Comments on various extensions of the Riemann–Liouville fractional derivatives : About the Leibniz and chain rule properties

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COMMENTS ON VARIOUS EXTENSIONS OF THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES : ABOUT THE LEIBNIZ AND CHAIN RULE PROPERTIES

by

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Abstract. — Starting from the Riemann-Liouville derivative, many authors have built their own notion of fractional derivative in order to avoid some classical difficulties like a non zero derivative for a constant function or a rather complicated analogue of the Leibniz relation. Discussing in full generality the existence of such operator over continuous functions, we derive some obstruction Lemma which can be used to prove the triviality of some operators as long as the linearity and the Leibniz property are preserved. As an application, we discuss some properties of the Jumarie's fractional derivative as well as the local fractional derivative. We also discuss the chain rule property in the same perspective.

Key words: fractional derivations, Leibniz property, chain rule, obstruction Lemma, Jumarie fractional derivative, local fractional derivative

AMS subject classification: 26A33, 49M25, 65Q30

1. Introduction

Recently, many authors have tried to define new operators acting on continuous functions starting from the well known Riemann-Liouville derivatives. The reason for such generalizations is usually the fact that the Riemann-Liouville fractional derivatives do not respect some interesting properties. As an example, the Riemann-Liouville derivatives of a constant function is non zero or they do not satisfy the Leibniz's relation. The Jumarie's derivative introduced by G. Jumarie in 2006 (see [**15**]) and in 2009 (see [**16**]) and the local fractional derivatives introduced by Kolwankar and Gangal in 1996 (see [**20**]) are examples of such a strategy.

The general question including all these work is then : **imposing some specific algebraic constraints on a given operator acting on continuous functions or a suitable subset of continuous functions, what type of operators can we construct ?**

In this article, we focus on two classical algebraic relations : the **linearity** and the **Leibniz property**. Precisely, we consider the following problem:

Leibniz and linear rigidity problem:*Let* $T : \mathcal{D}(T) \subset C^0(\mathbb{R}) \to Im(T) \subset C^0(\mathbb{R})$ *be a given operator. Assume that* T *is linear and satisfies the Leibniz property* $(\mathcal{L})$*, for all*

$f, g \in \mathcal{D}(T)$, $T(fg) = T(f)g + fT(g)$. What kind of operator can we obtain ? Can we precise there form ?

A partial answer was given by V.E. Tarasov in two articles [**31, 32**] dealing with operators satisfying the Leibniz or chain rule formula as long as $\mathcal{D}(T) = C^2(\mathbb{R})$ proving that in this case one obtain up to a given C^1 function only the classical derivative. He deduces from this result that operators defined over continuous functions such as some fractional derivatives can not satisfy the Leibniz or chain rule.

However, some sceptical authors continue to think that such operators can be obtained ([**17**],[**33**]) arguing that for some local version of the fractional derivative the Leibniz formula can hold and moreover that the aim of fractional derivative is precisely to deal with non-differentiable functions. However, some counter examples to the fact that the Jumarie fractional derivatives satisfies the Leibniz property were given in [**24, 26, 31, 32**]. The same questioning concerning the chain rule property was also discussed.

In the following, we discuss the previous problem using recent results obtained by König and Mellman ([**22**],[**23**]) who give a complete answer to the previous problem. In particular, we prove that any linear operator acting on the set of continuous functions and satisfying the Leibniz property is trivial (the obstruction Lemma). This result seems not to be known by many authors working on fractional calculus so that we provide a proof in this article. This result means that in order to generalize the classical derivative on continuous functions, one has to cancel one of the two previous relations. If not, one obtains an operator which gives zero for all functions. This result gives a complete proof of Tarasov's idea that the Leibniz property is too strong to be preserved when extending the derivative to continuous functions.

We then discuss the construction of alternative fractional derivatives proposed by G. Jumarie and by Kolwankar and Gangal respectively. These two operators are supposed to satisfy the linearity and the Leibniz's property and to be defined on continuous functions. In view of the obstruction Lemma, this is not possible if the properties which are usually attached to these operators are correct. We then look more precisely on the construction of these operators and their properties. We prove in particular that the Jumarie's fractional derivative can not satisfy the Leibniz's property by proving that this is only another name for the Caputo fractional derivative. This result has strong consequences as it invalidates many uses of this operator as for example to extend the classical calculus of variations [**1, 28**].

