Conservatism reduction for Nonlinear Takagi-Sugeno Observer : Interconnected System Approach

Hichem Arioui, Dalil Ichalal, Lamri Nehaoua, Saïd Mammar

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Abstract— **This work considers observer design for nonlinear systems by using Takagi-Sugeno (TS) models combined to the interconnected systems formalism. The approach is based, on a TS models of interconnected parts obtained from an adequate decomposition of the initial nonlinear system into sub-systems, and then transforming each one into Takagi-Sugeno's representation.**

The observer design conditions for this new representation are expressed as Linear Matrix Inequality (LMI) constraints obtained from a common quadratic Lyapunov functions for stability analysis. The proposed observer, Luenberger-like structure, aims to reduce the number of LMIs and then reduces the conservatism related to the huge number of verticies in the polytope. Numerical examples show the effectiveness of the proposed approach.

I. INTRODUCTION

A wide class of physical systems can be written as non-linear (descriptor or not) models. Since this type of systems often appears in control problems, a polytopic representation makes it possible to obtain a smaller number of Linear Matrix Inequalities (LMI) constraints [1]. This problem is crucial in polytopic systems's framework due to the conservatism induced by a judge number of vertexes in the polytop (sub-models). Indeed, from the beginning of the 1990 years, the use of LMI formalism became a tool of choice for studying the polytopic systems [2]. However, several difficulties appeared, among them, the conservatism related to the number of sub-models. The conservatism increases when the number of LMIs increases [3].

Designing observers for nonlinear systems is a challenging problem due to its importance in automatic control design such as control, fault diagnosis, monitoring and fault tolerant control. Polytopic and Takagi-Sugeno play an important role in such a problem and provide different and efficient way to design controllers and observers by using Lyapunov theory and LMI formalism. In the context of observer's design, several works have been provided. Among them, one can cite [3], [4], [5], for controller-based observer design. In [6], [7], [8], observers have been proposed for state estimation and state estimation in the presence of unknown inputs (faults, perturbations ...). Extensions have been proposed for state estimation of Takagi-Sugeno systems with state dependent premise variables [9], [10], [11] and differentiation techniques [12]. Notice that all these works transform the nonlinear system in a TS form and design the observer which leads, in several cases of strong nonlinear systems, to TS systems with a huge number of sub-models and then a huge number of LMIs to solve. This leads, often, too much conservatism property. This problem has been dealt with by different approaches such as polyquadratic and non-quadratic Lyapunov functions, considering the variation of the weighting functions [13] and using the Polya's theorem [14]. Recently, the descriptor approach has been used in order to reduce the conservatism for a certain class of descriptor system by letting them in the descriptor form [1].

As discussed before, reducing the number of LMIs may guarantee a less conservative design. Starting from this point, different directions can be explored. In this paper, the interconnected-based approach is considered. As in nonlinear systems, the interconnected representation is an interesting technique in stability analysis and control by using the small gain theorem [15]. Indeed, it is proven that for complex systems and large-scale systems, the decentralized approach for control is allows to reduce the complexity in computation point of view. Starting from this point, the idea proposed in this paper is based on the decomposition of a nonlinear system into an interconnection of several nonlinear sub-systems. Keeping in mind the reduction of sub-models in TS models, the interconnected systems can represent a nonlinear system into an interconnection of several TS systems with reduced number of vertexes. Then, the stability of each sub-system additionally to the small gain theorem may provide less restrictive stability conditions with a common Lyapunov storage function. The problem of designing state observers (or controller) is then transformed into a design of observers for equivalent interconnected Takagi-Sugeno sub-systems.

In order to deal with this problem, several approaches can be found. For example, exploit the nonlinear transformation in order to re-write the nonlinear system with less number of nonlinearities equivalently. In the proposed paper, a new approach is adopted. The idea is based on a decomposition of the nonlinear system into a suitable interconnected nonlinear sub-systems. To do so, each sub-system will contain less non-linearities compared to the whole nonlinear system. It will be proven that, using this representation, almost decoupled linear matrix inequalities constraints for each sub-system are obtained by common Lyapunov analysis (advanced functions may be considered).

The paper is organized as follows : after recalling some important lemmas, section III) states the problem and the main result and how to reduce number of sub-models (LMI conditions). Section IV discuss the synthesis of the observer and gives sufficient conditions for its convergence. In section

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V we illustrate the effectiveness of the proposed approach through an example. Section VI concludes the paper.

