Unpacking 9th grade students' algebraic thinking

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The purpose of the present study was to propose and empirically validate a model describing secondary school students' algebraic thinking. Based on a synthesis of the literature, a model for Grade 9 students' algebraic thinking was formulated. The major constructs incorporated in this model were "generalized arithmetic", "transformational ability" and "meta-algebra". The study involved one hundred fourteen students. Data analysis validated the hypothesized model and suggested a sequential effect between the three factors. Transformational ability had a direct effect on generalized arithmetic and the latter had a direct effect on meta-algebra.

Keywords: Algebra, algebraic thinking, secondary school algebra, functional thinking.

Introduction

Recently, there is a growing consensus that algebra is the gateway to school mathematics reform for the next century and school algebra should be reformulated as a K-12 strand of thinking (Kaput, 2008; National Council of Teachers of Mathematics, 2000; Stephens, Ellis, Blanton & Brizuela, 2017). Algebraic thinking can be coherently conceptualized as a synthesis of different content strands, concepts, processes or forms of reasoning that relate to the ideas of equivalence, generalized arithmetic, variable, proportional reasoning, modelling, and functional thinking (Blanton, Stephens, Knuth, Gardiner, Isler, & Kim, 2015; Chimoni, Pitta-Pantazi, & Christou, 2018).

A number of researchers provided diverse conceptualizations of algebraic thinking, focusing on different parameters (Drijvers, Coddijn, & Kindt, 2011). One of the most influential developments of the past decades in respect to conceptualizing the notion of algebraic thinking is Kaput's organizing framework (Stephens et al., 2017). Kaput (2008) suggested that algebraic thinking consists of two core aspects: (a) generalizing and representing generalizations and (b) syntactically guided reasoning and actions on generalizations represented in conventional symbolic systems (Stephens et al. 2017). Kieran (2004), adopting a different methodology, conceptualized algebra as a multifaceted activity that encompasses various types of tasks and ways of thinking. Kieran offered a slightly different view by arguing that algebraic thinking is not only about using symbols in order to express generality; algebraic thinking arises when individuals make use of any kind of representations when they try to manipulate quantitative situations in a relational way. Drijvers, Coddijn and Kindt (2011) provided a different conceptualization of algebraic thinking giving emphasis on the role of functional thinking and solving equations and in-equalities with reference to specific constraints.

Thus, researchers' efforts to describe algebraic thinking through several perspectives are characterized by diversity and there is not a consensus regarding the dimensions of students' algebraic thinking in secondary school (Carraher, Martinez & Schliemann, 2008). In addition, the existing models are based mainly on theoretical conceptualizations of students' algebraic thinking. The aim of this study is the development of a better understanding of the notion of secondary school students' algebraic thinking by proposing a model that takes into consideration existing and well-accepted theoretical frameworks (Kaput, 2008; Kieran, 2004). Thus, the proposed model is founded by utilizing aspects of algebraic thinking that are well-accepted in the existing literature. In addition, the proposed model will be validated based on empirical data.

Literature Review

Several researchers made efforts to analyse the nature and content of algebraic thinking. There are differing views on what constitutes algebraic thinking, but many agree that a fundamental element is generalization, that is the ability to see the general in the particular (Kaput, 2008; Kieran, 2004; Wilkie, 2016). Generalization is a cornerstone of mathematical structure, while symbolization is a catalyst of algebraic thinking development. In this paper, we examine Kaput and Kieran's models of algebraic thinking, which are considered of the most influential in recent literature (Stephens et al., 2017). Kaput (2008) asserted that generalizing and symbolizing are tightly linked in that symbols allow generalizations to be expressed in a stable and compact form, throughout three strands: (a) generalized arithmetic, (b) functional thinking and (c) modelling. Generalized arithmetic, as a way for applying algebraic thinking in arithmetical settings, involves the use of letters for generalizing rules about relations between numbers, manipulating numbers and operations properties, examining number structure, understanding the equals sign in number relations, notice relationships in operations on classes of numbers and reasoning with forms and representations of equivalence. The generalizations that students make in the realm of generalized arithmetic can serve as a context for developing students' abilities to represent mathematical ideas symbolically. Functional thinking refers to the identification and description of functional relations between independent and dependent variables in different forms of representation and the manipulation of covariance and correspondence relations. Finally, modelling involves the use of symbols for developing models, manipulating variables, and re-translating between models and situations. Emphasis is placed on exploring modelling problems that are derived from complex realistic situations (Blanton et al, 2015).

