# Framework for analysing children's ways of experiencing numbers

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*This study contributes to the field of arithmetic learning a framework based on a conjecture that learning is a change in ways of experiencing numbers – decisive for what is possible to do with numbers in arithmetic tasks. 309 task-based interviews with 5-6-year-old children were used to develop the framework. To validate its use for studies on arithmetic skills development we selected 7 children who between two interview occasions developed from scoring 0 in the first interview to >75% correct answers in the second interview. Based on the results of these children’s changing ways of experiencing numbers, we conclude that arithmetic skills development can be found empirically related to changes in the children’s ways of experiencing numbers, which validates the framework as being a useful tool to follow and describe children’s arithmetic skills over time.*

*Keywords: Arithmetic skills, early childhood education, mathematical development, variation theory of learning.*

## Background and aim

Early arithmetic development has raised an increasing interest during the last decades, providing the mathematics education field with observations of children’s counting strategies (Carpenter, Moser & Romberg, 1982) and learning trajectories (Clements & Sarama, 2009; Cross, Woods & Schweingruber, 2009). Multiple studies within this knowledge area have proven that learning arithmetic is complex but most children do in fact learn to solve simple arithmetic problems during their preschool years. Nevertheless, the development of arithmetic skills is a challenge to study, not least because arithmetic computation includes knowledge and awareness of several aspects of numbers and principles (Baroody, 2016). Even if children have strategies to solve a task, their strategies may not be based on conceptual understanding of relationship between and within numbers and arithmetical principles, but rather on procedural knowledge (see Fuson, 1992). Relying on procedural knowledge alone has not shown to facilitate further learning and may even induce mathematics difficulties when encountering more advanced arithmetic tasks (Gray, 1991; Gray & Tall, 1994; Neuman, 1987). Of educational and scientific interest is thereby to be able to interpret the way a child understands numbers and what s/he can do with numbers in arithmetic tasks.

Our approach to learning and development is experiential, based on Variation theory of learning (Marton, 2015), meaning that how a child acts when trying to solve an arithmetic task depends on the child’s way of experiencing numbers in the task. Variation theory explains learning failures in a specific way and spells out the conditions of learning; when learners fail to learn what was intended, they have not discerned aspects that are necessary to discern. So, the very core idea of Variation theory is that discernment is a necessary condition of learning. Furthermore, the way a learner is experiencing numbers is expressed in the way the learner is acting in solving arithmetic tasks. This approach adds to the field of research an alternative approach to understanding development and learning as changes in ways of experiencing numbers in arithmetic tasks, that is what is discerned by the learner and what is yet to be discerned, rather than learning to use more sophisticated strategies.

The aim of this paper is to describe a framework of children's ways of experiencing numbers. To validate its use for studying arithmetic skills, we used empirical data of children's acts and utterances in interviews from a smaller sample of the whole data set and discuss to what extent these children's increased arithmetical skills can be explained in terms of different ways of experiencing numbers.

## The study

This particular study is part of a larger project (FASETT) in which a pedagogical program to enhance children's number knowledge and arithmetic skills was developed. To assess this project we needed a framework to consolidate our conjecture of how children develop arithmetic skills that would allow us to study children's development and find theoretically and empirically sound explanations for their learning and possible impact of implemented teaching programs.

103 preschool children participated in researcher-designed interviews based on addition and subtraction tasks (number range 1–10). The interviews were conducted at three occasions: in the beginning of the last preschool year (5-yearolds), in the end of the last preschool year (8 months later) and after another year in pre-primary school (3 x 103 children = 309 interviews). Each interview was video-recorded, with the legal representatives' permission, to ensure repeated analyses. Ethical clearance has been provided to collect personal data for the longitudinal investigations of the children. Furthermore, we chose one group of children (n=7) who made substantial improvements in their interview scores between interview 1 and 2 (from 0 to >75% correct answers on arithmetic tasks) for thorough analyses of their answers in the interviews.

