# How concepts turn into objects: an investigation of the process of objectification in early numerical discourse

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*Mathematical discourse begins where the concrete ends. Despite the intangibility of mathematical objects, we still speak about them in a way that is quite similar to the way in which we speak about material objects. We speak of numbers, classes, and functions as if we were talking about self-sustained entities. But with what right? On the example of the use of number-words in early numerical discourse, I shed some light on the mechanisms by means of which mathematical objects obtain their peculiar ontological status. It turns out that any mathematical object refers back to a concept from which it originates. To better understand the process of objectification, I am thus asking: how do concepts turn into objects? By juxtaposing Piaget's notion of reflecting abstraction with Peirce's notion of hypostatic abstraction, I then propose an explanatory model of the transition from concepts to objects.*

*Keywords: Objectification, reflecting abstraction, Piaget, hypostatic abstraction, Peirce*

## I. Introduction

| | | |
|---|---|---|
| 1 | MH: | How many is two and one more? |
| 2 | Patrick: | Four. |
| 3 | MH: | Well, how many is two *lollipops* and one more? |
| 4 | Patrick: | Three. |
| 5 | MH: | How many is two *elephants* and one more? |
| 6 | Patrick: | Three. |
| 7 | MH: | How many is two *giraffes* and one more? |
| 8 | Patrick: | Three. |
| 9 | MH: | So how many is *two* and one more? |
| 10 | Patrick: | Six.[1] |

Although Patrick (4 years and 1 months) seems to have no problem in recognizing the number-words in the posed questions, he is for some reason unable to cope with them in the first and last question. How can it be that he does not see the pattern common to both questions? What seems unfamiliar to him are not the number-words themselves, but it is the way in which these words are used. In general, we can distinguish between two different uses of number-words (cf. Dummet, 2002, pp. 99–100): Number-words can either occur as adjectives, as in ascriptions of number such as 'There are *two* giraffes', or as nouns, as in most arithmetic propositions such as '*Three* is a prime number'. In their adjectival use, number-words function as predicates. They cannot stand for themselves, but they characterize something else. In that use, a number-word specifies how many there are of a certain kind and thus it must occur as the number of something, or more precisely as the number of a concept ('three *lollipops*', 'three *giraffes*', 'three *elephants*'). In contrast, number-words that occur as nouns stand for something that can be investigated in its own right. In this substantival use,

[1] 'MH' stands for Martin Hughes. The transcript is taken from Hughes (1986, pp. 47–48).

numbers are regarded as self-sustained entities without any intrinsic reference to the concepts which they are possibly ascribed to. Armed with this distinction, we are now able to formulate Patrick's difficulties in a more concise way: While Patrick seems to be familiar with the adjectival use of number-words (see lines 3–9), the substantival one appears foreign to him (see lines 1–2 and 9–10). Consequently, he does not see addition as an operation between numbers as abstract objects, but only as an operation performed on two concepts. To him, adding up means to join two collections of objects. Taking this into consideration, it becomes understandable why he shows difficulties grasping the first question. From an adjectival standpoint, it is of no importance which kinds of objects are conjoined (lollipops, giraffes, elephants, etc.) as long as there are some. An addition cannot be performed *in vacuo*. The question 'How many is two and one more?' is thus incomplete for Patrick, he feels that there is something missing, namely a pair of concepts with which the operation is executed. These observations gain in importance as soon as it is added that they are by no means a personal exception but rather are typical of many preschool children aged 3 to 4 years (cf. Hughes, 1986, p. 46; Sfard, 2010, p. 47). It is the result of an intricate process of successive steps of objectification that numerals and number-words can be treated as referring to the self--sustained entities we call numbers (cf. Sfard, 2010, p. 136). Within this process, the adjectival use of number-words precedes the substantival use, or bluntly put: *numbers are first concepts, and only become objects in a second step.*

This developmental sequence, however, is not restricted to the genesis of numbers. Rather, it repeats itself at all levels of mathematical sophistication. For example, Bakker (2007) observed in a teaching experiment on the concept of distribution in an 8th grade in the Netherlands that the students' discourse about the concept of spread took a similar turn:

> [M]any used the noun 'spread,' […], whereas students earlier used only predicates such as 'spread out.' […] Statistically, however, it makes a difference whether we say, 'the dots are spread out' or 'the spread is large.' In the latter case, spread is an object-like entity that can have particular aggregate characteristics that can be measured (for instance by the range or the standard deviation) (Bakker, 2007, p. 24)

Or, to provide a historical example: It took several centuries to move from the study of specific examples of functions to the study of functions as such, that is, to the investigation of functions as objects for which certain general characteristics such as continuity or differentiability can be formulated (cf. Kleiner, 1989; Sfard, 1991, pp. 14–16). The departure point of this article is the conviction that what we have observed in the case of number can be generalized: *At any level of mathematical sophistication, mathematical objects are products, i.e., they are produced in processes of objectification, and at any level of mathematical sophistication, the mathematical objects in question refer to concepts from which they originate.*

In this paper, I take early numerical discourse as an example to investigate the transition from concepts to objects as a key element of the *process of objectification* (in the sense of: Sfard, 2010). I attempt to provide a model of the mechanisms which underlie this last step in process of objectification. The rationale of the paper is thus a theoretical one and it is precisely here where the novelty of this contribution lies: There are plenty of works in mathematics education (cf. Bakker, 2007; Sfard, 1991) in which it is described *that* this transition happens, but I do not know a single one that provides an explanatory model of *how* it could actually work. In order to provide such a model, I pursue the following plan: At first, I clarify the use of the words ‘object’ and ‘concept’ in order to formulate the theoretical reference-problem as clearly as possible (II.); in the next two sections, I then successively explore the adjectival (III.) and the substantival use of number-words in more detail (IV.); this leads to an explanatory model of the transition from concepts to objects which is developed in the fifth section by juxtaposing Piaget's notion of reflecting abstraction with Peirce's notion of hypostatic abstraction (V.).

## II. On the distinction between concept and object

When speaking about material objects, we either use terms that tell us which individual is being talked about, or we use terms that characterize something as an instance of a certain kind. ‘Socrates’, ‘Berlin’, and ‘Donau’ are examples for the former kind of terms, while ‘human being’, ‘city’, and ‘river’ are examples for the latter. To adopt a usual phraseology, we will call these two kinds of terms *singular* and *general* terms. A singular term “purports to name one and only one object” (Quine, 1966, p. 205), while a general term does not name the individual object to which it refers at all but, instead, is said to be true of each of many objects of a certain kind (ibid.). The general term ‘human being’, for example, is true of the object *Socrates*, but it is not true of the object *Berlin*. Singular and general terms thus differ in their *way of reference*: While a singular term refers to one object and one object only, a general term divides its reference. It can always be applied to a multiplicity of objects. What a general term names is thus not any particular object, but at best its own rule of application, i.e., the operation used to decide whether or not the term is true of an object. We want to call these operations concepts: *A concept is any operation that is associated with a general term and that allows to decide whether or not the term is true of a particular object*.

Therefore, concepts are not to be found in the realm of being but belong to the realm of doing. We operate *upon* objects (as they are singled out for our investigation), while we operate *through* concepts (as it is the conceptual operation by means of which the general term is applied to a particular entity). In contrast to everyday language, objects to which mathematical terms refer are never components of our stream of direct experience. Mathematical discourse begins where the concrete ends. We cannot point at or see mathematical objects such as numbers, classes, or functions. Despite their intangibility, however, we still adopt a jargon that is quite similar to the way in which we speak about material objects. On the one hand, there are singular terms such as ‘$1+4$’, ‘$\sqrt{2}$’, or ‘$\pi$’, which purport to name one and only one object, on the other hand, there are general terms such as ‘prime number’, ‘continuous’, or ‘equilateral’, which characterize the objects in question as instances or non-instances of a certain concept. In mathematical discourse, the distinction between singular and general terms thus remains intact (see already Peano, 1891/1973, p. 156). The difference between mathematical terms and everyday terms is not the way of reference, *but the kinds of objects referred to*.[2] In the case of everyday language, the terms are mostly concrete in the sense that the objects referred to can be investigated as parts

[2] For a more detailed discussion of what it is that makes a concept a mathematical concept, see also: De Freitas, Sinclair & Coles (2017).

of our sensory material, while, in the case of mathematical discourse, they are not components of our direct experience and are thus abstract. Besides the division into the singular and the general, there is thus another division that must be taken into consideration, the one into the *concrete* and the *abstract*. On the expression layer, we then distinguish between singular and general *terms* on the one hand, and between concrete and abstract terms on the other hand, while, on the content layer, all singular terms name *objects* and all general terms stand for *concepts*.