We also discuss the Kolwankar-Gangal notion of local fractional derivatives. The situation in this case is different as this operator satisfies indeed the Leibniz property but only on the domain of definition of the Kolvankar-Gangal operator which is known to be a complicated subet of absolutely continuous functions. As a consequence, the obstruction Lemma does not applies in this case but one can nevertheless proved that the operator is trivial on a big function space containing Hölder functions of order $0 < \lambda < 1$. This result was already proved using different arguments in [**4**].

We end our discussion with the formulation of an interpolation problem for one parameter family of operators $\{T_\alpha\}_{\alpha \in \mathbb{R}}$ satisfying some algebraic constraints.

2. Rigidity property and the Leibniz property

In this Section, we consider the set $C^0([a,b])$ of real valued continuous functions defined on the interval $[a,b]$ and the set $C^1([a,b])$ of continuously differentiable real valued functions defined on $[a,b]$.

The classical derivative of a real valued function $f \in C^1([a,b])$ is usually denoted by f' and defined for all $t \in]a,b[$ as

$$f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}. \tag{1}$$

The classical derivative satisfies two fundamental properties:

- It is a *linear* operator: for all $(\lambda,\mu)\in\mathbb{R}^2$ and $(f,g)\in C^1([a,b])$, we have $(\lambda f+\mu g)' = \lambda f + \mu g$.
- The *Leibniz property* : for all $(f,g)\in C^1([a,b])$, we have

$$(f\cdot g)' = f'\cdot g + f\cdot g'. \tag{2}$$

These two properties have leaded to the algebraic notion of *derivations* which appears in many context.

Definition 1* (Derivations).** — *Let A be a ring over an algebraically closed field $\mathbb{K}$. A derivation on A is a mapping $D : A \to A$ such that D is $\mathbb{K}$ linear and satisfies the* **Leibniz property: for all $(x,y)\in A$, we have*

$$D(x\cdot y) = Dx\cdot y + x\cdot Dy. \tag{$\mathcal{L}$}$$

We refer to ([**14**],Chap.I,§.2,p.5) for more details.

Remark 1. — *The previous definition is the classical one concerning the meaning of the Leibniz property for an operator. Some authors distinguish between the Leibniz property of the first order (the previous one) and the Leibniz property of order n which corresponds to an iterative version of the previous one.*

In the following, we consider an **operator** T defined on some domain $\mathcal{D}(T)\subset C^0([a,b])$ and its image is denoted by $Im(T)\subset C^0([a,b])$. We recover a derivation when $\mathcal{D}(T)=Im(T)$.

We denote by D_{NL} the **Newton-Leibniz operator** defined over $C^1([a,b])$ and with image $C^0([a,b])$ and given for all $f\in C^1([a,b])$ by $D_{NL}(f)=f'$.

We have the following *rigidity* result:

***Lemma 1* (Leibniz and linear Rigidity over C^1).** — *Any operator D over $C^1([a,b])$ is of the form*

$$D = \gamma . D_{NL}, \tag{3}$$

where γ is a C^0 function.

This result is interesting when one is dealing with extending the classical derivative to more general function spaces. Indeed, the linearity and the Leibniz property completely characterize this operator.

The usual proof of this result when one has enough regularity (considering C^2 functions instead of C^1) used the Hadamard Lemma (see [**12**], Exercice 22, p.44-45 and also [**31**]).

However, for the readers which are not familiar with such a type of result, a good idea is first to look for a derivation acting on polynomials in order to see in a constructive way the role of the Leibniz property.

Let $\mathbb{R}[x]$ be the set of polynomial with real coefficients. Let us consider a derivation D on $\mathbb{R}[x]$. Let $P \in \mathbb{R}[x]$, $P(x) = \sum_{i=0}^{n} a_i x^i$, $a_i \in \mathbb{R}$, $i = 0, \dots, n$. We have $D(P) = \sum_{i=0}^{n} a_i D(x^i)$. Moreover, the Leibniz property implies that $D(x^i) = ix^{i-1}D(x)$. As a consequence, we have $D(P) = D(x)P'(x)$. The action of a derivation on polynomials is then fixed by the choice of $D(x)$.

The Leibniz property gives then huge constraints on the possible form of the operators satisfying the Leibniz as well as the linear property.

