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The following research design combines relational thinking as an aspect of algebraic thinking with the understanding of variables. In clinical interviews, 5 to 10 years old children were confronted with tasks that required the description of relations between known and unknown quantities. Their answers were examined regarding the ways in which relations are described and thereby variables are seen.

Keywords: Algebraic thinking, relational thinking, unknowns, primary school.

Introduction

This paper reports parts of a research focused on the abilities of young children to establish relationships between unknown quantities. The aim is to investigate how young children describe relations between known and unknown quantities that are represented with different colored boxes and marbles and how they understand the unknown quantities. First of all, relational thinking as an aspect of algebraic thinking will be described as well as important aspects of variables.

Theoretical Framework

Algebraic thinking and relational thinking

It is hard to give a comprehensive description of what algebraic thinking means. However, many researchers (e.g. Kieran, 2011) highlighted that relational thinking is seen as an important part of algebraic thinking.

> Relational thinking refers to students' recognition and use of relationships between elements in number sentences and expressions. When using relational thinking, students consider sentences as wholes (instead of a process to be carried out step by step), analyze them, discern some detail and recognize some relations, and finally exploit these relations to construct a solution strategy. (Molina, Ambrose, Castro, & Castro, 2009, p. 122)

In addition to the understanding of relational thinking regarding equations and formal expressions, this can also be extended to the recognition and use of relations between quantities. It enriches the learning of arithmetic and can be a foundation for smoothing the transition to algebra (Carpenter, Levi, Franke, & Koehler Zeringue, 2005).

Aspects of variables

Radford (2011) described algebraic thinking as dealing with indeterminate quantities conceived of in analytic ways. So dealing with indeterminacy is an important part of algebraic thinking. This indeterminacy leads to different aspects of variables. At least, three different kinds can be distinguished: *unknowns*, *variables*, and *general numbers*. *Unknowns* describe a specific, but undetermined number, whose value can be evaluated, i.e. the x in the equation $10 + x = 30$ (e.g. Freudenthal, 1973; Usiskin, 1988). *Variables* describe a range of unspecified values and a relationship between two sets of values, i.e. variables appear in statements about a functional relationship and indicate how one value depends on another value (e.g. Küchemann, 1981; Usiskin, 1988). *General numbers* describe indetermined numbers which appear in generalizations, such as descriptions of properties of a set, i.e. the commutative law of addition can be represented by the equation $a + b = b + a$. In contrast to the variable aspect of the unknown, this is not about specifying a value for the letters a and b. The meaning of general numbers lies precisely in the general expression of the relationship found (e.g. Freudenthal, 1973; Usiskin 1988). In addition to these three aspects of indeterminacy, empirical data provide further classifications. Children often use *quasi-variables* to express generality before they develop the capability to use algebraic language (e.g. Fujii & Stephens, 2001): general structures are expressed through the use of examples which stand for the general. For example, children use the number sentence $78 + 49 - 49 = 78$ to express that the number sentence $a + b - b = a$ is true whatever other number is taken away and added again. In this sense, they have an understanding of indeterminacy and can express a general number without the symbolic algebraic expression.

Research interest

Studies focused on relational thinking usually refer to relationships in formal representations (e.g. Carpenter, Levi, Franke, & Koehler Zeringue, 2005). In equations with an unknown, students have the opportunity to calculate. If there are two unknowns, this possibility is not given. It can be assumed that the use of several unknowns increases the use of relational thinking. A study by Stephens and Wang (2008) showed that 6th and 7th-graders use relational thinking in the following task: In each of the sentences below, you can put numbers in Box A and Box B to make each sentence correct. For example:

$$18 + \square = 20 + \square$$

Box A Box B

Based on students responses to tasks like above, they described three stages of relational thinking: *established relational thinking, consolidating relational thinking,* and *emerging relational thinking*. *Established relational thinkers* specify the relationship between the numbers in the two boxes with a clear reference to the numbers, including the magnitude and direction of the difference between them. These characteristics decrease in the other two categories. In terms of task design, they found that tasks with two unknowns served the purpose of moving students beyond computations to more in-depth thinking (Stephens & Wang, 2008). Research by Schliemann, Carraher, and Brizuela (2005) showed that even younger children aged 7 to 11 years old are capable of understanding equivalence and solve linear equations with unknown quantities. In that case, the tasks were represented with concrete objects or told as little stories. In contrast to formal representations, concrete materials have the chance to prompt even younger children such as kindergarteners to show their capabilities of relational thinking while handling unknown quantities. Melzig (2013) tested a task design with boxes and beans with 7th graders and found that the boxes can play a meaningful role in developing a viable idea of variables.

