{1}{n})^\theta} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By using as a test function $-u_n^-$, one has $u_n \geq 0$. Since the right hand side of (3) belongs to $L^\infty(\Omega)$, classical regularity results (see [12] or Théorème 4.2 of [16] in the linear case) imply that u_n belongs to $L^\infty(\Omega)$ (although its norm in $L^\infty(\Omega)$ may depend on n). $\square$

Lemma 2.2. *The sequence u_n is increasing with respect to n , $u_n > 0$ in Ω , and for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ (independent on n) such that*

$$u_n(x) \geq c_\omega > 0 \quad \text{for every } x \text{ in } \omega, \text{ for every } n \text{ in } \mathbb{N}. \quad (5)$$

Moreover there exists the pointwise limit $u \geq c_\omega$ of the sequence u_n .

Proof. Since $0 \leq f_n \leq f_{n+1}$ and $\gamma > 0$, one has (distributionally)

$$-\operatorname{div}(a(x)|\nabla u_n|^{p-2}\nabla u_n) + |u_n|^{r-1}u_n = \frac{f_n}{(u_n + \frac{1}{n})^\gamma} \leq \frac{f_{n+1}}{(u_n + \frac{1}{n+1})^\gamma},$$

so that

$$\begin{aligned} & -\operatorname{div}(a(x)(|\nabla(u_n)|^{p-2}\nabla(u_n) - |\nabla(u_{n+1})|^{p-2}\nabla(u_{n+1}))) + \\ & + |u_n|^{r-1}u_n - |u_{n+1}|^{r-1}u_{n+1} \leq f_{n+1} \frac{(u_{n+1} + \frac{1}{n+1})^\gamma - (u_n + \frac{1}{n+1})^\gamma}{(u_n + \frac{1}{n+1})^\gamma (u_{n+1} + \frac{1}{n+1})^\gamma}. \end{aligned}$$

We now choose $(u_n - u_{n+1})^+$ as test function. In the left hand side we use the monotonicity of the p -laplacian operator as well as the monotonicity of the function $t \rightarrow |t|^{r-1}t$. For the right hand side we observe that

$$\left[\left(u_{n+1} + \frac{1}{n+1} \right)^\gamma - \left(u_n + \frac{1}{n+1} \right)^\gamma \right] (u_n - u_{n+1})^+ \leq 0;$$

recalling that $f_{n+1} \geq 0$, we thus have

$$0 \leq \alpha \int_{\Omega} |\nabla(u_n - u_{n+1})^+|^p \leq 0.$$

Therefore $(u_n - u_{n+1})^+ = 0$ almost everywhere in Ω , which implies $u_n \leq u_{n+1}$.

Since u_1 belongs to $L^\infty(\Omega)$ (see Lemma 2.1), and there exists a constant (only depending on Ω and N) such that

$$\|u_1\|_{L^\infty(\Omega)} \leq C \|f_1\|_{L^\infty(\Omega)} \leq C,$$

one has

$$\begin{aligned} & -\operatorname{div}(a(x)|\nabla u_1|^{p-2}\nabla u_1) + |u_1|^{r-1}u_1 \\ &= \frac{f_1}{(u_1+1)^\gamma} \geq \frac{f_1}{(\|u_1\|_{L^\infty(\Omega)}+1)^\gamma} \geq \frac{f_1}{(C+1)^\gamma}. \end{aligned}$$

Since $\frac{f_1}{(C+1)^\gamma}$ is not identically zero, the strong maximum principle implies that $u_1 > 0$ in Ω (see [17]; observe that u_1 is differentiable by chapter 4 of [12]), and that (5) holds for u_1 (with c_ω only depending on ω , N , f_1 and γ). Since $u_n \geq u_1$ for every n in $\mathbb{N}$, (5) holds for u_n (with the same constant c_ω which is then independent on n). $\square$

3. A priori estimates

Let $k > 0$. We denote by $T_k(x)$ the function $\max\{-k, \min\{s, k\}\}$.

