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Double circulant self-dual and LCD codes over Galois rings ^{*}

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Abstract: This paper investigates the existence, enumeration and asymptotic performance of self-dual and LCD double circulant codes over Galois rings of characteristic p^2 and order p^4 with p and odd prime. When $p \equiv 3 \pmod{4}$, we give an algorithm to construct a duality preserving bijective Gray map from such a Galois ring to $\mathbb{Z}_{p^2}^2$. Using random coding, we obtain families of asymptotically good self-dual and LCD codes over \mathbb{Z}_{p^2} , for the metric induced by the standard \mathbb{F}_p -valued Gray maps.

Keywords: double circulant codes, self-dual codes, LCD codes

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1 Introduction

Double circulant self-dual codes over finite fields have been studied recently in [1]. Double circulant self-dual codes over a commutative ring can only exist if there is a square root of -1 over that ring [10]. Such a root does not exist over Galois rings of even characteristic which are not fields, but does exist over Galois rings of odd characteristic and even extension degree [8, Lemma 3.1, Lemma 3.2]. We study these codes in the case of Galois rings of characteristic p^2 and size p^4 , for p an odd prime. A recent topic related to self-dual codes is LCD codes. They are popular because of their connections with cybersecurity [2]. We also study LCD double circulant codes over the same Galois rings. For every such Galois ring we construct a duality preserving Gray map with image $\mathbb{Z}_{p^2}^2$, which maps self-dual (resp. LCD) codes to self-dual (resp. LCD) codes. Note that self-dual codes over \mathbb{Z}_{p^2} , for p an odd prime, have been studied by many authors [3, 5]. This alphabet \mathbb{Z}_{p^2} , in turn can be mapped into \mathbb{F}_p by the Gray map studied in [9]. For the two families of codes under scrutiny we give a complete enumeration formula in length $2n$ when n is coprime with p . This formula relies on the CRT approach to quasi-cyclic codes over rings [12, 10], and requires to count the number of solutions of certain algebraic equations over Galois rings and over their residue fields. From there, building on Artin conjecture in arithmetic progressions [11], we construct infinitely many odd primes n such that $x^n - 1$ has only three factors over \mathbb{F}_{p^2} the residue class ring of the Galois ring alphabet. Depending on the congruence class of n modulo 4, these irreducible factors are all three self-reciprocal, or consist of $x - 1$, and one reciprocal pair. When n varies in one of these two families of primes, we obtain two infinite families of double circulant codes, one self-dual, one LCD, of length $2n$. By expurgated random coding, we derive a lower bound on the relative Hamming distance of the \mathbb{F}_p image of both families. This shows, in particular, that both families are good.

The material is organized as follows. The next section collects the notions and notations needed in the forthcoming sections. Section 3 contains the main results on enumeration and Section 4 the main results on asymptotics. Section 5 displays some numerical examples. Section 6 concludes the article and points out some open problems.

2 Definitions and notation

2.1 Some rings

Throughout the paper, let p be an odd prime. The ring \mathbb{Z}_{p^s} is the ring of integers modulo p^s . A linear *code* of length N over \mathbb{Z}_{p^2} is a submodule of $\mathbb{Z}_{p^2}^N$. The *dual* C^\perp and C^{\perp_H} are understood with respect to the standard inner product and Hermitian inner product, respectively. A code is *self-dual* if it is equal to its dual. It is *LCD* (linear complementary dual) if it intersects its dual trivially. The Galois ring $GR(p^s, p^{ms})$ of order p^{ms} and characteristic p^s is the Galois extension of \mathbb{Z}_{p^s} with degree m . It is a local ring, with maximal ideal (p) . The *Teichmüller* set $\mathcal{T} = \{x \in GR(p^s, p^{ms}) \mid x^{p^m} = x\}$ is a set of representatives of the *residue field* $\mathbb{F}_{p^m} = GR(p^s, p^{ms})/(p)$. If $r \in GR(p^s, p^{ms})$, let \hat{r} denote its image in \mathbb{F}_{p^m} by reduction modulo (p) . It is known that $GR(p^s, p^{ms}) = \mathcal{T} \oplus p\mathcal{T} \oplus \cdots \oplus p^{s-1}\mathcal{T}$ (base p decomposition of $GR(p^s, p^{ms})$). See [13] for background and details.

2.2 Double circulant codes

Denote by R the ring $\frac{\mathbb{Z}_{p^2}[y]}{(h(y))}$, where $h(y)$ is a basic irreducible polynomial over \mathbb{Z}_{p^2} with $\deg(h(x)) = 2$. For any ring M , we denote by M^* the set of units in M . Assume that n is an integer with $\gcd(n, p) = 1$, and consider the code C of length $2n$ over R whose generator matrix has the form of (I_n, A) , where I_n is the identity matrix of order n and A is a circulant matrix over R . Note that C can be viewed as a submodule of $(\frac{R[x]}{(x^n-1)})^2$, with generator $(1, a(x))$ where the x -expansion of $a(x)$ is the first row of A .