Remark 2. — *The formulation of the "no violation of the Leibniz rule. No fractional derivative" by Tarasov is misleading. The statement is made for a one parameter $\alpha \in \mathbb{R}$ family of derivations D^α even if this parameter plays no role in the rigidity Theorem.*

The previous Lemma follows easily from a more general result proved by H. König and V. Milman in [**22**]:

Theorem 1 **(Leibniz rigidity over C^1-König-Milman,2011)**
If $T : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ is an operator satisfying the Leibniz property ($\mathcal{L}$) then there are continuous functions c, $d \in C^0(\mathbb{R})$ such that

$$T(f)(x) = c(x)f'(x) + d(x)f(x)\ln \mid f(x) \mid . \tag{4}$$

The main point is that the operator T is **not even assumed to be linear**. The Leibniz property is by itself very strong as it imposes already a very specific shape for the operators. If one imposes linearity then one obtains that any operators from $C^1(\mathbb{R})$ to $C^0(\mathbb{R})$ is of the form $c(x)f'(x)$.

Remark 3. — *The previous operator is well defined even if the function f takes the value 0 at some points by making a continuous extension. Precisely, let x_0 be a point such that $f(x_0) = 0$, one extend the previous expression taking the limit when x goes to x_0 and we obtain $T(f)(x_0) = 0$.*

3. What about derivations on continuous functions ?

The previous result does not apply to derivation acting on continuous functions. This remark was in fact at the basic of many comments against Tarasov's reasoning concerning the impossibility to extend the classical properties of the derivatives to continuous functions. The authors argues that the main object of fractional calculus is precisely to deal with non differentiable functions and as a consequence the rigidity result is not sufficient to conclude to the non existence of such operators. In this Section, we close the discussion proving that an operator defined on the set of continuous functions and satisfying the Leibniz and linearity properties is necessary trivial.

3.1. Leibniz and linear properties over continuous functions. — Let us consider an operator $T : C^0(\mathbb{R}) \to C^0(\mathbb{R})$ satisfying the linearity and Leibniz property $(\mathcal{L})$, i.e. a derivation on $C^0(\mathbb{R})$. We have:

***Lemma 2* (Leibniz and linear Obstruction).** — *There exists no non trivial derivations over* $C^0([a,b])$.

We give the proof for the convenience of the reader.

Proof. — First, let $C_c := c$, $c \in \mathbb{R}$ be a constant function. Then for all $c \in \mathbb{R}$ and any derivation D over $C^0([a,b])$ one has $D(C_c) = 0$. Indeed, the function C_1 is the neutral element of $(C^0([a,b],\cdot)$ so that for all $f \in C^0([a,b])$, we have $f \cdot C_1 = f$. As a consequence, we have using the Leibniz property

$$D(C_1) = D(C_1 \cdot C_1) = D(C_1) \cdot C_1 + C_1 \cdot D(C_1) = 2C_1 D(C_1), \tag{5}$$

from which we deduce that $D(C_1) = 0$. Using the linearity, we have

$$D(C_c) = D(c \cdot C_1) = c \cdot D(C_1) = 0. \tag{6}$$

Let us consider a continuous function $f \in C^0([a,b])$. We can always find a continuous function $g \in C^0([a,b])$ such that $f = g^3$. As a consequence, we have

$$D(f) = D(g^3) = 3g^2 \cdot D(g). \tag{7}$$

It follows that if there exists $t \in [a,b]$ such that $f(t) = 0$ then $D(f)(t) = 0$.

The end of the proof now goes as follows. Assume that for some $t \in [a,b]$, one has $f(t) \neq 0$. Let us define $g := f - f(t)$. We have $g(t) = 0$ and as a consequence $D(g)(t) = 0$. As $D(g) = D(f) - D(C_{f(t)}) = D(f)$, we deduce that $D(f)(t) = 0$.

Hence, for all $t \in [a,b]$, one has $D(f)(t) = 0$ and the derivation is trivial. □

3.2. Leibniz rigidity over continuous functions. — In order to define a non trivial operator on continuous functions which preserves the Leibniz property one must cancel the linearity property. However, even in this case, the possibilities are very stringent [**22**]:

***Theorem 2* (Leibniz Rigidity over $C^0(\mathbb{R})$-König-Milman-2011)**

If $T : C^0(\mathbb{R}) \to C^0(\mathbb{R})$ *is an operator satisfying the Leibniz property then there exists a continuous function* $d \in C^0(\mathbb{R})$ *such that* T *has the form*

$$T(f)(x) = d(x) f(x) \ln \mid f(x) \mid . \tag{8}$$

This kind of functions is called an *entropy function* in [**22**].

