Conceiving teaching trajectories in the form of series of problems: a step for the theoretical reconstruction of the Hungarian Guided Discovery approach

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In this article I introduce the term "series of problems" to give an account on a phenomenon existing in Hungarian mathematics education, the conception of long teaching trajectories based on problems. I will present a long research process aiming to reveal a "theory in act" hidden behind this phenomenon, and discuss the methodological tools developed in order to access to these hidden theoretical bases. The first part of the paper presents the historical-epistemological-didactical analysis of the origins of the Hungarian approach in question; the second part discusses the analysis of teachers' work, and introduces the idea of reverse engineering aiming to reveal the implicit principles to the teachers' design practice.

Keywords: Series of problems, Guided Discovery approach, Reverse engineering, Teaching trajectories, Hungarian mathematics education

Introduction

In Hungary, there exists a tradition of mathematics education through problem solving and mathematical inquiry called "felfedeztető matematikaoktatás" that I translate as "Guided Discovery" (GD) approach. This approach was originally developed and practiced for the education of gifted students, but the reform movement of T. Varga, realized in the 1960s and '70s, transformed it into a basis of general mathematics education and a guiding principle of the official curriculum for primary and lower secondary school (Halmos & Varga, 1978). The approach is still recognized by Hungarian specialists of mathematics education as relevant and coherent with modern educational trends. At the same time, it is practiced in ordinary classes only by a narrow circle of teachers, and its dissemination poses problems. Furthermore, the approach exists only "in act;" its theoretical description has never been developed. Although Varga actively participated in the international discourse on mathematics education in the '60s and '70s, and developed his conception at the same time as Brousseau or Freudenthal[1], his publications were mostly focused on experimentation and curriculum development, and less on the theoretical description of the underlying conception.

This article treats a step of a longer research project aiming to reconstitute the underlying theory of Varga's approach. I consider this reconstruction necessary for several reasons. On the one hand, it is an inevitable first step for its large-scale implementation, for the development of more adapted resources and teacher education programs. On the other hand, it is necessary to introduce Varga's work to the current international discussion on Inquiry Based Mathematics Education: to enrich

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[1] Varga and Freudenthal met quite late when both of their conceptions were already well developed, but they mutually recognized each other's work and perceived conceptual proximities (Varga, 1975).

both the international discourse by the specificities of Varga's approach and the Hungarian mathematics education by the dialogue with other approaches.

In this paper, I focus on one important aspect of the Hungarian approach: the long and complex teaching trajectories shaped by series of problems. The concept of Series of Problems appears to be crucial in this approach, both for curriculum and resource-developers and for teachers (Gosztonyi, 2015a). However, a theoretical understanding of this phenomenon is missing. Originally, the term "series of problems" was introduced in the frame of an interdisciplinary history of science project as a more or less undetermined methodological tool to analyze the structure of historical texts (Bernard, 2015). In this specific Hungarian context, by "series of problems" (SoP) I mean a list of problems with a conscious ordering which is relevant for an (or several) educational purpose(s). In what follows I will discuss the relevance of this notion for the Hungarian context, the difficulties to develop theoretical frames and methodological tools for the analysis of SoP, and the research process of this development.

In the first part of the article, I briefly summarize how a combined, historical-epistemological-didactical analysis of Varga's reform contributed to reveal the main principles of the GD approach. In the second part, I present an ongoing analysis of nowadays teachers' work, developed in the frame of a larger research project focused on Varga's legacy (MTA-ELTE project)[2]. I suggest to describe the methodology of this analysis as *reverse engineering*[3]: a progressive elucidation of the hidden principles of this "theory in act", starting from teacher's actual designs.

Back to history

The first step for the reconstruction of the GD approach was to go back to its origins, in the frame of a comparative study between Varga's reform and the contemporary French "Mathématiques modernes" reform, carrying out their combined, historical, epistemological and didactical analysis, based on written sources (Gosztonyi, 2015b). The analysis of the historical context of the reforms pointed out, among other things, the crucial role of mathematicians in defining the epistemological foundations of mathematics and its teaching: a "bourbakist" epistemology in the French case, and a "heuristic" one in the Hungarian case. The didactical analysis of different sources (curricula, textbooks, teachers' handbooks) showed the epistemology be one of the main driving forces for realizing the reforms.