Kieran's (2004) model for conceptualizing algebraic activity denotes that algebra is not just a topic in mathematics curriculum, but a multifaceted activity that encompasses various types of tasks and ways of thinking. Kieran asserted that algebraic thinking is considered as an approach to quantitative situations that seeks to look for relationships and structure with means that are not strictly letters-symbolic. Kieran, compared to Kaput, emphasized that algebraic thinking is a way for introducing students to the more abstract aspects of formal algebra and pointed out that her model tries to unfold the kinds of meaning that secondary students make when they are engaged with algebraic tasks. This model involved three types of activities: "generational" activities, "transformational" activities, and "global, meta-level" activities. The generational activities refer to the generation of equations and expressions in various situations and involve exploration of problem situations and numerical and geometrical patterns that lead to the formulation of generalization, and exploring numerical relationships. The transformational activities refer to the transformation of expressions by applying specific rules and involve conceptual understanding of algebraic objects. The global, meta-level activities are not strictly algebraic in nature, but algebraic tools are needed to be investigated and involve general mathematical processes, such as proving, studying functional relationships and identifying structure.

In conclusion, Kaput and Kieran’s models examine algebraic thinking under a different perspective. Kaput (2008) emphasized the role of generality and symbols by explaining algebraic reasoning under two core lenses: (a) symbolization that serves purposive generalization and (b) reasoning with symbolized generalizations. Thus, in Kaput’s model the use of symbols is a fundamental aspect, while in Kieran’s model the use of symbols is not a prerequisite. For instance, Kieran’s meta-level activities may be solved without using algebraic symbols at all. In this study, we will synthesize the two frameworks and propose a model that describes students’ algebraic thinking in an explicit and parsimonious way. The synthesis of the two frameworks makes possible the combination of the salient parameters of each model to provide a comprehensive description of students’ algebraic thinking. In addition, it provides a more accurate conceptualization of students’ algebraic thinking by adopting the parameters of each model that are described in a more concrete way. For instance, transformational activities are implicitly integrated in the strands of Kaput’s model, while in Kieran’s model they are described as a separate type of algebraic activity. In the following section, we explain the rationale of including each parameter in the proposed model.

The Present Study

The purpose of the present study was to propose and empirically validate a model describing Grade 9 students’ algebraic thinking. In particular, the aims of the study were: (a) to propose a theoretical model describing the dimensions of 9^{th} grade students’ algebraic thinking based on a synthesis of the literature, (b) to examine the validity of the proposed model by using empirical data and (c) to examine the relations between the dimensions of algebraic thinking.

The Proposed Model

The development of the proposed model describing students’ algebraic thinking dimensions took into consideration Kaput’s (2008) and Kieran’s (2004) frameworks. To do so, we thought appropriate to include in the model the common types of thinking of each model and the parameters of each model that are involved to a greater extent in 9^{th} grade algebra. Based on a synthesis of the literature, we hypothesized that 9^{th} grade students’ algebraic thinking consists of four distinct, but interrelated factors: (a) Generalized arithmetic, (b) Functional thinking, (c) Transformational ability, and (d) Modelling-meta-algebra. The dimension of generalized arithmetic is a common component of Kieran’s (2004) and Kaput’s (2008) models. Functional thinking is one of the main components of Kaput’s model, while it is a sub-component of meta-algebra in Kieran’s model. We included functional thinking as a separate component of the model because functions are a top priority topic in secondary school algebra. Transformational ability is a synthesis of transformational activities in Kieran’s model and formalizations of Kaput’s model and a fundamental type of activity in secondary school. Finally, the dimension of modelling-meta-algebra is a synthesis of components from both models. It includes Kieran’s global, meta-level activities and Kaput’s modelling dimension. In particular, it was hypothesized that generalized arithmetic involves applying algebraic thinking in arithmetical settings, by manipulating numbers and operations and exploring their properties, understanding the equal sign in numerical relations and becoming aware of the structure of arithmetic. Functional thinking refers to the identification and description of functional relationships between independent and dependent variables, by manipulating the concept of change and variation and generalizing patterns. Transformational ability refers to the transformation of numeric and algebraic expressions and solving equations by applying specific rules. Modelling-meta-algebra conceptualizes problem solving by using modelling, proving, manipulating variables in problem-solving situations and using symbols to represent situations.