### Developing a theoretically informed framework

The interview data, consisting of 3 x 103 interviews (0% falling-off), was used as empirical basis for developing a framework of the outcome space of different ways of experiencing numbers. The children's acts and utterances regarding eight items were the unit of analyses. Several researchers participated in this thorough work, both individually and collectively to fine-tune the characteristics for each category. Six different ways of experiencing numbers have been found. They are distinct in that each category forms a qualitatively different way of seeing and thus understanding what is possible to do with numbers in an arithmetic task. The framework is generated from the analysis of answers to each task, not as a general expression of a specific child. The ways of experiencing numbers are thus analysed as unique observations of each item in the interviews, which gives us a sample of approximately 2400 observations (309 interviews x 8 items).

**Validating the framework**

The purpose of this paper is furthermore to validate the framework as a tool for studying children's arithmetic skills development. We thereby compared a group of children (n=7) who in the first interview scored low concerning correct answers (0 of 8 possible) and high in the second interview (>75%), eight months later. Our concern is to what extent the improvement can be explained by their different ways of experiencing numbers. These children's ways of experiencing numbers are, in line with the framework, analysed as unique observations of each item in the interviews, which gives us a sample of 56 observations (7 children x 8 items).

## The framework

The result from analysing the interviews constitutes six distinct ways of experiencing numbers (Table 1, left column). These are described in terms of criteria for inclusion (Table 1, middle column).

| ***Ways of experiencing numbers*** | ***Criteria for category inclusion*** | ***Strategies enacted when experiencing numbers in respective ways*** |
|---|---|---|
| As words | - Number words are used in quantitative contexts but without numerical meaning | - Random number words are used solely or as a sequence, *or*<br>- Number words from the given task are repeated |
| As names | - Numbers are ordered in a sequence<br>- Number words describe "the n^th^" object, no cardinal meaning<br>- Names cannot be added or subtracted from other names, because they lack cardinal meaning | - Ascribing number names to objects<br>- Saying the following number word in the counting sequence as an answer to an arithmetic task<br>- Static way of representing a number with fingers |
| As extent | - Numbers are approximate with a cardinal meaning<br>- Some sense of plausible quantities related to the task | - No attempts to count single units<br>- Numbers are used to give a plausible description or comparison of quantities (however not exact) |
| As countables | - Strong influence of the ordinal aspect of number<br>- Added units of 'ones' | - Counting all parts and the whole<br>- Exceeding the subitizing range imposes counting to determine a set |
| As structure | - Numbers are composite sets<br>- The part-part-whole relation is constituted in the act | - Using finger patterns to structure numbers' parts and whole<br>- Operating on the relation between parts and/or the whole |
| As known number facts | - Numbers are instantly recognized as a part-whole relation | - Giving an instant (correct) answer<br>- Retrieving from known facts |

**Table 1. Overview of the criteria for category inclusion and empirical expressions**

Furthermore, the way to experience numbers is conjectured to be related to what is possible to do with numbers, which means that how children use numbers and express their understanding in words and in gestures are keys to interpreting their ways of experiencing numbers (Table 1, right column). In the following these different ways of experiencing numbers and the implications for solving arithmetic tasks will be described with examples from the interviews.

### Numbers as words

Children expressing their experiencing numbers as words say random numbers, either solely as an answer to a task, or as a sequence of words (not in an orderly fashion). This category includes instances when a child repeats one of the number words appearing in the task. Thus, numbers are words used in certain contexts but there is nothing you can *do* with numbers to solve the task at hand.

Task: Seven marbles are hidden in two hands. How many could be in each hand?
Child: Five, eight, nine [taps her finger at the right hand of the interviewer, once for each number word], ten [points at both hands], six [points at the right hand] and seven [points at the left hand]. (PFÅ09-1)

Number words are said as an answer to a question with a numerical content, but as shown in the excerpt above, there is hardly a numerical reasoning guiding the child's way of using numbers. Number words are in this sense experienced as a type of word in a certain context. They refer to quantitative situations, but the child does not know *what* numbers they refer to (Wynn, 1992).

### Numbers as names

Children seeing numbers as names have been observed by Neuman (1987), and described as a procedure of 'word tagging' by Brissiaud (1992). The child considers number words as names given to a counted item and the last uttered word denotes only that single object.