II. Mathematical background

Before giving the formulation of our problem, recall the following basic results which will be used in the proof of our main results :

Lemma 1: (Xie Lemma, [16])

Given matrices X, Y and G of appropriate dimension with G symmetric matrix (G=$G^T > 0$), the property below is true:

$$X^TY + Y^TX \leq X^TGX + Y^TG^{-1}Y \tag{1}$$

Lemma 2: (Schur Lemma, [2])

Consider X, Y and Z with appropriate dimensions with $X = X^T$ and $Z = Z^T$, hence:

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} < 0 \Leftrightarrow \begin{cases} Z < 0 \\ X - Y^TZ^{-1}Y < 0 \end{cases} \tag{2}$$

III. Problem statement, motivations and notations

In this section, we discuss the decomposition of the affine nonlinear systems and the intricacy (in term of conservatism) they induce by comparing it with a prior decomposition into two (or more) interconnected systems before writing the Takagi-Sugeno form for each sub-model. To the knowledge of the authors, this work is precursor.

A. Interconnection of TS Systems

Lets consider the following nonlinear system:

$$\begin{cases} \dot{\nu}(t) &= f(\nu(t)) + g(\nu(t))u(t) \\ y(t) &= C\nu(t) \end{cases} \tag{3}$$

where $\nu(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ the control input, and $y(t) \in \mathbb{R}^l$ the output vector; $f(\cdot)$ and $g(\cdot)$ are nonlinear functions.

The previous system can be represented according to the number r of sub-models by the Takagi-Sugeno structure.

As said before, a TS model is composed of a finite set of a weighted linear systems, used to achieve a trade-off between the accuracy and complexity of the model. The mathematical formulation of the TS model of system (3) is given by:

$$\begin{cases} \dot{\nu}(t) &= \sum_{k=1}^{p} \mu_k(\rho(t))(A_k\nu(t) + B_ku(t)) \\ y(t) &= C\nu(t) \end{cases} \tag{4}$$

The r nonlinearities are captured via membership functions. These functions satisfy the convex-sum property in the compact set of the state space, i.e.

$$\sum_{k=1}^{p} \mu_k(\rho(t)) = 1 \quad \text{with} \quad \mu_k(\rho(t)) \geq 0 \tag{5}$$

with $p = 2^r$, and $\rho(t)$ is the permise variable vector depending on states system (measured or not). In this work the premise variables are assumed to be known at real time.

In order to illustrate the proposed approach based on system decomposition into some nonlinear interconnected subsystems (here we suppose a two sub-systems decomposition), and under some conditions (there is a dynamic coupling between the interconnected subsystems), the previous system is expressed by:

$$\begin{cases} \dot{x}(t) &= f_1(x(t)) + g_1(x(t))z(t) + h_1(x(t))u(t) \\ y_x(t) &= \bar{C}x(t) \\ \dot{z}(t) &= f_2(z(t)) + g_2(z(t))x(t) + h_2(z(t))u(t) \\ y_z(t) &= \check{C}z(t) \end{cases} \tag{6}$$

where : $\nu(t) = [x(t) \;\; z(t)]^T$ and $C = [\bar{C} \;\; \check{C}]^T$. $x(t) \in \mathbb{R}^{\bar{n}}$ and $z(t) \in \mathbb{R}^{n-\bar{n}}$ are the state vector, and $y_x(t) \in \mathbb{R}^{\bar{m}}$ and $y_z(t) \in \mathbb{R}^{m-\bar{m}}$ the output vector; $f_i(\cdot)$ and $g_j(\cdot)$ are nonlinear functions. The mathematical formulation of the TS model of system (6) is given by:

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^{p_1} \sigma_i(x(t))(\bar{A}_ix(t) + \bar{D}_iz(t) + \bar{B}_iu(t))) \\ y_x(t) &= \bar{C}x(t) \\ \dot{z}(t) &= \sum_{j=1}^{p_2} \eta_j(z(t))(\check{A}_jz(t) + \check{D}_jx(t) + \check{B}_ju(t)) \\ y_z(t) &= \check{C}z(t) \end{cases} \tag{7}$$

where matrices $\bar{A}_i$, $\bar{B}_i$, $\bar{C}$, $\check{A}_i$, $\check{B}_i$ and $\check{C}$ represent the i^{th} linear right-hand side of each sub-dynamics of model (7).

Respectively, the r_1 and r_2 nonlinearities of the two (or more) sub-dynamics are captured via membership functions with : $\sum_{i=1}^{p_1} \sigma_i(\cdot) = 1$, $\sigma_i(\cdot) \geq 0$ (respectively $\sum_{j=1}^{p_2} \eta_j(\cdot) = 1$, $\eta_j(\cdot) \geq 0$), with $p_i = 2^{r_i}$.

Since, $r_1 + r_2 \leq r$, because of the vanishing nonlinear terms between $x(t)$ and $z(t)$ under the interconnected form, hence, $p_1 + p_2 \leq p$. For example, if the initial number of nonlinearities are about $r = 5$, this implies $p = 2^r = 32$ sub-models. Otherwise, $r_1 = r_2 = 2$ this implies $p_1 = p_2 = 4$ sub-models for each sub-dynamics. This means that the system under the interconnected decomposition needs only 8 LMI constraints instead 32 LMI conditions to be verified for the classical TS model. In the previous example, only one coupling nonlinearity was removed.