Note that the two divisions cross each other: There are concrete and abstract singular terms such as 'Socrates' and 'π', and concrete and abstract general terms such as 'human being' and 'prime number' (cf. Quine, 1966, p. 204). An investigation of the relations between these different kinds of terms unveils that we can find correspondences. The abstract singular term 'redness' corresponds to the general term 'red thing'; 'sweetness' is the abstract singular term to the general term 'sweet', and the general term 'round' can be matched to the abstract singular term 'roundness'. This series of correspondences is a systematic feature of our language (cf. Quine, 1966, pp. 204–205): For every general term, be it concrete or abstract, we can come up with a corresponding abstract singular term which names the (abstract) attribute that is 'shared' by all the objects which fall under the concept in question (or, respectively of which the associated general term is true). The term 'roundness', for example, names the attribute that all round things have in common. It names the particular form or gestalt that we recognize in our sensory experience from case to case when encountering a round thing. That also leads us to a hypothesis as to why general terms precede their corresponding abstract singular terms in cognitive development: *If the corresponding abstract singular term is what all of the objects that fall under a particular concept have in common, it is hardly possible to recognize this common attribute if the concept in question has not yet been learned.*

## III. Numbers as concepts

Equipped with this terminology, it is worth returning to the introductory example, which made obvious that number-words can either function as general or singular terms, meaning they can stand for concepts or objects. Let us take a closer look at the adjectival use of number-words first since it precedes the substantival use in the developmental sequence. As a reminder, our introductory thesis was: *numbers are first concepts, and only become objects in a second step*. Since we have clarified at this point what is meant when speaking of a concept, we can now ask for the operation or rule of application which is associated with a number-word when it functions as a general term. However, before we can do that, there is one more thing that needs to be clarified in advance: *the relation of number-words to the sensuous world*. It is important not to conceive number-words, not even in their adjectival use, as referring to certain complexes of our direct stream of experience. One and the same sensory impression can always give rise to a majority of equitable ascriptions of number:

> While looking at one the same external phenomenon, I can say with equal truth both 'It is a group of trees' and 'It is five trees', or both 'Here are four companies' and 'Here are 500 men'. Now what changes here from one judgement to the other is neither any individual object, nor the whole, the agglomeration of them, but rather my terminology. But that is

> itself only a sign that one concept has been substituted for another. This suggests [...] that the content of a statement of number is an assertation about a concept (Frege, 1970, §46)

Whether a certain sensory impression gives rise to one application of a number-word or another, hence solely depends on the way in which we *conceptually* organize or structure this experience. If a number-word is used as a general term, what falls under the associated concept are again also concepts. What the number-word ‘four’ stands for is then a *second order concept* (cf. Frege, 1970, §53), a concept that comprises all of those concepts with exactly four objects. Such a concept does not collect the counted objects or certain aggregates of them, but only the concepts structuring that material. In their adjectival use, number-words are, therefore, abstract general terms. Since a general term cannot stand on its own but is always predicated of something else, in this case of other concepts, it becomes clear why Patrick cannot make sense of the questions without any reference to other concepts. His behavior indicates that he treats the number-words as standing for second order concepts. In this use, there must some concepts upon which the addition is executed. Now we can come back to the question of what the operation or rule of application might look like that is associated with the term ‘four’.