At this point, we have a partial answer to the extension problem of the classical derivative to continuous functions. As we will see, it will be sufficient to discuss some proposed extensions in fractional calculus.

It must be pointed out that the obstruction Lemma is valid for all function space $F \subset C^0([a,b])$ such that for all $f \in F$ one can find $g \in F$ such that $f = g^3$.

4. About the Jumarie's fractional derivative

In a series of papers, G. Jumarie introduces a "new" fractional derivative which is now called the *Jumarie's fractional derivative*. This derivative knows some success due to the fact that it satisfies unusual properties like the Leibniz's property or the Chain rule property and is moreover defined on the set of continuous functions.

Let $(a,b)\in\mathbb{R}^2$ such that $a<b$. We denote by L^1 the usual Lebesgue space.

The *left fractional Riemann-Liouville integrals* of order $\alpha>0$ of $f\in L^1$ is defined by

$$I^{\alpha}_{a+}[f](t) := \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}ds,\ t\in[a,b], \tag{9}$$

provided that the right-hand side term is well defined.

G. Jumarie (see [**15**], Definition 2.2, p.1369) gives the following definition:

Definition 2. — *Let* $0<\alpha<1$*, the Jumarie's fractional derivative denoted by* D_J *is defined by*

$$D_J[f] = \frac{d}{dt}\left[I^{1-\alpha}_{0+}[f-f(0)]\right]. \tag{10}$$

The previous definition is assumed to be define on continuous functions (see [**15**],Definition 2.1 p.1368 and top of p.1369). The author insists on the fact that continuous but non differentiable functions can be considered (see [**15**], Introduction p.1367). We will return to these points in the following which will be fundamental.

4.1. Triviality of the Jumarie's fractional derivative ?— In ([**15**] p.1371, Equation (3.11) and [**16**] p.382, Corollary 4.1, Equation (4.3)) the author states without proof that his operator satisfies the Leibniz property

$$D^{\alpha}_J(f\cdot g) = D^{\alpha}_J(f)\cdot g + f\cdot D^{\alpha}_J(g), \tag{11}$$

over the set of continuous functions (see [**16**], discussion on the validity of each properties after Corollary 4.1, p.382).

Using the obstruction Lemma, we deduce :

Lemma 3. — *Let us assume that the Jumarie fractional derivative* D^{α}_J*,* $0<\alpha<1$*, satisfies the linearity and the Leibniz relation on the set of continuous functions then* D^{α}_J *is trivial, i.e.* $D^{\alpha}_J := 0$.

A consequence of this result is that **any work dealing with the Jumarie's fractional derivative and using the Leibniz property leads to an** *empty* **theory**.

However, the Jumarie's fractional derivative is known to be non zero on some special functions as for example monomial functions t^{γ}. What is the problem ?

4.2. Jumarie's fractional derivative versus the Caputo derivative. — As the Jumarie's fractional derivative is obviously non trivial over some functions, it means that the assumptions of Lemma 3 are not satisfied. The linearity being evident, only the Leibniz property has to be questioned. In fact, one can easily proves the following Lemma :

Lemma 4. — *The Jumarie's fractional derivatives does not satisfy the Leibniz property.*

This Lemma is a consequence of the fact that the Jumarie's fractional derivative can be rewritten using the Caputo fractional derivative.

Precisely, let us denote by $AC := AC([a,b],\mathbb{R}^n)$ the space of absolutely continuous functions on $[a,b]$.

Let $t \in [a,b]$ and $\alpha \in (0,1]$ then we define the **left Riemann-Liouville fractional derivative of order** α by

$$D^{\alpha}_{a+}f(t) = \left(\frac{d}{dt} \circ I^{1-\alpha}_{a+}\right) f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^{\alpha}} ds. \tag{12}$$

The **fractional Caputo differential operator of order** $\alpha > 0$ is given by

$$ {}_cD^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \frac{d^m}{dt^m} f(s) ds, \tag{13}$$

where $\Gamma(\cdot)$ is the Euler gamma function and $m = \lceil \alpha \rceil$ defines the smallest integer larger then α.