Moving to the interconnection of two or more dynamics can significantly reduce the number of sub-models as well as the number of LMIs and inevitably increase the feasibility set; therefore, it reduces the conservativeness of the optimisation problem [5]. The following example highlights these remarks.

B. Motivating Example

Consider the nonlinear descriptor system :

$$\begin{cases} E(\nu)\dot{\nu}(t) &= A(\nu(t)) + Bu(t) \\ y(t) &= C\nu(t) \end{cases} \tag{8}$$

with matrices

$$E(\nu(t)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -a\cos(\nu_1) & 0 \\ 0 & 0 & 1 & 0 \\ -b\cos(\nu_1) & 0 & 0 & 1 \end{bmatrix},$$

$$A(\nu(t)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ ag\sin(\nu_1)/\nu_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -b\sin(\nu_1) & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{2c} \\ 0 \\ \frac{1}{c} \end{bmatrix}, \text{ and } C^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Writing the example in the form (8) gives $r_e = 2^1 = 2$ due to the nonlinear term $\cos(v_1(t))$ in $E(v(t))$ and $r = 2^2 = 4$ due to $\sin(v_1(t))/v_1(t)$ and $sin(v_1(t))$ and about $2 \times 4 = 8$ sub-models. In this case, the inversion $E(v(t))$ has no effect on the increase of the number of sub-models. Indeed, the new system

$$\left\{ \begin{array}{lcl} \dot{v}(t) & = & E^{-1}(v(t))A(v(t)) + E(v)^{-1}Bu(t) \\ y(t) & = & Cv(t) \end{array} \right. \tag{9}$$

$$E^{-1}(v)A(v) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ l_1(v) & 0 & 0 & a\cos(v_1) \\ 0 & 0 & 1 & 0 \\ 0 & l_2(v) & 0 & -1 \end{bmatrix}$$

and $E^{-1}(v)B = B$, where

$$l_1(v) = ag\sin(v_1)/v_1, \quad l_2(v) = -b(\cos(v_1) - \sin(v_1))$$

To rewrite the previous example (8) into the classical Takagi-Sugeno representation it is necessary to invert the matrix $E(v)$ resulting in (9). This gives $r = 2^3 = 8$ sub-models.

On the other side, splitting the system (8) into two dynamics as follows:

$$\Sigma_1 : \left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & ag\sin(x_1) + a\cos(x_1)z_2 + \frac{1}{2c}u \\ y_x & = & x_1 \end{array} \right. \tag{10}$$

and,

$$\Sigma_2 : \left\{ \begin{array}{lcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & -z_2 + bx_2(\cos(x_1) - \sin(x_1)) + \frac{1}{c}u \\ y_z & = & z_1 \end{array} \right. \tag{11}$$

Sub-system Σ_1 gives $r_x = 2^2 = 4$ sub-models due to the nonlinear terms $\cos(x_1(t))$ and $\sin(x_1(t))$. System Σ_2 becomes linear with respect to $z(t)$. The nonlinear part depending on $x_1(t)$ and $x_2(t)$ is considered as input to the second dynamics.

Remark 1: The splitting of the overall system into two or more subsystems is neither straightforward nor unique. For an optimal choice, the subsystems must be chosen in such a way that they have as little interaction as possible between them. This can involve cases, when the states of a subsystem are measurable, where the observer design is not necessary. This is the case of the motivating example above with subsystem Σ_1. Also, the number of sub-models is determined by the number of nonlinearities that can be aggregated into a sector function.

Remark 2: It is interesting to note that in some cases, following a decomposition of the descriptor systems results in an interconnection of several non-descriptor sub-systems. This can lead to a greater reduction in the number of sub-models of interconnected Takagi-Sugeno systems.

Remark 3: For the rest of the developments and for simplicity, only the case of measurable decision variables is considered. This means that the membership function, in systems and observers, are depending only on measured states. In the same way, the outputs are linearly depending to the system states.

IV. Observer design

The objective of this section is to design a nonlinear observer for interconnected Takagi-Sugeno systems based on a common quadratic Lyapunov functions. The extension of the proposed approach to poly-quadratic Lyapunov functions is straightforward.