One of the theoretical frameworks used for the didactical analysis was Brousseau's *Theory of Didactical Situations* (1998). This theory proved to be powerful to reveal important characteristics of each reform – but also insufficient to explain some specificities of Varga's one. One of the most important aspects is the long teaching trajectories shaped by SoP.

Brousseau developed his theory from the 1970s, in the French context of the debates following the "Mathématiques modernes." Thus, the emergence of the theory goes back to the historical context I

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[2] MTA-ELTE Complex Mathematics Education Research Group, working in the frame of the Content Pedagogy Research Program of the Hungarian Academy of Sciences (ID number: 471028).

[3] I am grateful to Arthur Bakker for suggesting this term during the conference. See also (Calderon, 2010).

studied. The comparison of an early experimentation of Brousseau with Varga's works contributed to understand their conceptual differences.

Characteristics of the Hungarian Guided Discovery approach and the importance of SoP

The analyses described above helped to identify some main characteristics of the GD approach. In the background of Varga's reform project, a quite coherent "heuristic" epistemology of mathematics and of mathematics education can be outlined in the 20th century's Hungarian mathematical culture (Gosztonyi, 2016), represented by mathematicians living in Hungary as well as by Pólya and Lakatos. These thinkers considered mathematics to be a continuously progressing and changing human creation, and as a social, dialogic activity. The development of mathematics appeared as a collective process based on a diversity of experience, driven by problems, attempts of solutions, and new questions emerging from these attempts. They posited that mathematical knowledge only could be attained through the experience of mathematical creation, and thus students should be guided through similar discovery processes.

The idea of organizing problems into ordered series appears often in the writings of these authors. For example, R. Péter's book popularizing mathematics, *Playing with infinity* (Peter, 1961) is written as an interesting inquiry story where one question invites the other. Pólya's work, especially later books like *Induction and analogy* (1954), also present inquiry processes shaped by series of problems, and analyses their structure.

This epistemology deeply influenced Varga's "Complex Mathematics Education Reform" conception (Gosztonyi, 2015b). The coherence of his curriculum is assured by a parallel treatment of different mathematical themes. Elements of mathematical knowledge are introduced through long, progressive processes (taking often several years), based on numerous small, concrete problem situations. Problems crossing different themes contribute to the variety of experiences. In the treatment of these problem situations, collective discussions led by the teacher play a crucial role. The use of various manipulative tools and representations, playfulness, students' autonomy and creativity are also in the focus of the reform project.

In the teacher's handbooks related to Varga's reform, researchers can find numerous commentaries on the importance of organizing problems into series. Primary school teachers' handbooks describe a number of activities with ideas for their repetitions, variations, and ordering. However, the principles of the organization are dispersed in the books and rarely generalized. The handbooks insist on the importance of teachers' autonomous work on conceiving of long teaching processes on their own. In middle-school textbooks, new knowledge is often introduced by fictive dialogues of students around short series of problems.

Analysis of published examples

Although SoP play a central role in Varga's conception, they are seldom theorized. Problem collections used to be presented without commentaries about tasks' connections and their possible roles in a teaching process. Furthermore, mostly fragments of SoP are published; long-term series are rarely presented. Some specific historical documents, however, present more detailed examples and furnish enough commentary to understand the logic and the purpose of their construction.

One example is the above mentioned book of R. Péter, describing in one of the chapters a classroom situation driven by a dialectic of problems, attempts of solutions and new problems motivated by these attempts. The text is written in a literary style; thus, its structuration by problems is not obvious at the first glance, but the rich meta-discourse makes possible to identify the problems and to understand the rationale behind the structuring of the text (Gosztonyi, 2015a).

In the documentation of Varga's reform, one of the most transparent examples is the introduction of combinatorics in primary school. Here, a double structuration of the series can be identified. On one hand, a sub-series is related to a given material, varying several variables and progressively introducing restrictions (e.g., building colored towers, building colored towers with a given height, finding all towers with a given height, varying the number of colors and levels, using a color only once, using the same color several times but not for neighbor levels, and so on). On the other hand, same or similar tasks are repeated with different materials (e.g., building towers with Cuisenaire rods, coloring flags or drawing houses, creating words with given letters, writing melodies with musical notes, and so on). The ordering is not strict here, but a progression in the abstraction should be respected by teachers. The observation of the analogies between these tasks serves as a basis for generalization and abstraction.