Note that, for every $0 < \alpha < 1$ and $f \in AC([a,b],\mathbb{R}^n)$ the above derivatives are defined almost everywhere on the interval $[a,b]$. Moreover we have the following relations between Caputo and Riemann-Liouville definitions :

$$D^{\alpha}_{a+}[f] = {}_cD^{\alpha}_{a+}[f] + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a). \tag{14}$$

An easy consequence is then the following Lemma (see [**18**]):

Lemma 5. — *For every* $0 < \alpha < 1$ *and* $f \in AC([a,b],\mathbb{R})$*,* $D^{\alpha}_{0,+}[f]$ *and* ${}_cD^{\alpha}_{0+}[f]$ *are defined almost everywhere on* $[a,b]$ *and the following equality holds:*

$$ {}_cD^{\alpha}_{0+}[f] = D^{\alpha}_{0+}[f - f(0)]. \tag{15}$$

As a consequence, we have:

Lemma 6. — *For every* $0 < \alpha < 1$ *and every* $f \in AC([a,b],\mathbb{R})$*, we have* $D^{\alpha}_{J}[f] = {}_cD^{\alpha}_{0+}[f]$.

Proof. — By definition of the Jumarie fractional derivative, we have

$$D^{\alpha}_{J}[f] = D^{\alpha}_{0+}[f - f(0)]. \tag{16}$$

Using Lemma 5, this concludes the proof. □

This result implies the fact that the Jumarie fractional derivative does not satisfy the Leibniz rule (as usual for the Caputo's derivative) and also that the Jumarie's derivative is not a new fractional derivative.

Remark 4. — *This result implies also that all the articles dealing with the Jumarie fractional derivative and using the asserted Leibniz property of the derivative are wrong. This is the case for example for papers dealing with an extension of the calcul of variation using the Jumarie fractional derivative (see for example* [**1**] *and* [**28**]*).*

5. The Kolwankar-Gangal local fractional derivative

In [**19**], Kolwankar and Gangal introduce the idea of *local fractional derivative* giving rise to the local fractional calculus. The idea is to localize the Riemann-Liouville derivative with respect to its base point in order to recover a local operator. In [**3**], Ben Adda and one of the authors have given an alternative representation of this theorem using what is now called α difference quotient local fractional derivative(1). This result can be used to deduce that the Kolwankar-Gangal (KG) fractional derivative satisfies some special properties which are unusual on some subset of continuous functions. In particular, one can prove that the KG fractional derivative satisfies the Leibniz property. As for the Jumarie fractional derivative, this will have very strong consequences, in particular that this derivative is trivial over a very big set and almost everywhere trivial generically.

5.1. Definitions and properties. — We denote by $\mathrm{D}^{\alpha}_{KG,\sigma} f$ the *Kolwankar-Gangal local fractional derivative* (KG-LFD) defined by

$$\mathrm{D}^{\alpha}_{KG,\sigma}[f](y) = \lim_{x \to y^{\sigma}} D^{\alpha}_{y,\sigma}\left[\sigma(f - f(y))\right](x), \ \sigma = \pm. \tag{17}$$

When $\alpha = 1$, and $f \in AC([a,b])$, we have $D^1_{y,\sigma}[f](x) = \sigma f'(x)$ and then $D^1_{y,\sigma}(\sigma(f - f(y)))(x) = \sigma^2 f'(x) = f'(x) \in L^1$. We deduce that for $f \in C^1$, $D^1_{KG,\sigma}[f](y) = f'(y)$.

The domain of existence of the Kolwankar-Gangal fractional derivative is difficult to describe. First, the left Riemann-Liouville derivative of a function f is defined as long as f belongs to the function space $\mathrm{E}^{\alpha}_{a,+}([a,b])$ defined by

$$\mathrm{E}^{\alpha}_{a,+}([a,b]) = \left\{ f \in \mathrm{L}^1, \ I^{1-\alpha}_{a,+}[f] \in AC([a,b]) \right\}. \tag{18}$$

This space is defined in Samko and al. (see [**30**], Definition 2.4,p.44).