In the case of the number-word ‘four’, this operation might be depicted as following: We coordinate one after the other all the objects that fall under the concept in question with all terms of the number-word sequence starting from ‘one’ until we come by the word ‘four’. If the counting procedure stops at ‘four’, that is, if all objects were counted up to this point, we say that the given concept falls under the concept *four* (or, respectively that the term ‘four’ is true of the particular concept). If the procedure already stops before or if we exceed ‘four’ without having finished the counting procedure, we will say that the given concept does not fall under the concept *four* (or, respectively that the term ‘four’ is *not* true of the particular concept). Here it is important to emphasize that there is no need at all to suppose an entity such as an attribute that is common to all the concepts comprising exactly four objects. Without a doubt, we can learn and apply the operation described above without supposing the number-words to refer to a separate abstract object of any kind. This observation holds true quite generally: the “use of the general term does not of itself commit us to the admission of a corresponding abstract entity into our ontology” (Quine, 1950, p. 630). Therefore, we should consider the final step of objectification, leading to the treatment of numbers-words as singular terms, as an additional step that requires the use of number-words as general terms already been learned. So, how, we must ask, do numbers become objects?

## IV. Numbers as objects

Let us assume we have a box in front of us which contains eleven marbles and we are supposed to determine how many marbles there are in the box. Wanting to communicate our counting result, we have at least two options: (1) ‘There are eleven marbles in the box’; (2) ‘The number of marbles in the box is eleven’. These two options correspond to our two forms of number-word use: in (1) ‘eleven’ functions as general term that is predicated of the concept *marbles in the box*, while in (2) ‘eleven’ can be said to stand for the same abstract entity as ‘the number of marbles in the box’, indicating that it functions as an abstract singular term. We argued above that, for a given general term, the corresponding abstract singular term can name the attribute that is shared by all the things which fall under the concept in question.

However, for number-words functioning as general terms the situation is somewhat different. It appears to be quite difficult to come up with a feature or attribute that all of the concepts have in common to which a number word such as ‘eleven’ can be successfully applied. What is the distance in meters between the penalty spot and the goal? How many corner points does the maple leaf on the Canadian flag have? How many criminals are there in Danny Ocean’s gang? How many academy awards did the film Titanic win? And, last but not least, what do these things have to do with our marbles? At the level of content, all these different manifestations of the number eleven obviously have nothing in common. So, what exactly is it that repeats itself from one manifestation to the other?

No matter if we count our marbles, the vertices of the maple leaf of the Canadian flag, or the meters from the penalty spot to the goal, in each of the cases we perform a counting procedure. The sameness does not lie at the level of the concepts upon which the counting procedure is executed, but it lies at the level of the procedure itself, that is, at the level of our operations with these concepts. What repeats itself from one counting act to the other is the fact that all of them end up with the number-word ‘eleven’. But this is itself only a sign that one and the same concept is applied: the second order concept associated with the number-word ‘eleven’. To go one step further in the process of objectification, a child capable of the adjectival use must thus recognize all the successive counting acts ending up with the same number-word as applications of the same concept. This requires a minimal reflexive loop: In order to sense this kind of ‘operational’ sameness, the child must make her own operations the object of her own operations. She must relate several successive applications of a number-word to each other and analyze them into sameness and difference of the counting results. However, in order to be able to identify several counting acts that end up with the same number word, a series of *factual* (the acts might be executed on very different concepts), *temporal* (the acts are most likely executed one after the other) and *social* (the acts might be executed by different people) *differences* between the individual acts must be left out simultaneously.[3]

## V. Piaget versus Peirce: What is abstraction?

What we are facing here is a two-sided process in which something is *retained* and at the same time something else is *left out*. Traditionally, this two-sided process is referred to as *abstraction* (cf. Locke, 1836, book 2, chapter 11, §9). In any process of abstraction there is a certain kind of material or substance upon which the abstraction is carried out. We cannot simply abstract something but rather we always abstract *from* something. Now, what is unusual in our case is this very substance of abstraction. The substance upon which the abstraction is carried out does not consist of objects of a certain kind but of our own operations. To take account of this reflexivity, Piaget has introduced the notion of *reflecting abstraction* (cf. Piaget, 2014, pp. 317–323; Glasersfeld, 2003, pp. 103–105). In contrast to what Piaget calls *empirical abstraction*, a reflecting abstraction is an operation that runs on operations:

[3] This distinction between three *meaning dimensions* – a factual, a temporal, and a social –, in which we can seek for commonalities and differences, stems from the German sociologist Niklas Luhmann (cf. Luhmann, 1995, pp. 59–102).