A. State estimation

Based on the developments above (3)-(7), the following nonlinar observer is proposed [17]:

$$\left\{ \begin{array}{lcl} \dot{\hat{x}} & = & \sum_{i=1}^{p_1} v_i(y_x)(\bar{A}_i\hat{x} + \bar{D}_i\hat{z} + \bar{B}_iu - \bar{L}_i(y_x - \hat{y}_x)) \\ \hat{y}_x & = & \bar{C}\hat{x} \\ \dot{\hat{z}} & = & \sum_{j=1}^{p_2} \eta_j(y_z)(\check{A}_j\hat{z} + \check{D}_j\hat{x} + \check{B}_ju - \check{L}_j(y_z - \hat{y}_z)) \\ \hat{y}_z & = & \check{C}\hat{z} \end{array} \right. \tag{12}$$

By considering $e_x = x - \hat{x}$ and $e_z = z - \hat{z}$, observers errors dynamics are given by:

$$\left\{ \begin{array}{lcl} \dot{e}_x & = & \sum_{i=1}^{p_1} v_i(y_x)(\bar{\Phi}_ie_x + \bar{D}_ie_z) \\ \dot{e}_z & = & \sum_{j=1}^{p_2} \eta_j(y_z)(\check{\Phi}_je_z + \check{D}_je_x) \end{array} \right. \tag{13}$$

where $\bar{\Phi}_i = (\bar{A}_i - \bar{L}_i\bar{C})$ and $\check{\Phi}_j = (\check{A}_j - \check{L}_j\check{C})$. The following Theorem provides LMI conditions that ensure asymptotic convergence and allows to compute the gains of the interconnected observer.

Theorem 1: The state estimation error between the system and the interconnected observers converges asymptotically to zero if there exists two symmetric and definite matrices P and Q, two diagonal positive matrices Ω_1 and Ω_2 and vectors gains $\bar{K}_i$, $i = 1, \ldots, p_1$ and $\check{K}_j$, $j = 1, \ldots, p_2$ such that the LMI conditions are :

$$\begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i - \bar{K}_i\bar{C} - \bar{C}^T\bar{K}_i^T + \Omega_2 & P\bar{D}_i \\ \bar{D}_i^T P & -\Omega_1 \end{bmatrix} < 0, \quad i = 1, \ldots, p_1 \tag{14}$$

$$\begin{bmatrix} \check{A}_i^T P + P\check{A}_i - \check{K}_i\check{C} - \check{C}^T\check{K}_i^T + \Omega_1 & Q\check{D}_j \\ \check{D}_j^T Q & -\Omega_2 \end{bmatrix} < 0, \quad j = 1, \ldots, p_2 \tag{15}$$

hold. The gains of the interconnected observer are obtained from the equations $L_i = P^{-1}\bar{K}_i, \quad i = 1, \ldots, p_1$ and $\check{L}_i = Q^{-1}\check{K}_i, \quad i = 1, \ldots, p_1$.

Proof: To study the convergence of the first observer, the quadratic Lyapunov function is used:

$$V(e(t)) = e_x(t)^T Pe_x(t) + e_z(t)^T Qe_z(t) \tag{16}$$

The time-derivative of the Lyapunov function (16) is:

$$\begin{array}{l} \dot{V}(e) = \sum_{i=1}^{p_1} v_i(y_x)((\bar{\Phi}_ie_x + \bar{D}_ie_z)^T Pe_x + e_x^T P(\bar{\Phi}_ie_x \\ \quad + \bar{D}_ie_z)) + \sum_{j=1}^{p_2} \eta_j(y_z)((\check{\Phi}_je_z + \check{D}_je_x)^T Qe_z \\ \quad + e_z^T Q(\check{\Phi}_je_z + \check{D}_je_x)) \end{array} \tag{17}$$

Considering $\bar{\Gamma}_i = \bar{\Phi}_i^T P + P\bar{\Phi}_i$ and $\check{\Gamma}_j = \check{\Phi}_j^T Q + Q\check{\Phi}_j$, we have:

$$\begin{array}{l} \dot{V}(e(t)) < 0 \Leftrightarrow \\ \sum_{i=1}^{p_1} \nu_i(y_x)(e_x^T \bar{\Gamma}_i e_x + e_x^T P\bar{D}_i e_z + e_z^T \bar{D}_i^T P e_x) + \\ \sum_{j=1}^{p_2} \eta_j(y_z)(e_z^T \check{\Gamma}_j e_z + e_z^T Q\check{D}_j e_x + e_x^T \check{D}_j^T P e_z) < 0 \end{array} \tag{18}$$

By considering Lemma (1), inequality (18) yields:

$$\begin{array}{l} \sum_{i=1}^{p_1} \nu_i(y_x)(e_x^T(\bar{\Gamma}_i + P\bar{D}_i G_1 \bar{D}_i^T P)e_x + e_z^T G_1^{-1} e_z) + \\ \sum_{j=1}^{p_2} \eta_j(y_z)(e_z^T(\check{\Gamma}_j + Q\check{D}_j G_2 \check{D}_j^T Q)e_z + e_x^T G_2^{-1} e_x) < 0 \end{array} \tag{19}$$