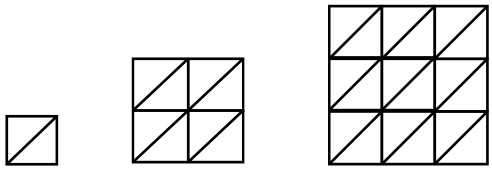
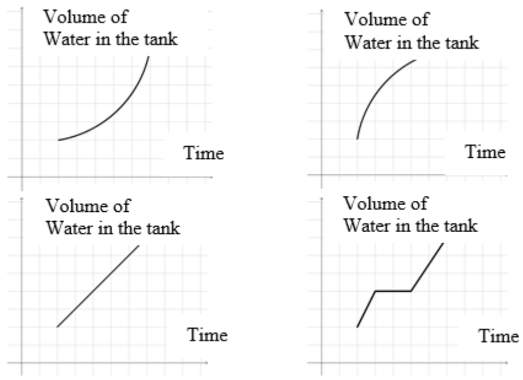
<table>
<tr><th>Type of Task</th><th>Example</th></tr>
<tr><td>Patterns</td><td>How many triangles are needed to construct the 10th figure?
</td></tr>
<tr><td>Relation between variables</td><td>Which of the following equations corresponds to the relation of the variables in the table?
$y=2x$ $y=2x+1$
$y=x^2+1$ $y= x^2+x$
<table><tr><td>x</td><td>y</td></tr><tr><td>0</td><td>1</td></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>5</td></tr><tr><td>3</td><td>10</td></tr><tr><td>4</td><td>17</td></tr></table></td></tr>
<tr><td>Relation between variables in a graph</td><td>Which graph corresponds to the following situation?
“Water is being poured into a tank with a constant rate. The faucet is closed for a while and then it is opened again. The rate that the task is now being filled is slower than the initial one”.

</td></tr>
</table>

Table 1: Functional Thinking Tasks

Measures

The test items were adopted or developed based on previous research studies. The test items were evaluated by three experts in mathematics education who provided feedback on the content validity. The multiple choice tasks were corrected as correct or incorrect, while the open tasks were given partial marks for incomplete correct answers. Three types of tasks were used to measure the factor “generalized-arithmetic” (Kaput, 2008): (a) Properties and relationships of numbers and operations (Tasks 1-3), (b) Structure of numbers and numerical expressions (Tasks 4-6), and (c) Equality and inequality (Tasks 7-9). In Tasks 1-3 students had to use number and operation properties to calculate numerical expressions (e.g., find the result of $-1245 \cdot 15 + 245 \cdot 15$). In Tasks 4-6 students had to treat numbers as placeholders and attending the structure of numbers rather than relying on computations (e.g., find the remainder of the division (946 + 950 + 952 + 960)÷ 950, and examine whether the number 3^{400} is divisible by 9). In Tasks 7-9 students explored equality and inequality situations (e.g., for what value of *a* is the inequality $-10 > (-5)a$ valid). Three types of tasks were used to measure the "transformational ability" factor: (a) Numerical transformations (Tasks 10-11), (b) Algebraic transformations (Tasks 12-13), and (c) Solving equations (Tasks 14-15). In Tasks 10 and 11 students had to make complex calculations with fractions and roots (e.g., $A = 2\sqrt{2} - 3\sqrt{3} + 7\sqrt{2} + 4\sqrt{3} + 5$). In tasks 12 and 13 students had to simplify algebraic expressions (e.g., $A = (a - 2b)(a + b) - (a + b)(a - a) + b(b + a)$). Finally, in Tasks 14 and 15 students had to solve fractional equations. Three types of tasks were used to measure functional thinking factor (see Table 1): (a) Finding the remote or general term of a pattern (Tasks 16-18), (b) Finding the relation between variables (Tasks 19-20), and (c) Finding the relation between variables in a graph (Tasks 21-22). Finally, we used two types of tasks to measure "modelling-meta-algebra" factor. In the modelling tasks (Tasks 23-25) students had to construct a mathematical model to solve a real-life problem, translate a word-situation to an algebraic expression using symbols and to represent a numerical relation using a bar-model (see Table 2). In the meta-algebra tasks (Tasks 26-27) students had to solve a complex problem involving inequalities and to make a proof (see Table 2).

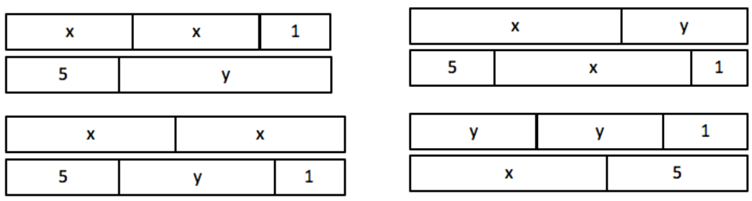
Type of Task	Example
Modelling	Which of the following corresponds to the relation: "One less than the double of a number is equal to five more than a second number". 
Meta-Algebra	Prove that the product of an even number by an odd number is an even number.