Lemma 3.1. *Let $k > 0$ be fixed. The sequence $\{T_k(u_n)\}$, where u_n is a solution to (3), is bounded in $W_0^{1,p}(\Omega)$.*

Proof. It is sufficient to take $T_k(u_n)$ as a test function in problems (3). $\square$

Theorem 3.2. *Assume $r > \frac{(1-\theta)(p-1)N}{N-p}$, $p \geq 1 + \frac{1-\theta}{r}$. The sequence of solutions $\{u_n\}$ to (3) is bounded in $W_0^{1,q}(\Omega)$, with $q = \frac{p}{1+\frac{1-\theta}{r}}$. Moreover the sequence $\{u_n^{r+\theta}\}$ is bounded in $L^1(\Omega)$.*

Proof. We use $(u_n + h)^\theta - h^\theta$ as a test function in (3): we get

$$\alpha \int_{\Omega} \frac{|\nabla u_n|^p}{(u_n + h)^{1-\theta}} + \int_{\Omega} u_n^r [(u_n + h)^\theta - h^\theta] \leq \int_{\Omega} f.$$

At the limit as $h \rightarrow 0$ we get

$$\alpha \int_{\Omega} \frac{|\nabla u_n|^p}{u_n^{1-\theta}} + \int_{\Omega} u_n^{r+\theta} \leq \int_{\Omega} f.$$

Then

$$\begin{aligned} \int_{\Omega} |\nabla u_n|^q &= \int_{\Omega} \frac{|\nabla u_n|^q}{u_n^{\frac{q}{p}(1-\theta)}} u_n^{(1-\theta)\frac{q}{p}} \leq \left[\int_{\Omega} \frac{|\nabla u_n|^p}{u_n^{(1-\theta)}} \right]^{\frac{p}{q}} \left[\int_{\Omega} u_n^{(1-\theta)\frac{q}{p} \frac{p}{p-q}} \right]^{\frac{p-q}{p}} \\ &\leq \left[\frac{1}{\alpha} \int_{\Omega} f \right]^{\frac{p}{q}} \left[\int_{\Omega} u_n^{(1-\theta)\frac{q}{p-q}} \right]^{\frac{p-q}{p}}. \end{aligned} \tag{6}$$

Observe that q is such that $(1 - \theta) \frac{q}{p-q} = r$. Thus

$$\int_{\Omega} |\nabla u_n|^q \leq \left[\frac{1}{\alpha} \int_{\Omega} f \right]^{\frac{p}{q}} \left[\int_{\Omega} f \right]^{\frac{p-q}{p}} = \left[\frac{1}{\alpha} \right]^{\frac{p}{q}} \int_{\Omega} f.$$

□

Lemma 3.3. *Let u_n be a solution to problem (3). Then*

$$\int_{\{k < u_n\}} u_n^r \leq \frac{1}{k^\theta} \int_{\{k < u_n\}} f$$

and $\lim_{|E| \rightarrow 0} \int_E u_n^r = 0$ uniformly with respect to n .

Proof. Let $k > 0$ and ψ_i be a sequence of increasing, positive, $C^\infty(\Omega)$ functions, such that

$$\psi_i(s) \rightarrow \begin{cases} 1, & s \geq k \\ 0, & 0 \leq s < k. \end{cases}$$

Choosing $\psi_i(u_n)$ in (3), we get

$$\int_{\Omega} u_n^{r-1} u_n \psi_i(u_n) \leq \int_{\Omega} \frac{f_n}{(u_n + \frac{1}{n})^\theta} \psi_i(u_n).$$

The limit on i gives

$$\int_{\{k < u_n\}} u_n^r \leq \int_{\{k < u_n\}} \frac{f}{(u_n + \frac{1}{n})^\theta}.$$

Therefore we have

$$\int_{\{k < u_n\}} u_n^r \leq \frac{1}{(k + \frac{1}{n})^\theta} \int_{\{k < u_n\}} f.$$

This implies that

$$\int_E u_n^r \leq k^r |E| + \int_{E \cap \{u_n > k\}} u_n^r \leq k^r |E| + \frac{1}{k^\theta} \int_{\{u_n > k\}} f.$$

By the above theorem, the sequence $\{u_n\}$ is bounded in $L^{r+\theta}(\Omega)$ and therefore in $L^1(\Omega)$. This implies that there exists a constant $C > 0$ such that

$$k\mu(\{u_n \geq k\}) \leq \int_{\{u_n \geq k\}} u_n \leq C.$$

Since $f \in L^1(\Omega)$ for any given $\varepsilon > 0$, there exists k_ε such that $\int_{\{|u_n| > k_\varepsilon\}} |f| \leq \varepsilon$.