If this last condition holds then $\dot{V}(e(t)) < 0$. This condition leads to the following optimization problem:

$$\left[\begin{array}{cc} \bar{\Gamma}_i + P\bar{D}_i G_1 \bar{D}_i^T P + G_2^{-1} & 0 \\ 0 & \check{\Gamma}_j + Q\check{D}_j G_2 \check{D}_j^T Q + G_1^{-1} \end{array} \right] < 0 \tag{20}$$

or

$$\left[\begin{array}{c} \bar{\Gamma}_i + P\bar{D}_i G_1 \bar{D}_i^T P + G_2^{-1} \end{array} \right] < 0 \tag{21}$$

and

$$\left[\begin{array}{c} \check{\Gamma}_j + Q\check{D}_j G_2 \check{D}_j^T Q + G_1^{-1} \end{array} \right] < 0 \tag{22}$$

The previous inequalities are connected by matrices gains G_1 and G_2. Using Schur Lemma (2), inequality (20) yields to:

$$\left[\begin{array}{cc} \bar{\Gamma}_i + G_2^{-1} & P\bar{D}_i \\ \bar{D}_i^T P & -G_1^{-1} \end{array} \right] < 0, \quad i = 1, \ldots, p_1 \tag{23}$$

$$\left[\begin{array}{cc} \check{\Gamma}_j + G_1^{-1} & Q\check{D}_j \\ \check{D}_j^T Q & -G_2^{-1} \end{array} \right] < 0, \quad j = 1, \ldots, p_2 \tag{24}$$

Finally, by using the definitions of the matrices $\bar{\Gamma}_i$ and $\check{\Gamma}_j$ and change of variables $\bar{K}_i = P\bar{L}_i$, $\check{K}_j = Q\check{L}_j$, $\Omega_1 = G_1^{-1}$ and $\Omega_2 = G_2^{-1}$ where Ω_1 and Ω_2 are diagonal and positive definite matrices. Hence, the gains of the interconnected observer are computed from the LMI conditions given in the following theorem. ■

Remark 4: The previous optimization problem is a set of two independent conditions from the dynamics point-of-view. The sufficient condition given above is less conservative than the Small Gain Theorem [18].

Remark 5: As for affine nonlinear systems case, the previous optimization problem is a set of two independent conditions and may be rewritten in separate optimisation problem for each sub-system. The Small Gain Theorem is not needed.

Remark 6: Notice that the approach requires a number of outputs greater than one. But it is possible to decompose the system into different sub-systems, some of them are observable and others detectable. In such a case it is also possible to design the observer.

Remark 7: The presented result in this paper aims to illustrate the approach based on interconnected systems for observer design. The stability is then studied by a common Lyapunov function. However, one can use different Lyapunov functions in order to establish less conservatism LMI conditions.

These approaches are illustrated in the following numerical examples, where a comparison between both observers (TS and interconnected one) is given.

V. Simulation results

A. Numerical example

Consider the example of double inverted pendulum given by equation (8). A simulation behavior of the system is illustrated by Fig. (1).

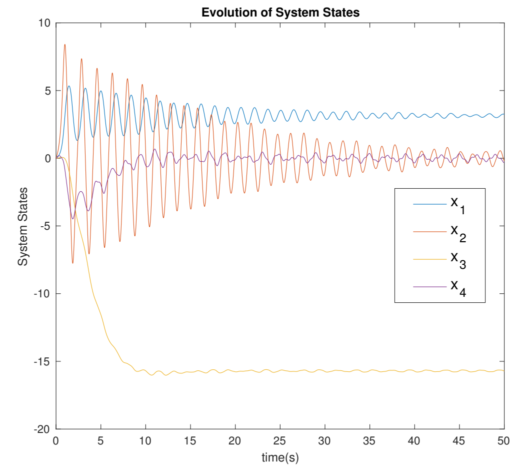

Fig. 1. Evolution of System States

In order to express the Takagi-Sugeno form of the system, the number of nonlinearities is analyzed. In fact, number of nonlinearity of $E(x)$ is one, hence sub-models are about $p_e = 2^1$. In the same way, number of nonlinearity of $A(x)$ is two, hence sub-models are about $p = 2^2$. Indeed, the system dynamics accepts about $p_1 = p \times p_e = 2^1 \times 2^2 = 8$ sub-models or LMI problems to solve. The previous system is separated into two interconnected dynamics as follows (Fig. 2):

$$\Sigma_1 : \left\{ \begin{array}{ccc} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & ag\sin(x_1) + a\cos(x_1)z_2 + \frac{1}{2c}u \\ y_x & = & x_1 \end{array} \right. \tag{25}$$

and,

$$\Sigma_2 : \left\{ \begin{array}{ccc} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & -z_2 + bx_2(\cos(x_1) - \sin(x_1)) + \frac{1}{c}u \\ y_z & = & z_1 \end{array} \right. \tag{26}$$

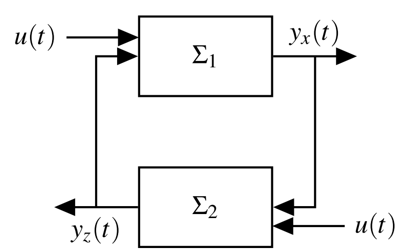

Fig. 2. Interconnected systems Σ_1 and Σ_2.