The following study addresses the question: *How do young children describe relations between known and unknown quantities that are represented with concrete materials? How do they understand the represented unknown quantities (e.g. as unknowns, as variables…)?*

Methodology

The following interview-study was performed on 82 children aged from 5 to 10 years old. The sample consisted of 5 to 6 years old kindergarteners (n=27), 7 to 8 years old 2nd graders (n=28) and 9 to 10 years old 4th graders (n=26). The children did not receive any special algebra lessons or instructions before.

In order to capture children's competencies describing relations between known and unknown quantities, an exploratory design was created based on the representation of boxes and marbles (e.g. Melzig, 2013; Affolter et al., 2003). The concept of the tasks was to translate different kinds of equations with one or two unknowns into a representation that young children could handle. Known quantities were represented as marbles and unknown quantities were represented as boxes which contained an unknown number of marbles (see table 1). A story was told: "Here you see two children: Tino and Anna. They are playing with marbles. Some marbles are packed up in different colored boxes and some marbles are separate. Boxes with the same color always contain the same amount of marbles." The study includes 12 tasks of four different kinds. Kindergarteners got 10 of the 12 tasks, 2nd and 4th graders got all of them.

Type of task	Examples
A: "The same?": Children have to decide if both children have the same amounts of marbles or not.	A1: boy: $x + 1$ girl: $x + 2$
B: "How many?" (known): The children have to specify how many marbles are in a box so that both children have the same number of marbles. Children can answer with a specific number.	B2: boy: 4 girl: $x + x$ 
C: "How many?" (unknown): In type C a relationship between two unknowns can be stated. Task C1: How many marbles have to be in the boy's box so that both children have the same amounts of marbles?	C1: boy: y girl: $x + 1$ 
D: "Make them equal": Both children have the same amounts of marbles. The interviewer makes a transformation. Children have to decide what amounts of marbles they have to give to one child or take away to the interviewer, to make the quantities equal again.	D1: boy: $x + x$ girl: $y + y + 2$ → The boy gives one of his boxes (x) to the interviewer. How much does the girl have to give to the interviewer?

Table 1: Overview of the types of tasks used in the study. The variables in the last column represent the unknowns, which were represented as colored boxes. Each variable was represented with another color

To get an insight into children’s thinking, clinical interviews (Selter & Spiegel, 1997) were conducted. Various semi-standardized questions aimed to let children think more about the tasks. Each child worked on each of the tasks in an individual interview. The request “How did you get your answer?” helped to get an insight into children’s way of thinking. The interviews were videotaped and transcripted.

Analysis and results

After the transcription of children's answers, the diversity of children's solutions to the individual tasks was compiled. The tasks of type A were answered correctly by nearly all of the children. The correct answers of type B tasks varied between 40 to 72 % by kindergarteners, 72 to 97 % by 2nd graders and 92 to 97 % by 4th graders. Particularly interesting was the evaluation of the tasks of type C, where relations between two unknowns had to be established. The following presentation of results refers to the answers of the children to the task C1 (see table 1).

Categories of the first answers

In task C1, the children were confronted for the first time with two unknown quantities. In contrast to the last task, no concrete value can be given. Therefore, the first spontaneous answers of children to task C1 are of interest. Table 2 presents the categories of children’s answers.

Category:	Kinder-garten	2nd grade	4th grade
State a relationship: Children describe a relationship between the amounts of marbles in the boxes.	**0%**	**4%**	**11,5%**
Describe a dependency: Children describe that the amounts of marbles in both boxes depend on each other.	**0%**	**12,5%**	**30,8%**
Numerical values: Children mention values for the amounts of marbles in one or both boxes.	**96%**	**58%**	**38,5%**
… as examples: The values are meant as examples.	12,5%	57%	90%
… as fixed values: The values are meant as fixed numbers.	41,6%	29%	10%
… not to classify exactly: At this point of analysis, it is not clear if the mentioned values are meant as examples or not.	45,8%	14%	0%
Unintended answers: The children interpret the task differently. E.g. children want to change the amounts of given marbles in the task to give an answer.	**0%**	**12,5%**	**0%**
No answer: Children give no answer.	**4%**	**8%**	**7,7%**
Not possible to evaluate: The answers of the children cannot be evaluated more precisely at this point.	**0%**	**4%**	**11,5%**