The comparison of this example with Brousseau's above mentioned teaching progression revealed important differences between the two authors' conceptions, and contributed to understanding the role of SoP in the Hungarian case. For Brousseau, the progression is based on one "fundamental problem situation," and a decontextualization (through phases of institutionalization) is necessary for the passage to other situations. For Varga, the progression is based on a diversity of problems and their convenient ordering to favor progressive generalization through the recognition of links between the problems. (Gosztonyi, 2017).

In summary, the analysis of written documents representing the heritage of the Hungarian GD approach, confirms the importance of SoP in the planning of teaching by GD, identifies some key principles concerning the role and the structuration of SoP, and highlights tools to analyze and represent SoP (e.g., by graph representations or the structuration by main and sub-series).

Analysis of teachers' work designing Series of Problems

Subsequent research steps involve analyzing the GD approach of expert teachers, drawing on the key principles and analytic tools described above. Several of these teachers reacted vividly when I proposed the term "Series of Problems" to them. It resonated with the kind of planning they considered crucial. However, they rarely make these practices explicit.

The documents created by these expert teachers typically contained a list of problems, with or without solutions. But the documents are only a skeleton of their planned teaching process. Essential elements remain implicit: the main teaching goals, the organization of classroom work, the planned discussions, and choices and adaptations they consider to make 'on the spot', depending on the students' reactions (e.g., change of order, omission of some problems or introduction of new ones). Furthermore, these projects are conceived as long-term developments, even for several years, and only parts of them appear in written documents. Expert teachers confirm having a complex network of problems in mind that they can mobilize in order to create problem series and to adapt them to their aims and to the students' needs or interest–during preparation or in the classroom.

Gaining access to these complex implicit networks is necessary but quite challenging. Its reconstruction is an iterative process, combining two approaches: (1) an a priori analysis on the principles revealed by the analysis of historical documents, and (2) a study of the expert teachers' documentational processes (in the sense of Gueudet & Trouche, 2010).