In order to express the Takagi-Sugeno form of the system, number of nonlinearities is analyzed. One can state that the E_i matrices of each sub-models are identity matrix. In

Parameters		
g	=	9.81
m_{ch}	=	0.195
L_b	=	0.4
m_b	=	0.095
a	=	$3/(4 * L_b)$
b	=	$m_b * L_b/(m_b + m_{ch})$
c	=	$1/(m_b + m_{ch})$;

TABLE I
SYSTEM PARAMETERS

addition, the interconnection of TS sub-models gives the exact nonlinear system.

Also, following the interconnection of the two systems we remark that the system is still nonlinear with respect to x allowing 4 sub-models (2 nonlinearities). However, the second sub-model is linear with respect to z. Hence, the whole system dynamics presents, under interconnection outline, 5 sub-models and the corresponding LMI constraints to solve.

1) Observer analysis: It is easy to see that the previous example shows that the system is completely observable.

By using *Sedumi* & *Yalmip* [19] LMI toolbox, one can solve the problem and find the observer gains as follows:

$$L = \left[\begin{array}{c} L_1 \\ L_2 \\ L_3 \\ L_4 \\ \hline L_5 \end{array}\right] = \left[\begin{array}{cc} 5.8081 & 46.5631 \\ 5.8081 & 18.9725 \\ 5.8081 & 46.5631 \\ 5.8081 & 18.9725 \\ \hline 1.6400 & 2.3675 \end{array}\right]$$

Matrices G_i, equations (23) and (24), are:

$$G_1 = \begin{bmatrix} 1.0007 & 0 \\ 0 & 1.3880 \end{bmatrix}, \text{ and } G_2 = \begin{bmatrix} 1.0007 & 0 \\ 0 & 0.0256 \end{bmatrix}$$

Taking $x_0 = [0.5\ 0.3]^T$ and $z_0 = [0.3\ 0.6]^T$, simulation results are shown in Fig. 3.

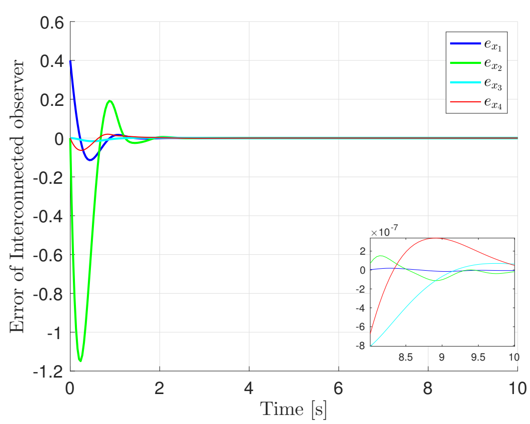

Fig. 3. State estimation error with the Interconnected TS Luenberger observer

Fig. (4) shows results of a classical Luenberger TS observer (gains are given in the appendix VII-A), synthesized under the same conditions as before, with observer gain of dimension 4×16. The new approach presents almost the same performance with a faster convergence than the classical technique.

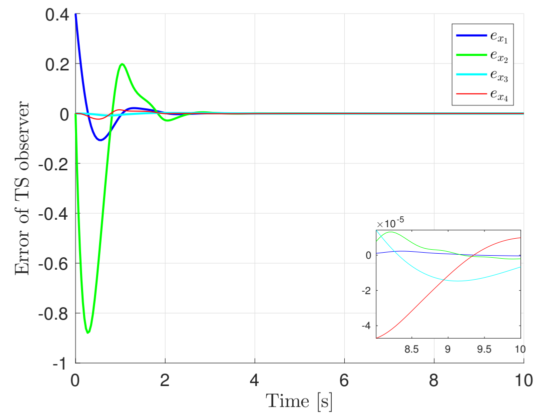

Fig. 4. State estimation error with TS Luenberger observer

B. Conservatism study

The observability of the exemple above was verified with theorem 1 and for $a \in [0 \quad 200]$, $b \in [0 \quad 200]$. Fig. (5) shows that the feasibility region of the optimisation problem (15) is larger than the region given by a classic TS observer. Thus, the results presented here are less conservative.