Second, the Kolwankar-Gangal local fractional derivative makes sense as long as $\mathrm{D}^{\alpha}_{a,+}[f - f(a)]$ is well defined on a given interval $]a, a+\delta[$ where $\delta > 0$. In the following, the *existence* of the KG-LFD will be always understood as f being such that $f - f(a) \in E^{\alpha}_{a,+}([a, a+\delta])$. This condition implies that $\mathrm{D}^{\alpha}_{a,+}[f - f(a)]$ is well defined at least almost everywhere in $]a, a+\delta]$. In particular, if $f - f(a)$ has a left RL fractional derivative in the usual sense, i.e. $I^{1-\alpha}_{a,+}[f - f(a)]$ is differentiable at every point, then $f - f(a)$ belongs to $E^{\alpha}_{a,+}([a, a+\delta])$ (see [**30**],Remark 2.2 p.44).

(1)The proof of the representation Theorem in [**3**] was incorrect as pointed out for example by [**7**] and [**6**]. A correct proof was given in [**7**] and later under more general conditions in [**4**].

In [**3**] another quantity which generalizes directly the representation of the classical derivative as limit of a difference-quotient is defined by :

$$\mathrm{D}^{\alpha}_{BC,\sigma}[f](y) = \Gamma(1+\alpha) \lim_{x\to y^{\sigma}} \frac{\sigma(f(x)-f(y))}{\mid x-y\mid^{\alpha}},\ \sigma = \pm.$$

This quantity was introduced by G. Cherbit in [**8**] and is denoted BC-LFD in the following. This quantity is called *difference-quotient* local fractional derivative[(2)] in [**7**] or local fractional derivative in the sense of Ben Adda-Cresson in [**10**].

In [**3**], we state the following Theorem :

Theorem 3. — *Let* $0<\alpha<1$ *and* $f\in C^0([a,b])$ *and* $y\in]a,b[$ *be such that* $\mathrm{D}^{\alpha}_{KG,\sigma}[f](y)$ *exists, then*

$$\mathrm{D}^{\alpha}_{KG,\sigma}[f](y) = \mathrm{D}^{\alpha}_{BC,\sigma}[f](y). \tag{19}$$

For a complete and corrected proof of this theorem, we refer to [**4**]. This equality leads easily to the following result :

Lemma 7. — *For all* (f,g) *such that* $\mathrm{D}^{\alpha}_{KG,\sigma}[f]$ *and* $\mathrm{D}^{\alpha}_{KG,\sigma}[g]$ *exist, we have*

$$D^{\alpha}_{KG,\sigma}[f\cdot g] = D^{\alpha}_{KG,\sigma}[f]\cdot g + f\cdot D^{\alpha}_{KG,\sigma}[g]. \tag{20}$$

As for the Jumarie's fractional derivative, the fact that the Kolwankar-Gangal operator or BC- operator satisfy the Leibniz property seems to indicate that these operators must be trivial on some sets.

5.2. Triviality of the Kolwankar-Gangal fractional derivative ?— As already pointed out, the fact that the Kolwankar-Gangal fractional derivative satisfies the Leibniz property and is a linear operator gives some indications that this operator is trivial at least on some functional sets where it is defined. This is indeed the case, as proved in [**4**]:

Let $1\geq\lambda>0$; a function $f:[a,b]\to\mathbb{R}$ is said to satisfy the Hölder condition of order λ on $[a,b]$ if $\mid f(x)-f(y)\mid < c\mid x-y\mid^{\lambda}$ for any $x,y\in[a,b]$ where $c>0$ is a constant. The scalar λ is called the Hölder exponent. We denote by $H^{\lambda}([a,b])$ the set of λ-Hölderian functions (see [**30**],p.2).

Theorem 4. — *Let* $f\in H^{\lambda}([a,b])$*, then for all* $0<\alpha<\lambda\leq 1$*, we have* $\mathrm{D}^{\alpha}_{KG,+}[f](x)=0$ *in* $[a,b]$*.*

Proof. — This follows from a direct computation. As $f\in H^{\lambda}([a,b])$, $0<\lambda\leq 1$, we have for all $\alpha<\lambda$ and all $x\in[a,b[$ (see [**30**],p.239,Lemma 13.1 and page 242, Corollary of Lemma 13.2) that

$$D^{\alpha}_{x,+}[f](y) = \frac{f(x)}{\Gamma(1-\alpha)\,(y-x)^{\alpha}} + \psi_f(y), \tag{21}$$

where $\psi_f\in H^{\lambda-\alpha}([x,x+\delta])$ for a certain $\delta>0$ such that $\psi_f(x)=0$. As a consequence, $D^{\alpha}_{x,+}[f-f(x)](x)=\psi_{f-f(x)}(x)=0$ which concludes the proof. □

Remark 5. — *The previous result is proved in (*[**6**]*,Lemma 3.1 p.69) when* f *is at least* C^1*.*

(2) Up to the constant factor $\Gamma(1+\alpha)$.