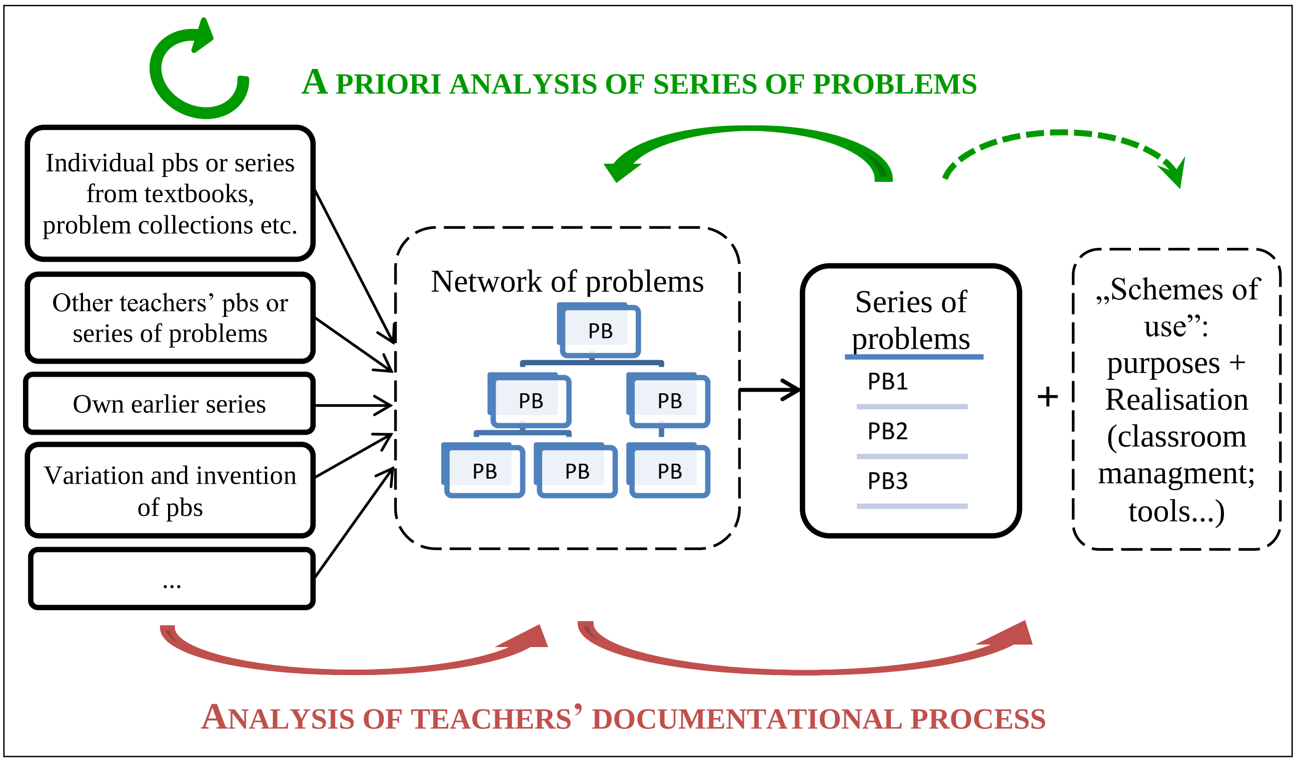

Figure 1: Analysis of Series of Problems

(the visible elements are marked with bold, the hidden ones with broken outline)

For the MTA-ELTE project, a mixed group of researchers, teacher trainers and expert teachers of the GD approach was created in order to fulfill a double aim: analyzing the expert teachers' practices (with a special focus on SoP), and fostering dissemination of the approach by developing resources and teacher education programs. However, complex negotiations were necessary to meet the interest and expectations of each participant. While I felt necessary to focus first on the analysis of teachers' practices, this was not motivating and relevant enough for the participating teachers. Thus, we decided to develop a collection of commented examples of SoP. The purpose of this collection under preparation was not to offer ready-made resources for teaching, but generic examples, which introduce the reader into the practice of using and preparing SoP. In addition, preparing commented examples also served the project's research aim: for teachers to explicate their teaching purposes and principles of structuration of SoP. I call this procedure *reverse engineering*. Starting from teachers' existing designs of SoP, researchers reconstruct tacit principles driving teachers' choices during the design of these trajectories.

In the first step, teachers were asked to provide examples from their own practice of what they consider SoP. The received examples were of quite different nature: SoP of short and long-term sequences; SoP to introduce new mathematical notions, SoP to explore heuristic strategies, and SoP to serve diagnostic purposes. Confronting the new and previously analyzed examples led to discussion on various questions: classification of purposes, different structuration principles, and forms of representation. These discussions evoked new examples. For example, most SoP consisted of short sequences at the beginning, because this was easier to document for teachers; but during the discussions, they all agreed about the importance of long-term planning, leading to the description of long trajectories built by SoP.

In the next step, teachers were asked to write commentaries to their SoP, addressing colleagues unfamiliar with the GD approach, focusing on the didactical aims of their trajectories and on explaining the SoP-structure. The group is currently comparing these first writings in order to define the main elements to be included in each commentary.

In the further phases of the project, the MTA-ELTE project team will conduct experiments; giving commented examples to non-expert teachers, and asking them to use the examples as inspiration to develop their own teaching trajectories, and offer them consultation with expert teachers of the group. This part of the project will give evidence of how well the created resources may mediate the GD-approach and provoke further exploring of the expert teachers' practices.

Eszter's case

Eszter is an expert teacher of our group, a high school teacher for 20 years. As a long-term example, she proposed an example taken from geometry[4], built around the concepts of incidence and distance in the Euclidean plane/space, equidistant sets (loci) of points, and especially a "core" problem: finding the equidistant points from three different lines in the plane. This series covers a very long period: It starts at the first year of high school, and some parts of it concern the last year's curriculum. Although some notable theorems of higher level mathematics appear in the series (in a problematized form), the aim of the SoP, according to Eszter, is not to learn these theorems, but to familiarize students with the above mentioned concepts, related mathematical techniques and problem solving skills through the exploration of interesting geometrical questions.