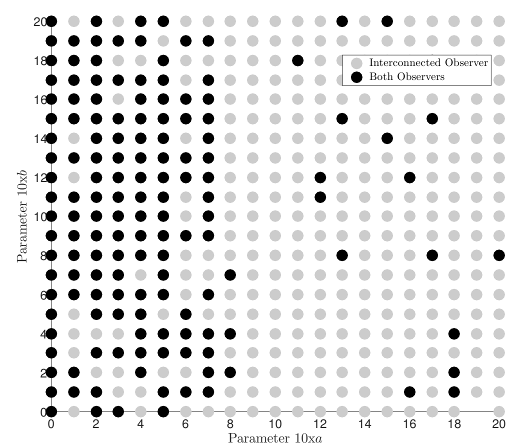

Fig. 5. Conservatism analysis with both obsrervers : TS observer (black circles) and Inerconnected observer (black and gray circles).

C. Result Discussion and Analysis

Results of Fig. 3 and 4, show the states estimation for the both observer approaches. Under the interconnected formalism, the performances are much better: number of gains is six times less than for the classical design, less conservatism as shown previously and rapid convergence. For many cases, the optimisations problem is not feasible for Takagi-Sugeno representation due to the huge number of LMI's.

The estimated states with both observers are quite different, and we state cannot which method is more accurate. Table V-C gives some objective comparisons based on the sensitivity of the estimation approach. Three simulations were performed with different initial conditions ($x_{01} = [0.5\ 0\ 0\ 0]$ and $x_{02} = [-0.5\ 0.8\ 1\ -1]$) and some of parameters set ($a_1 = 1.87,\ b_1 = 3.13,\ a_2 = 10,\ b_2 = 40$). In this last case, the optimisation problem for the classical TS observer is not feasible.

The estimated states are compared with their corresponding data by means of the Root Mean Square Error (RMSE) to show how close are the estimated values to the real states. The mathematical formulas, are defined below:

$$\mathrm{RMSE_j} = \sqrt{\frac{\sum_{j=1}^{n} (x_j - \hat{x}_j)^2}{n}} \tag{27}$$

where x_j is the real state containing n data and $\hat{x}_j$ is its estimate provided by both observers.

The resulting RMSE for the conducted simulations are shown in table (1). One can state that the RMSE for both observers are close and generally better for the proposed approach.

States	RMSE	a_1, b_1 and x_{01}	$a_1 = 1.87$, b_1 and x_{02}	a_2, b_2 and x_{01}
x_1	TS Obs.	$5.36\ 10^{-2}$	$9.75\ 10^{-2}$	unfeasable
	Inter. Obs.	$3.69\ 10^{-2}$	$7.78\ 10^{-2}$	12.38
x_2	TS Obs.	$5.9\ 10^{-1}$	1.02	unfeasable
	Inter. Obs.	$7.42\ 10^{-2}$	1.01	0.58
x_3	TS Obs.	$7.3\ 10^{-3}$	$2.09\ 10^{-1}$	unfeasable
	Inter. Obs.	$1.45\ 10^{-2}$	$1.4\ 10^{-1}$	23.94
x_4	TS Obs.	$6.22\ 10^{-4}$	1.99	unfeasable
	Inter. Obs.	$2.36\ 10^{-2}$	1.908	1.55

TABLE II

RMSE PERFORMANCE STUDY

VI. CONCLUSION

In this paper, an approach based on transforming a nonlinear system into interconnected Takagi-Sugeno sub-systems is proposed for state observer design. The idea is to reduce the number of vertices in the polytope and to reduce the conservatism related to the number of LMIs. As illustrated in the example, expressing a nonlinear system may lead to interconnected Takagi-Sugeno sub-systems with less number of vertices for each one. Almost decoupled LMI conditions have been proposed in order to ensure the asymptotic state estimation error convergence for a number of sub-systems with reduced number of vertices for each one. Numerical results confirm the simplification offered by the proposed result compared to the classical approach. Future results will concern the use of the descriptor modeling and extension to uncertain and unknown input nonlinear systems.

REFERENCES

[1] T. Guerra, V. Estrada-Manzo, and Z. Lendek, "Observer design for takagi-sugeno descriptor models: An LMI approach," *Automatica*, vol. 52, no. 0, pp. 154 – 159, 2015.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, P. SIAM, Ed. SIAM ed., 1994.

[3] K. Tanaka, T. Ikeda, and H. Wang, "Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs," *IEEE Transactions on Fuzzy Systems*, vol. 6, no. 2, pp. 250–265, 1998.

[4] T.-M. Guerra, A. Kruszewski, L. Vermeiren, and H. Tirmant, "Conditions of output stabilization for nonlinear models in the Takagi-Sugeno's form," *Fuzzy Sets and Systems*, vol. 157, no. 9, pp. 1248–1259, May 2006.