Table 2: Categories of children’s first spontaneous answers to task C1

The overview shows that most children first enter numerical values for the content of the green box. But there are also clear differences between different age groups. While the majority of kindergarten children indicate numerical values, some children in grades 2 and 4 are already able to describe relationships or dependencies. Of particular importance at this point are the interviewer's following requests, which encourage children to overlook different numerical values and consider the relationship of possible numbers of marbles in the boxes. This is especially the case with some children, who initially cannot answer, but often show a deep understanding of relationships in the process of the interview. It can be assumed that the children were unaware that "a marble more" may be an appropriate answer.

Further data analysis

The answers of the children after the requests of the interviewer were very different. Since the task design suggests making relationships between unknowns, the children's responses can be categorized in two directions: establishing relationships and dealing with unknowns, whereas the boxes had to be interpreted as variables. Regarding relational thinking, there are three ways to recognize relationships (children describe a relationship or display the dependency or neither describe a relationship nor a dependency). Regarding the question how children handle the unknown quantities, there were two ways to get the categories: first in a deductive way, because the theoretical framework gave various categories about aspects of variables (as unknowns, as a general number, as a variable, and as a quasi-variable); secondly, the categories are found in an inductive way, because the data also revealed categories that could not previously be found in the theoretical framework. These were interpretations of the boxes as an absolute number and undeterminable.

Children's answers regarding relational thinking

Some children directly describe a relationship between the two unknown quantities as in types of task C (table 1). The 4th grader Luca said: "In the green box is always one marble more than in the orange box". Other children describe that the two unknowns depend on each other, but without describing the relation more precisely. The 4th grader Kathy answered: "It depends on how many marbles are in the orange box". Other children neither describe a relationship nor describe a dependency between the amounts of marbles in the boxes. There are very different answers in this category. Some children mentioned specific quantities or wanted to shake the boxes to find out how many marbles are in them.

Children's answers regarding the understanding of the represented unknown quantities

Children have very different answers by describing the unknowns in tasks of type C. These different types of understanding the unknown quantities are the following:

The unknown amounts of marbles in the boxes as a *general number*: children see the amounts of marbles in the boxes as an unknown, indeterminate number, which cannot be determined and therefore has to be stated as a general relationship. The 4th grader Luca says: "In the green box is always one marble more than in the orange box". The unknown amounts of marbles in the boxes as *quasi-variables*: children describe the general relationship between the amounts of marbles in the boxes with the aid of examples. The difference to the former is less in the recognition of the relationship than in the linguistic expressiveness. The 6 years old kindergartener Adam says: "...if there are eight or nine marbles in the orange box, then I take one marble more, that's nine or ten marbles for the green box". The unknown amounts of marbles in the boxes as *variables*: children describe the dependency of the amounts of marbles between the different colored boxes. Children say that there is a relation, but they do not specify it. In contrast to the interpretation as a general number, the dependence of both values is indeed recognized, but cannot be described as a fixed relation. Although the children are able to assume different values for the number of marbles in the boxes, they are not able to give a general relationship. The 4th grader Anton says: "There are various possibilities because I don't know how many marbles are in the orange box". The unknown amounts of marbles in the boxes as an *absolute number*: the children give a specific number of marbles, sometimes without considering the amount of marbles in the other box. The kindergartener Clara says: "Four. Because the box is so small, there just fit four marbles in". The unknown amounts of marbles in the boxes as an *undeterminable*: This category results from the empirical data and finds no equivalent in the theoretically elaborated aspects of variables. Children who see the unknown amounts of marbles in the boxes as an undeterminable answer that the amounts of marbles in the boxes can't be determined without to open it. The 2nd grader Rob answers: "I have to open the box".