The first version consisted of a "core" series of four problems with increasing complexity about the loci of points (red dots in Figure 2). In this "core sub-series", the order of the problems is strict. This order is intended to make the latter, more difficult problems progressively available for students. Several other sub-series are conceived, based on this core, exploring different mathematical themes (blue dots), including space

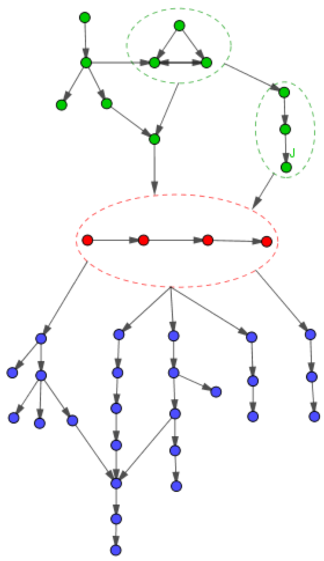

Figure 2: Eszter's graph

(Simplified version)

[4] The complete collection of problems can be found here:
https://drive.google.com/open?id=1Nhe5DbX3Wgs29l75GhTlrGN8_5rs1-yl

analogies, angles and similarities, and the boundaries of geometry and combinatorics. Eszter immediately proposed a graph to represent the structure of the SoP. Her arrows showed complex mathematical or didactical interdependences between the problems, but also illustrated the flexibility of the structure which is not visible in the simple numerated list of problems. Some problems or sub-series are sequential, while for other problems, the order can be freely chosen. The representation of this structure helps the reader to understand which problems could be deleted without impacting a longer trajectory, or where new problems could be inserted.

After a first discussion, we agreed that it would be useful to complete this SoP with a preparatory part which treated the lower secondary school level prerequisites. Thus, Eszter added a new part to her graph (green dots in Figure 2). As she explained, the problems of this part serve for diagnostic purposes in her high school work. They also fill possible gaps in students' knowledge. The problems presented illustrate different types of tasks which have to be explored in order to prepare students for the "core." More tasks can be inserted, depending on the students' needs. In order to make this "typology" more accessible, Eszter added a coding system to the graph (e.g. D stands for distance, L for loci of points, S for space analogies). Some subgroups of problems are related to some of these codes, but many problems have multiple codes. The network of problems is less hierarchic here than in other parts leaving more flexibility to the teacher. However, the discussion also revealed some further hidden interdependences of the ordering. This suggests that Eszter's didactical choices are even more complex than what was in her current version of the graph. Another iterative process of discussions and the preparation of representations is necessary to reveal implicit choices.[5]

The discussions with Eszter led to a progressive exploration of her design and its guiding choices, while the graph representation plays the role of a mediator. Her case also shows individual specificities beyond the common principles shared with the colleagues (e.g. Eszter accords even more importance to the flexibility in her SoP than most of her colleagues).

Conclusion

This paper presented a research process aiming to make explicit a "theory in act" existing in the Hungarian context of mathematics education and to develop theoretical bases and methodological tools to analyze the work of the teachers who follow this approach. The first phase of this process consisted in the analysis of the historical sources of the approach with complex research tools from history, epistemology and didactics, while the current second phase is focused on teachers work. The process is essentially iterative; the analysis of the epistemological background offered some first steps to reveal main principles of the hidden theory. The use of existing didactical theories in the analysis of Varga's reform documents helped not only to reveal its characteristics but also to point out the limits of the use of those theories, specificities of the GD approach which needs new theoretical development to be analyzed. These studies helped to choose the aspects to focus on while analyzing teachers' work—the structuration of teaching trajectories by SoP being one of them. The principles revealed by these analyses and the tools developed by case studies of

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[5] A more detailed version of the graph was presented in form of a poster at CERME 11.

representative written examples contribute to interpret the teachers' designs but also to provoke teachers to make their guiding principles explicit. Collective discussion and interpretation of the teachers' designs led to further elucidation of the fundamental principles of the approach and to the development of tools to investigate their design process. The term "reverse engineering" is used to describe the iterative process of exploring teachers' designs and describing their guiding design principles. Through reverse engineering, "theories in act" about SoP will be gained, so they may contribute to the discourse on Inquiry Based Mathematics Education (Artigue & Blomhøj, 2013).

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