Table 2: Modelling-Meta-Algebra Tasks

Participants, Procedure and Data Analysis

One hundred fourteen 9th grade students (55 males and 59 females) were the subjects of the study from one private secondary school in Athens, Greece. The tasks of the study were randomly split into two parts. Each part was administered in the form of a written test during one school period. The two parts were administered in two successive weeks. The instructions were provided in written and verbal form. Confirmatory factor analysis was used to examine the validity of an a priori model, based on past evidence and theory. CFA was conducted by using MPLUS (Muthén & Muthén, 1998-2007). To evaluate model fit, three widely accepted fit indices were computed: x^2/df should be <2; the Comparative Fit Index should be >.9; and the root mean-square error of approximation (RMSEA) should be <.08. The Cronbach's alpha index of internal consistency was very good (a=.83).

Results

Confirmatory factor analysis (CFA) was used to evaluate the construct validity of the model; by validating that the a-priori model matched the data set of the present study and determined the "goodness of fit" of the hypothesized latent construct. The results of the study showed that the fit-indices were not satisfactory and the hypothesized model could not be supported (χ^2/df>2, CFI<.95, και RMSEA=.08). Examining the results of the study, we noticed that the correlation between functional thinking and modelling-meta-algebra factors was too high. Thus, we decided to examine the validity of an alternative model hypothesizing students' variances in functional thinking and modelling-meta-algebra tasks compose a unified factor. Analysis showed that the fit-indices of the alternative model were excellent (χ^2/df=1.07, CFI=.97, and RMSEA=.03), validating empirically the fit of the structure of the alternative model to the empirical data. CFA showed that the factor loadings of the tasks employed in the present study were statistically significant and most of them were rather large (see Figure 1). The factor loadings ranged from .38 to .84, giving support to the assumption that all latent factors were adequately measured by the observed variables. Thus, in accordance with our theoretical assumption, all algebraic thinking measures were clustered into three first-order factors in the expected factor-loading pattern.

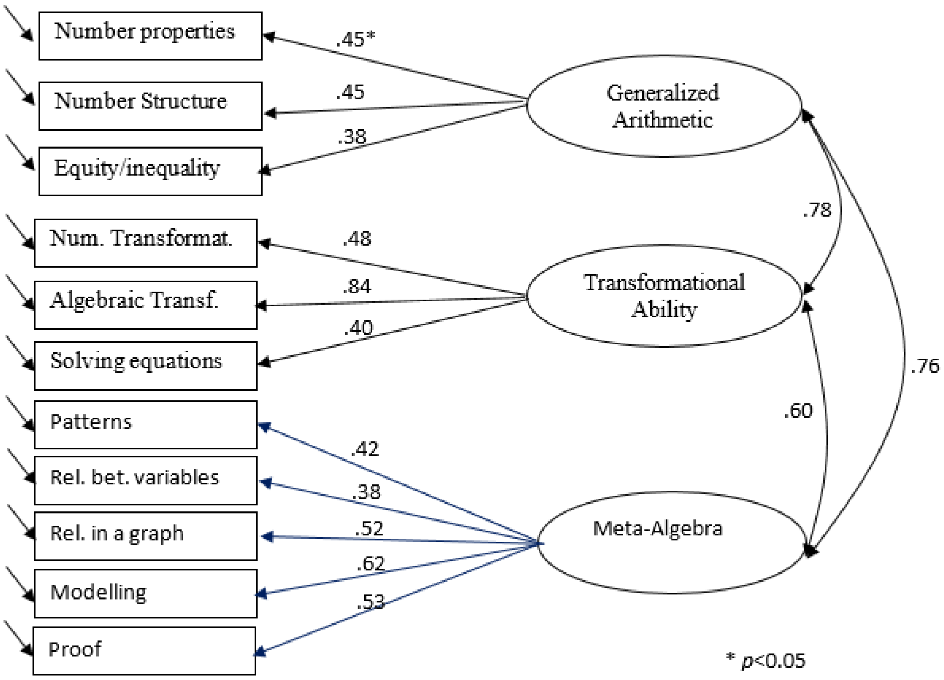


Figure 1: The Algebraic Thinking Model

Thus, analysis showed that algebraic thinking consists of three interrelated factors that is (a) generalized arithmetic, (b) transformational ability and (c) meta-algebra. The factor "meta-algebra" is a synthesis of students' variances in functional thinking tasks, modelling and proving tasks. The correlations between the three factors were significant. In particular, the correlation between the factors "generalized arithmetic" and "transformational ability" was 0.78 (p<0.05), the correlation between "transformational ability" factor and "meta-algebra" factor was 0.60 ($p<0.05$) and the correlation between "generalized arithmetic" factor and "meta-algebra" factor was 0.76 ($p<0.05$).