Putting both together: Relational thinking and understanding unknown quantities

The design of the tasks allows putting both dimensions together: relational thinking and the understanding of unknown quantities. The following evaluation schema shows all the combinations found in the data. They indicate that the two dimensions are intertwined. Children who describe relationships between the unknown quantities understand them as either a general number or a quasi-variable. Children who describe dependency take the unknown quantities as a variable. Children who neither establish a relationship nor describe a dependency understand the unknown quantities as either an absolute number or an undeterminable. The interviewer's requests encourage the children to think more profoundly about the tasks[1]. After requests of the interviewer, 12% of the children in kindergarten describe a relationship between the unknown quantities and describe them as a general number. This increases to 17% in the second grade up to 73% in the fourth grade. 12% of kindergarten children describe a relationship with the help of quasi-variables. This is also the case for almost 17% of 2nd graders and almost 4% of 4th graders.

[1] Case studies and their application to the evaluation scheme can be viewed in Lenz (2016) and in Steinweg, Akinwunmi, & Lenz (2018).

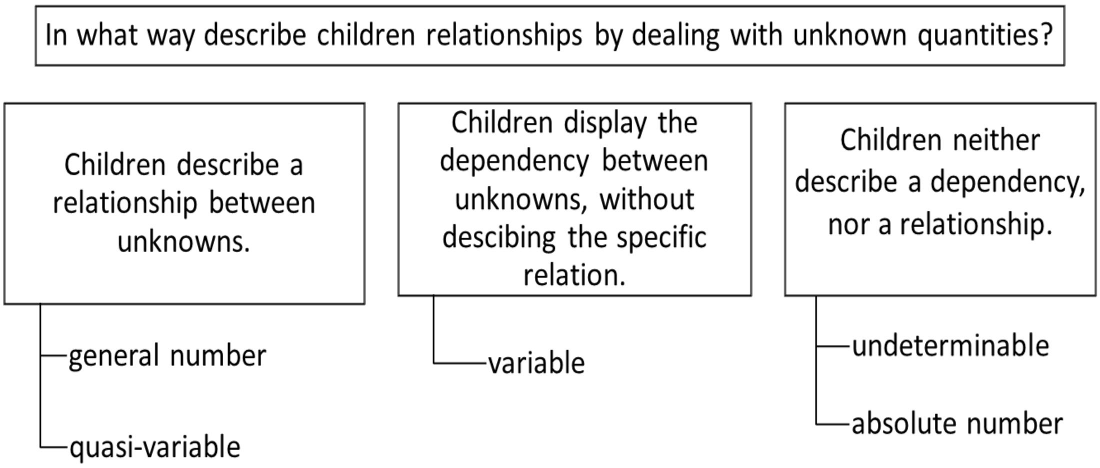

Figure 1: Evaluation scheme

It should be mentioned that the fields shown in the evaluation scheme are not to be seen as a fixed classification of individual children. Rather, the analysis scheme describes a single answer in the course of the interview and is not seen as a stable attribution to a child.

Conclusion

The analysis indicates that the task design is actually designed to encourage children to establish relationships between unknown quantities. Particularly interesting is the transition between task types B and C which seems to be a special breaking-point in the use of variables. The role of the unknown amounts of marbles in the boxes changes from an unknown that can be determined to a variable whose value cannot be known but can be described as a relation. It must be distinguished in which way the children interpret the unknown number of marbles in the boxes.

The evaluation scheme allows two dimensions of algebraic thinking to be linked: relational thinking and understanding unknown quantities. The analysis has shown that children from the age of 5 years old as well as primary school children are able to show relational thinking about unknown quantities. However, there are also children who understand the number of marbles in the boxes as an absolute number. A possible reason for this might be due to the concrete representation of the boxes. It is important to investigate whether this would be still the case of representing photos of the tasks. A difficulty is to see in the limitation of the linguistic expressiveness of the children. In particular, the children of the kindergarten found it difficult to express their thoughts in words. They often made gestures and facial expressions to help, which requires further investigation. On an active level, all children have the opportunity to get started with the tasks. Although the task design looks very simple, it allows for variation in primary school and secondary level. For example, tasks with limit value determination or case distinctions can also be created. The design allows continuing these tasks on a formal level. Placeholders, symbols, or letters can replace the boxes. This study can be seen as a starting point to develop classroom lessons that allow relational thinking in elementary school mathematics lessons and a first approach to dealing with different variable aspects. The common exchange in the classroom can be seen as particularly profitable.

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