Therefore

$$\int_E u_n^r \leq k_\varepsilon^r |E| + \frac{\varepsilon}{k_\varepsilon^\theta}$$

and the statement of this lemma is thus proved. $\square$

Proposition 3.4. *Let u_n be a solution to problem (3). Let $q = \frac{p}{1+\frac{1-\theta}{r}}$. Then*

$$\lim_{|E| \rightarrow 0} \int_E |\nabla u_n|^q = 0 \text{ uniformly with respect to } n.$$

Proof. From (6) and Lemma 3.3 we infer

$$\begin{aligned} \int_{\{k < u_n\}} |\nabla u_n|^q &= \int_{\{k < u_n\}} \frac{|\nabla u_n|^q}{u_n^{(1-\theta)\frac{q}{p}}} u_n^{(1-\theta)\frac{q}{p}} \leq \left[\int_\Omega \frac{|\nabla u_n|^p}{u_n^{(1-\theta)}} \right]^{\frac{p}{q}} \left[\int_{\{k < u_n\}} u_n^{(1-\theta)\frac{q}{p} \frac{p}{p-q}} \right]^{\frac{p-q}{p}} \\ &\leq \left[\frac{1}{\alpha} \int_\Omega f(x) \right]^{\frac{p}{q}} \left[\int_{\{k < u_n\}} u_n^{(1-\theta)\frac{q}{p} \frac{p}{p-q}} \right]^{\frac{p-q}{p}} \leq \left[\frac{1}{\alpha} \int_\Omega f(x) \right]^{\frac{p}{q}} \left[\int_{\{k < u_n\}} f(x) \right]^{\frac{p-q}{p}} \end{aligned}$$

which gives the equintegrability of $|\nabla u_n|^q$, with the same technique as in the previous result. $\square$

4. Proof of the main theorem

To prove Theorem 1.1 we are going to pass to the limit in problems (3). By Lemma 3.1 $T_k(u_n) \rightarrow T_k(u)$ weakly in $W_0^{1,p}(\Omega)$, where u is the pointwise limit of u_n (see Lemma 2.2). We will use the a.e. convergence of ∇u_n to ∇u , stated in theorem 2.3 of [11], that we recall:

Proposition 4.1. *Let u_n be a sequence of functions such that $T_k(u_n) \rightarrow T_k(u_n)$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \rightarrow u \leq u_n$ a.e. in Ω . Assume that $-\operatorname{div}(a(x)|\nabla u_n|^{p-2}\nabla u_n) \geq 0$. Then $T_k(u_n) \rightarrow T_k(u)$ in $W_0^{1,p}(\Omega)$. In particular $\nabla u_n \rightarrow \nabla u$ a.e. in Ω .*

We are now in position to prove Theorem 1.1.

Proof. It is easy to pass to the limit in the right hand side of problems (3): indeed $u_n \rightarrow u$ a.e. and $u_n \leq u \in W_0^{1,1}(\Omega)$. Therefore one can apply the Lebesgue theorem.

By the same argument, the second term of the left hand side converges to $\int_\Omega u^r \varphi$ (recall that by Theorem 3.2 the sequence $\{u_n^{r+\theta}\}$ is bounded in $L^1(\Omega)$).

For the first term, we have that $a(x)|\nabla u_n|^{p-2}\nabla u_n$ converges almost everywhere in Ω to $a(x)|\nabla u|^{p-2}\nabla u$ by Theorem 4.1. Furthermore, this term is dominated by

$\beta|\nabla u_n|^{p-1}$. Observe that $p-1 \leq q$. The limit follows by Vitali's theorem, which can be applied thanks to Proposition 3.4. $\square$

Remark 4.2. We now compare the regularity found in [11] (that is the same problem without lower order term) with q , as defined in Theorem 1.1.

The solution found in [11] belongs to $W_0^{1,\tilde{p}}(\Omega)$, $\tilde{p} = \frac{N(p+\theta-1)}{N+\theta-1}$, for $2-\theta + \frac{\theta-1}{N} \leq p < N$. One has $\tilde{p} \leq q$ if $r > \frac{N(p+\theta-1)}{N-p}$ which is larger than the lower bound $\frac{(1-\theta)(p-1)N}{N-p}$, $p \geq 1 + \frac{1-\theta}{r}$ for r of Theorem 1.1.

Remark 4.3. We observe that the uniqueness of the solutions cannot be inferred as in [14], since we are not dealing with finite energy solutions. We recall that even in the linear case the distributional solutions are not unique, as J. Serrin showed in [15].

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1. Sapienza Università di Roma - Istituto Lombardo

E-mail: boccardo@mat.uniroma1.it

2. Normandie Univ, France; ULH, LMAH, F-76600 Le Havre; FR CNRS 3335, 25 rue

Philippe Lebon 76600 Le Havre, France

E-mail: gisella.croce@univ-lehavre.fr