Task: You have ten candies and eat six of them. How many are left?
Child: [shows all the fingers on the left hand and the thumb on the right raised] Ate six. This is six. And then I took the thumb away. So, it's five. (ASJ10-1)

Numbers as names means that each item within a group is given a specific number name. Taking away one (the thumb) of the six fingers and answering "five" to this specific task, is a reasonable answer if the thumb is seen as "the six". Such named items cannot be added to or subtracted from anything in a true sense, since they do not have a cardinal meaning. What characterizes this way of experiencing numbers is thus the ordinal aspect, which means that for example when fingers are used to count on, each finger is given a specific number name and this name cannot be altered because the other fingers are given their unique number names. The lack of cardinal meaning of these number names is also expressed when children answer with the following number word in the counting sequence. This makes sense to the child since the order of the number words implies that the addition of a quantity will include (at least) the next word in the sequence (the $n^{th}$ item).

**Numbers as extent**

Experiencing numbers as extent means that children have an awareness of numbers in that they use quantities as approximate (Neuman, 1987).

| | |
|---|---|
| Task: | Seven marbles are divided in two closed hands, how may the marbles be divided? |
| Child: | [points on the closed right hand] One in that one [points at the closed left hand] five in that one. |
| Interviewer: | How did you figure that out? |
| Child: | [points at the left hand] Eight in that one and two in that one [points at the right hand]. (PLS05-1) |

To experience numbers as extent are in a sense cardinal: children give plausible approximations of smaller and larger numbers that together are close to the correct answer. The lack of ordinality means that children do not make any attempts to count in order to determine the quantity of a set. This also makes it impossible to add or subtract in other ways than giving an approximate estimation of more or less than the given numbers in the task. The answer may though be quite close to the correct answer.

**Numbers as countables**

Experiencing numbers as countables is strongly influenced by the ordinal aspect of numbers, as children count to make a number, often using their fingers to count on. Numbers are thus seen as added units of 'ones' which may be difficult to coordinate in an arithmetic task.

| | |
|---|---|
| Task: | You have two shells and your friend five. How many do you have together? |
| Child: | One, two [points at his little and ring finger] wait, I have to start over, 1, 2, 3, 4, 5 [pointing at fingers starting from long finger, index, thumb and thumb, index on the other hand. Then starts again from the little finger] 1, 2, 3, 4, 5. Five! (AVT09-1) |

This child knows how to count fingers as representations for the items mentioned in the task. However, he does not seem to discern any structure that would help him solve the task in an easier way – he is able to create parts but has trouble coordinating them (c.f. Steffe, Thompson & Richards, 1982). Numbers as countables further implies that numbers are not comprehended as a composite set, they are created by adding 'ones'. This becomes visible when using the strategy "counting all". Some children do however experience small sets by subitizing for example three fingers (that will not be counted), but when the quantity exceeds the subitizing range the numbers have to be counted in ones.

**Numbers as structure**

A structural approach in arithmetic problem solving is assumed to promote children in developing their conceptual knowledge of numbers (Davydov, 1982; Lüken, 2012; Schmittau, 2004).

| | |
|---|---|
| Task: | You have two shells and your friend five. How many do you have together? |
| Child: | [puts her left hand with all fingers unfolded on the table, then the right hand with thumb and index finger unfolded, looks at her hand shortly] Seven! [with a confident smile] (HNV02-2) |

The child knows the finger pattern for five and two, but needs to create the whole by seeing the parts simultaneously both as parts and as a whole, related to each other. This means that some parts may need to be counted, while other parts are experienced for example as finger patterns, but the parts and the whole are experienced as related to each other and are thus possible to handle as a triad (see Baroody, 2016). The difference, compared to experiencing numbers as countables, is the way children experience the parts as composite sets that relate to another set and/or the whole. This opens up for a different way of handling arithmetic tasks, for example in "seeing the five in the eight". This has also shown to be a key feature of some young children's spontaneous ways of successfully solving arithmetic problems (Björklund, Kullberg, & Runesson Kempe, 2019).

### Numbers as known facts

The last category of experiencing numbers reflects an advanced understanding of number relations. Either, the number relations are known to the child and s/he does not need to calculate, or s/he has an advanced way of reasoning and using retrieved facts to come up with the answer.