An alternative proof of Theorem 4 is to see that the arguments of the obstruction Lemma holds when one is dealing with a functional set F such that for all $f \in F$, we can find $g \in F$ such that $f = g^3$. This is indeed the case for Hölderian functions of order $1 > \lambda > \alpha$.

The previous result does not implies that the Kolwankar-Gangal fractional derivatives is always trivial. One can indeed find some explicit class of functions for which this derivative can be effectively computed. See for example [**21**].

Two results are important here and related to the validity of the Leibniz property for this operator and its domain of definition.

First, in order to apply the obstruction Lemma, the operator must satisfy the Leibniz property over a functional set where it is always defined. However, for Hölderian functions in the class $H^\alpha([a,b])$ the RL fractional derivative of $f - f(x)$ is not always defined, even almost everywhere. Indeed, when $\lambda = \alpha$, we have $I_{x,+}^{1-\alpha}[f - f(x)] \in H^{1,1}([x, x+\delta])$ (see [**30**],Theorem 3.1 p.53-54) which corresponds to Log-Lipschitz continuous functions instead of $H^{1+(\lambda-\alpha)}([a,b])$ when $\lambda > \alpha$.

Remark 6. — *An example of a function f in $H^\alpha([a,b])$ for which the RL fractional derivative of order α $D_{x,+}^\alpha[f - f(x)]$ is not defined is given by the Weierstrass function* $W_\alpha(x) = \sum_{n=0}^{\infty} q^{-\alpha n}\cos(q^n x)$ *with $q > 1$ (see* [**29**] *Theorem 2 p.150 and Remark 4 p.155).*

Second, assuming that the Kolwankar-Gangal fractional derivative is well defined on an interval and continuous, one can proved directly that it is trivial (see for example [**9**],§.3, Theorem 4.1 and Corollary 4.1 p.4923)(3).

Remark 7. — *This negative result was in fact one of the reasons that has leaded one of the authors not to continue the exploration of the use of local fractional derivatives in Physics as first studied in* [**5**].

Even with a less stronger condition, existence almost everywhere on a given interval, we have the following result proved in [**7**]:

Theorem 5. — *Let $f : [a,b] \to \mathbb{R}$ be a locally α-Hölder continuous function in $[a,b]$ for $0 < \alpha < 1$. Assume that the Kolwankar-Gangal fractional derivative exist almost everywhere in $[a,b]$. Then, the Kolwankar-Gangal fractional derivative is zero almost everywhere in $[a,b]$.*

As a consequence, the Kolwankar-Gangal fractional derivative is not trivial but almost everywhere trivial.

6. Rigidity property and the chain rule property

The previous analysis proves that the Leibniz property is indeed very strong. What about the chain rule property ? The Jumarie fractional derivative was assumed to satisfy some kind of chain rule formula (see [**16**],Corollary 4.1 formula (4.4) and (4.5)). However, as this fractional derivative corresponds exactly to the Caputo derivatives this can not be true.

(3) See also [**25**].

Tarasov has in particular discussed this problem restricting his attention to the action of an operator satisfying a specific chain rule formula on monomial functions (see [**32**]). In fact, a complete answer is given in [**2**]. But first, we give the following result states in [**22**]:

Theorem 6 **(Leibniz and chain-rule rigidity, König, Milman, 2011)**
Suppose $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ satisfies the Leibniz rule and the chain rule functional equations

$$\begin{array}{rcl} T(f\cdot g) & = & Tf\cdot g + f\cdot Tg, \\ T(f\circ g) & = & (Tf)\circ g\cdot Tg, \end{array}$$

for all $f, g \in C^1(\mathbb{R})$. Then T is either identically 0 or the derivative, $Tf = f'$.