To investigate the relations between the three algebraic thinking factors, we examined the fit to the data of alternative structural models, hypothesizing a direct sequential path between the three factors. The model that had the best fitting indices (χ^2/df=1.04, CFI=.98, and RMSEA=.02) showed that "transformational ability" factor has a direct effect on "generalized arithmetic" factor and the latter predicts "meta-algebra" factor (see Figure 2). The regression coefficient of transformational ability factor on generalized arithmetic factor was 0.79 ($p<0.05$), while the regression coefficient of generalized arithmetic on meta-algebra was 0.78 ($p<0.05$).

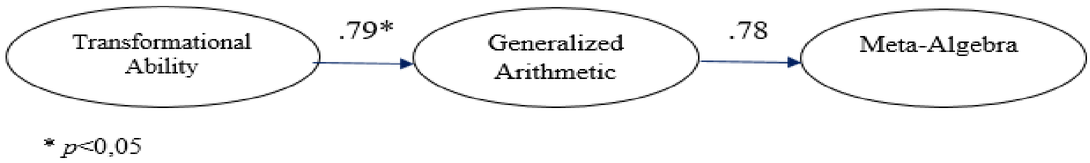


Figure 2: The Relation between Algebraic Arithmetic Factors

Discussion

The contribution of the study lies on the empirical evaluation of a proposed model that unpacks the dimensions of 9th grade students' algebraic thinking. The results of the study showed that 9th grade students' variances in algebraic situations might be modelled by three distinct and interrelated latent factors. The first factor involves students' capacity in generalized arithmetic tasks, the second factor in transformational situations, while the third factor reflects students' capacity in meta-algebra and functional tasks. The structure of the validated model is in line with the fundamental types of algebraic tasks suggested by Kieran (2004) and integrates ideas from Kaput's model (2008). In particular, the validated model showed that meta-algebra factor is a synthesis of algebraic thinking parameters suggested by Kieran (2004) and Kaput (2008) and consists of functional thinking, modelling in various situations and proving. The inclusion of functional thinking in meta-algebra factor can be interpreted by the fact that in problem solving situations modelling activities prerequisite understanding the implied functional relations. Analysis showed that functional thinking can be described by adopting Kaput's (2008) conceptualization that is generalization of patterns and manipulation of relations of variables in different representational forms. In conclusion, the empirically validated model that synthesized existing models in mathematics education could be a valid measurement model of 9th grade students' algebraic thinking.

Analysis showed that there is a sequential effect between the three factors. Students' capacity in transformational situations has a direct effect on their capacity in generalized arithmetic tasks and the latter affects directly meta-algebra. This is in line with research findings suggesting that modelling and meta-algebra tasks are the most difficult algebraic activities (Blanton et al, 2015). The finding that students' capacity in transformational activities predicts generalized arithmetic might be explained by the fact that in 9th grade transformational tasks are mostly procedural and manipulating algebraic structures in a flexible way may help students conceptualize and express arithmetic structures in a generalized way more efficiently. Students' advancement in transformational tasks might enhance their further development in generalized arithmetic by enhancing awareness of the structure of numeric and algebraic procedures, algebraic language and rules and applying generalizations strategically. Then, students' advancement in generalized arithmetic might contribute in further enhancing their capacity in meta-algebra tasks, by developing generalization processes and manipulating numbers and quantities relations strategically in different forms of representation. The aforementioned sequential relation might indicate a possible learning trajectory based on the fact that transformational ability and the ability to use algebraic language are essential so the students are successful in using algebraic structures for generalizing arithmetic tasks. Then students could succeed in meta-algebra that consists of more demanding tasks. Teachers should give students the opportunity to have systematic experience with transformational and generalized-arithmetic activities that lay the foundation to work with symbols and algebraic expressions that build up to an understanding of more abstract tasks. Moreover, teachers should take into consideration the aforementioned learning trajectory, which suggests a specific instructional sequence and the identification of key tasks designed to promote learning for each dimension of algebraic thinking. A future research could examine alternative learning trajectories in different populations and grades to get a further insight of students' difficulties and progression.

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