[5] K. Tanaka and H. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. John Wiley and Sons, 2001.

[6] A. Akhenak, M. Chadli, J. Ragot, and D. Maquin, "Design of sliding mode unknown input observer for uncertain Takagi-Sugeno model," in *15th Mediterranean Conference on Control and Automation, MED'07*, Athens, Greece, 2007.

[7] ——, "Fault detection and isolation using sliding mode observer for uncertain Takagi-Sugeno fuzzy model," in *16th Mediterranean Conference on Control and Automation Congress Centre*, Ajaccio, France, June 25-27 2008.

[8] A. Akhenak, "Conception d'observateurs non linaires par approche multi-modle: application au diagnostic," Ph.D. dissertation, Institut National Polytechnique de Lorraine, Nancy, France, 2004.

[9] M. E.-H. Dabladji, D. Ichalal, H. Arioui, and S. Mammar, "Unknown-input observer design for motorcycle lateral dynamics: Ts approach," *Control Engineering Practice*, vol. 54, pp. 12–26, 2016.

[10] D. Ichalal, S. Mammar, and J. Ragot, "Auxiliary dynamics for observer design of nonlinear TS systems with unmeasurable premise variables," in *4th IFAC International Conference on Intelligent Control and Automation Sciences, ICONS 2016*, vol. 49, no. 5, Reims, France, Jun. 2016, pp. 1–6.

[11] D. Ichalal, B. Marx, J. Ragot, and D. Maquin, "Advances in observer design for takagi-sugeno systems with unmeasurable premise variables," in *20th Mediterranean Conference on Control Automation (MED)*, july 2012, pp. 848 –853.

[12] L. Nehaoua, D. Ichalal, H. Arioui, J. Davila, S. Mammar, and L. M. Fridman, "An unknown-input hosm approach to estimate lean and steering motorcycle dynamics," *IEEE Transactions on Vehicular Technology*, vol. 63, no. 7, pp. 3116–3127, 2014.

[13] A. Sala and C. Arino, "Relaxed stability and performance LMI conditions for Takagi-Sugeno fuzzy systems with polynomial constraints on membership function shapes," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 5, pp. 1328–1336, 2008.

[14] A. Sala and C. Arinõ, "Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem," *Fuzzy Sets and Systems*, vol. 158, no. 24, pp. 2671–2686, 2007.

[15] Z. Duan, L. Huang, L. Wang, and J. Wang, "Some applications of small gain theorem to interconnected systems," *Systems & Control Letters*, vol. 52, no. 3-4, pp. 263 – 273, 2004.

[16] L. Xie, "Output feedback H_∞ control of systems with parameter uncertainty," *International Journal of Control*, vol. 63, no. 4, pp. 741–750, 1996.

[17] M. Abbaszadeh and H. Marquez, "Robust H_∞ observer design for a class of nonlinear uncertain systems via convex optimization," in *American Control Conference, ACC*, 2007.

[18] M. Green and D. J. N. Limebeer, *Linear Robust Control*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1995.

[19] J. Lofberg, "YALMIP : a toolbox for modeling and optimization in MATLAB," in *IEEE International Conference on Robotics and Automation*, Sept 2004, pp. 284–289.

VII. APPENDIX

A. Takagi-Sugeno Observer gains

$$L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_7 \\ L_8 \\ L_9 \\ L_{10} \\ L_{11} \\ L_{12} \\ L_{13} \\ L_{14} \\ L_{15} \\ L_{16} \end{bmatrix} = \begin{bmatrix} 4.7450 & 35.5068 & 0.0053 & -1.0263 \\ -0.0044 & -0.0537 & 0.4653 & 1.6914 \\ 4.7450 & 7.9162 & 0.0053 & -1.0263 \\ -0.0044 & -0.0536 & 0.4653 & 1.6914 \\ 4.7364 & 35.4386 & -0.0320 & 0.8057 \\ -0.0045 & -0.0539 & 0.4655 & 1.6835 \\ 4.7364 & 7.8478 & -0.0320 & 0.8056 \\ -0.0044 & -0.0536 & 0.4655 & 1.6837 \\ 4.7561 & 35.5496 & 0.0053 & 0.0053 \\ -0.0293 & -0.1492 & 0.4654 & 1.6921 \\ 4.7561 & 7.9588 & 0.0053 & -1.0266 \\ -0.0295 & -0.1497 & 0.4654 & 1.6922 \\ 4.7476 & 35.4814 & -0.0321 & 0.8054 \\ -0.0293 & -0.1490 & 0.4656 & 0.4656 \\ 4.7476 & 7.8907 & -0.0321 & 0.8054 \\ -0.0294 & -0.1493 & 0.4656 & 1.6843 \end{bmatrix}$$