The chain rule property is even a stronger algebraic constraint than the Leibniz rule. Indeed, we have (see [**2**],Proposition 3):

Theorem 7 **(Chain-rule rigidity, Artstein-Avidan, König, Milman, 2010)**
Assume that $T : C(\mathbb{R}) \to C(\mathbb{R})$ satisfies the functional equation $T(f \circ g) = (Tf) \circ g \cdot Tg$, for all f,$g \in C(\mathbb{R})$ and that there exists $g_0 \in C(\mathbb{R})$ and $x_0 \in \mathbb{R}$ with $(Tg_0)(x_0) = 0$. Then T is zero on the class of half-bounded continuous functions.

This Theorem implies the following result :

Theorem 8 **(Obstruction-chain rule)**. — *There exists no non trivial operator $D : C(\mathbb{R}) \to C(\mathbb{R})$ which satisfies the chain rule property and such that D is zero on constant functions.*

The reason for formulating such a result is that most of the modifications of the fractional Riemann-Liouville derivatives are done in order to satisfy the fact that the derivative of a constant function is zero. We see that imposing this condition and the chain rule property lead to a trivial operator.

We return now to the "proof" proposed by G. Jumarie in [**17**] of the Leibniz rule for fractional derivatives as long as one considers non-differentiable functions. We have already seen that this is not possible but the previous Theorem invalidate also the approach given in [**17**]. Indeed, G. Jumarie assumes in [**17**] that his fractional derivative is zero on constant functions and satisfies the chain rule property (see [**17**],§.1,p.50). As a consequence, even if one can make some computation assuming the existence of such an operator and to deduce some properties satisfied by this operator, all the theory is empty due to the fact that no such operators exist.

7. Conclusion and perspectives

The set of results presented in this article implies that looking directly for an operator defined on the global set of continuous functions and satisfying the linearity and Leibniz property is hopeless.

One can replace the previous problem by changing a little bit the way that we look for such an operator by considering a one parameter family of operators:

Interpolation problem: *Let T_α, $\alpha \in \mathbb{R}$ be a family of operators defined on $\mathcal{D}(T_\alpha) \subset C^0(\mathbb{R})$ and with image in $Im(T_\alpha) \subset C^0(\mathbb{R})$. Assume that the family $\{T_\alpha\}_{\alpha\in\mathbb{R}}$ satisfies the following properties:*

– *For all* $\alpha \in \mathbb{R}$, T_α *is linear.*
– *For* $\alpha \in \mathbb{N}$, *we have* $T_n = \dfrac{d^n}{dx^n}$.
– *For* $\alpha = 0$, $T_0 = Id$ *where* Id *is the identity mapping on* $C^0(\mathbb{R})$.
– *For* $\alpha = -n$, $n \in \mathbb{N}$, *we have* $T_n = I^n$ *where* $I_0^1[f](x) = \int_0^x f(s)ds$ *is the antiderivative of* f *vanishing at* $x = 0$ *and* $I^n = I^1 \circ \cdots \circ I^1$ *(n-times).*

Can we describe the set of operators satisfying the four constraints above ? Can we give the explicit form of such a family of operators.

Of course, one can obtain some special examples of family of operators which satisfy the previous interpolation problem as for example the (left) Riemann-Liouville and Caputo fractional derivatives. This is precisely one of the motivation at the beginning of the story of fractional calculus. However, this does not give an answer to the previous problem.

The main objective of the previous problem is to understand to which extent one can generalize the classical notion of fractional derivative as defined by Riemann-Liouville or Caputo, etc. In particular, if the previous algebraic conditions are sufficient to characterize a specific fractional calculus then a generalization can only be obtained by canceling one of these conditions. This will give a systematic way of looking for such new family of operators. As pointed out to us by a referee, such a program "**fits into a broader discussion concerning a more rigorous mathematical definition of fractional derivative**". We refer to [**27, 11**] and in particular to the work of R. Hilfer and Y. Luchko [**13**] for close connection with this problem.

Of course, the previous interpolation problem can be modified by changing the algebraic conditions one is looking for. It depends mainly on the applications that people are considering as fundamental for some mathematical problems. As an example, one can impose some invertibility relations like $T_\alpha \circ T_{-\alpha} = Id$ or a composition rule like $T_\alpha \circ T_\beta = T_{\alpha \star \beta}$ for a certain group action $\star$ on $\mathbb{R}$.

A discussion of all these problems will be given in a forthcoming paper.

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