Task: Seven marbles are divided in two closed hands, how may the marbles be divided?
Child: Four in that one and three in that one. 'Cause, if you have three and one more it makes four. You have three in that one [points at the left closed hand] and three in that one and one more, makes seven [points at the right closed hand]. (HRM01-2)

In this example the child knows how three, four (seen as 'three and one more') and seven are related and is able to retrieve from memory known facts ('three and three makes six') that he can operate with to find a plausible solution. The difference between numbers as structure and numbers as known facts lies in the child's approach to the arithmetic task. The former is focused on structuring numbers in that the missing addend will be possible to discern (it appears in the structure), while this latter category means that the child sees number relations instantly and is often able to describe how s/he retrieves the answer from known number relations.

## Validating the framework

The categories outlined above derives from the 309 interviews and 8 tasks given in each interview. Our theoretical conjecture has a pedagogical purpose, which directs our attention towards the extent to which the framework can be used to understand children's developing arithmetic skills. Thus, we analysed the target group who made substantial improvements during an eight months period. Table 2 shows the number of observations within each category in the first and the second interview.

| words | names | extent | countables | structure | known facts |
|---|---|---|---|---|---|
| 10 / 0 | 11 / 0 | 10 / 6 | 1 / 9 | 0 / 22 | 0 / 16 |

**Table 2: Occurrences of ways of experiencing numbers, interview 1 / interview 2**

In Table 2 we can see that children in the first interview are with only one exception found in the three first categories: *Numbers as words*, *Numbers as names* and *Numbers as extent*. From earlier research we know that a simultaneous awareness of numbers' cardinal and ordinal aspects are important for the child's ability to operate with numbers. Without these aspects coordinated in the arithmetic problem solving process it is not possible to use any counting strategies or arithmetic principles (Davydov, 1982; Fuson, 1982). Experiencing numbers in these ways do not include this crucial aspect of arithmetic skills and the children's low achievement on the arithmetic tasks in interview 1 is thereby no surprise.

The second interview includes most observations within the categories *Numbers as countables, Numbers as structure* and *Numbers as known facts*. Children's way of experiencing numbers change, in this group, towards ways that allow them to handle numbers in prosperous ways in arithmetic problem solving. However, there are differences in these categories concerning what they enable the children to do with numbers. In some addition tasks the children who experience numbers as countables manage to solve the problems by 'counting all' and thus working with added 'ones'. This way of experiencing numbers stand in bright contrast to numbers as structure or as known facts, in which the parts to be added or taken away already from the beginning are seen as related to each other as composite sets. Because of this latter relational approach to a task, the child usually avoids cumbersome strategies such as working with one unit at a time. In other words, they can handle arithmetic tasks in a more prosperous way when numbers are experienced as structure.

## Discussion

The framework was developed first through qualitative interpretations of a large sample of children's different ways of experiencing numbers in simple arithmetic tasks. In this study, we then used the six categories derived from the large data, to test the extent to which it was possible to use the framework to explain differences in arithmetic skills. The results suggest a confirmation of basic principles of Variation theory that failing to solve a task is due to the learner not being able to discern necessary aspects within the task and the strategies children use reflect what aspects they discern. Consequently, the children who score 0 on their initial interview have not discerned necessary aspects of numbers. This is also shown in our definition of the different ways of experiencing numbers. The validation of the framework further shows a hierarchical structure of the categories: when more necessary aspects are discerned there is also an advancement in what the child can do with the numbers in the tasks.

When analysing children's ways of experiencing numbers, their strategies for solving arithmetic tasks become possible to understand, what works and why? Their ways of experiencing numbers change in most cases from primitive ways (that do not enable them to operate with arithmetic tasks) to quite advanced ways of experiencing numbers (that allow them to solve arithmetic tasks as part-part-whole relations). The framework opens for interpreting how new experiences may advance children's arithmetic skills that does not depend on correct answers, or learnt strategies, instead it offers an alternative: what it means to learn about numbers and how to do simple addition and subtraction.

The analysis of the changes in ways of experiencing numbers seems to be solid enough to explain why some children have difficulties solving arithmetic tasks. Since our target group contained children who scored 0 points and improved to almost perfect score, we would suggest that changes in ways of experiencing numbers are possible over time and the ways to experience numbers have significant impact for children's possibilities to solve arithmetic tasks. We would argue that more attention should be directed towards what children are able to discern in arithmetic situations, than strategies to come up with a correct answer. However, more research on a larger sample is needed to confirm our statement.

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