> [W]hen we are acting upon an object, we can also take into account the action itself, or operation if you will, since the transformation can be carried out mentally. In this hypothesis the abstraction is drawn not from the object that is acted upon, but from the action itself (Piaget, 1971, p. 16)

We can now see that the process of relating several counting acts and analyzing them into the sameness and difference of the counting results can be reconstructed as a reflecting abstraction. It runs upon several applications of the second-order concept associated with the number-word 'eleven'. The child must retain what all of the counting acts have in common while simultaneously leaving out all the features in which these successive acts differ from each other. But what is the outcome of this process? Have we already arrived at the substantival use of number-words? To cut it short: yes and no.

In general, we can say that the 'minimal' attribute shared by all things that fall under a certain concept is this purely formal feature, the fact that the things fall under that very concept. What all concepts that fall under the second order concept *eleven* have in common, is that they bear the same relation to this concept. They fall under it because in each case the counting procedure ends with the word 'eleven'. And to identify this formal sameness, we must perform a reflecting abstraction. In this way, the notion of reflecting abstraction allows extending the notion of a common property or a shared attribute far beyond the immediate, sensuous world. At best, we might then say that in the process of reflecting abstraction the successive mental or communicative acts are 'condensed' into an object-like entity, an attribute, which is the common feature of all these different acts. But, in the end, that is mere speculation. How, one might ask, is this mysterious object-like entity anchored if we obviously cannot perceive it?

In order to provide an answer to this question, we have to move away from the content layer, the series of operations on operations, onto the expression layer, the layer of the use of signs to fix these volatile operations (cf. Sfard, 1991, p. 21). It is precisely here where the notion of reflecting abstraction comes into trouble. It can explain how we get to sense the operational sameness described above, however, what it cannot explain is how this sameness is objectified. We therefore want to complement Piaget's account on abstraction with Peirce's considerations on the very same topic. Peirce has introduced the notion of *hypostatic abstraction* to describe the exact process we are interested in, the transition from a general term to an abstract singular term or in Peirce's own words:

> That wonderful operation of hypostatic abstraction by which we seem to create *entia rationis* that are, nevertheless, sometimes real, furnish us the means of turning predicates from being signs that we think or think *through*, into being subjects thought of. We thus think of the thought-sign itself, making it the object of another thought-sign. (CP 4.549)

It is noticeable that Peirce gives a beautiful account on the reflexive loop we have already described above: Whenever we apply a general term or a predicate we think *through* the sign or the concept associated with it in precisely that sense that we always predicate it of something else, while in any attempt to transform it into an object, we must reflect on this very use and make it the subject of another thought. That is to say, we must think *of* the sign *through* which we have thought before. So far nothing new, but in the second part of the quotation Peirce goes beyond what we have already described before. We “think”, Peirce writes, “of the thought-sign itself, making it the object of another thought-sign” (ibid.). As noticed before, what stays the same from application to application of number words in their adjectival use is nothing but the purely formal feature that the number-words are true of all the concepts to which they are successfully applied. We can now add that, what the corresponding abstract singular term names, is ultimately nothing other than the sign thought through before. Number-words functioning as singular terms name the sign, i.e., the *relation* between the general term and the particular counting procedure associated with it. This way, the conceptual operation becomes crystallized or frozen. When a number-word is used as a singular term, the operation must not be performed anymore but is rather only implied or pointed at. We say that we ‘know’ what the numeral ‘1,098,7877’ stands for because it “points to the last element in a familiar counting procedure” (Glasersfeld, 1995, p. 100), a procedure that no longer needs to be executed.

Concluding, I want to generalize this thesis. We do not need to posit some sort of ontological realm to answer the question of how we can speak about numbers, classes, and functions as self-sustained entities. What a singular term in mathematical discourse stands for is ultimately no more than another sign, it is the relation between the general term (from which the singular term in question originates) and the associated conceptual operation.

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