For all p 's, we know there exists $w \in R$ such that $w^2 = -1$, by [8]. When $p \equiv 3 \pmod{4}$, the polynomial $y^2 + 1$ is irreducible over \mathbb{F}_p , hence over \mathbb{Z}_{p^2} . We may write $R = \mathbb{Z}_{p^2}[y]/(y^2 + 1)$, and take $w = y$.

2.3 Gray map

Recall Lagrange's four-square theorem, also known as Bachet's conjecture [7].

Lemma 1. *Every natural number can be represented as the sum of four integer squares.*

We will also need the sum of two squares theorem, and the Diophantus identity [7].

Lemma 2. *An integer greater than one can be written as a sum of two squares if and only if its prime decomposition contains no prime congruent to 3 (mod 4) raised to an odd power.*

Lemma 3. *The product of two sums of two squares is a sum of two squares in following two different ways.*

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 \quad (1)$$

$$= (ac + bd)^2 + (ad - bc)^2. \quad (2)$$

Assume $p \equiv 3 \pmod{4}$. By Lemma 1, we define a Gray map from R to $\mathbb{Z}_{p^2}^2$ as follows:

$$\phi : R \rightarrow \mathbb{Z}_{p^2}^2$$

$$a + by \mapsto (ka + sb, ta + rb),$$

where $3p^2 = k^2 + s^2 + t^2 + r^2$. Then extend it in the obvious way to R^N . This Gray map is bijective, as the next result shows.

Theorem 1. *The matrix $\begin{pmatrix} k & t \\ s & r \end{pmatrix}$ is nonsingular.*

Proof. Suppose that, looking for a contradiction, a linear dependence between the two rows of the matrix. Let μ, ν be two integers satisfying $\nu(k, t) = \mu(s, r)$. Substituting k and s by their values in $3\nu^2 p^2$ we get

$$3\nu^2 p^2 = (\mu^2 + \nu^2)(r^2 + s^2).$$

By Diophantus identity (Lemma 3) the RHS is a sum of two squares. This contradicts Lemma 2, because the integer 3 times a square will always contain 3 to an odd power in its primary factors decomposition. \square

Theorem 2. *Assume $p \equiv 3 \pmod{4}$. For all codes C over R , we have $\phi(C)^\perp = \phi(C^\perp)$. If C is a self-dual (resp. LCD) code of length N over R , then $\phi(C)$ is self-dual (resp. LCD) of length $2N$ over \mathbb{Z}_{p^2} .*

Proof. For any vector $\mathbf{u} = (u_1, u_2, \dots, u_N), \mathbf{v} = (v_1, v_2, \dots, v_N) \in R^N$, where $u_i = a_{i_1} + b_{i_1}y, v_i = a_{i_2} + b_{i_2}y \in R$ ($i = 1, 2, \dots, N$). Suppose $\mathbf{u}\mathbf{v} = 0 \pmod{p^2}$, and noting that $p \equiv 3 \pmod{4}$, we have $y^2 + 1 = 0$ over R . Considering the standard inner product, we then obtain

$$\sum_{i=1}^N (a_{i_1} + b_{i_1}y)(a_{i_2} + b_{i_2}y) = \sum_{i=1}^N [(a_{i_1}a_{i_2} - b_{i_1}b_{i_2}) + (a_{i_1}b_{i_2} + a_{i_2}b_{i_1})y] = 0,$$

which is equivalent to

$$\begin{cases} \sum_{i=1}^N (a_{i_1}a_{i_2} - b_{i_1}b_{i_2}) \equiv 0 \pmod{p^2}, \\ \sum_{i=1}^N (a_{i_1}b_{i_2} + a_{i_2}b_{i_1}) \equiv 0 \pmod{p^2}. \end{cases}$$

We then naturally obtain

$$\begin{aligned} \phi(\mathbf{u})\phi(\mathbf{v}) &= \sum_{i=1}^N \phi(u_i)\phi(v_i) \\ &= \sum_{i=1}^N (ka_{i_1} + sb_{i_1}, ta_{i_1} + wb_{i_1})(ka_{i_2} + sb_{i_2}, ta_{i_2} + wb_{i_2}) \\ &\equiv \sum_{i=1}^N [(k^2 + t^2)(a_{i_1}a_{i_2} - b_{i_1}b_{i_2}) + (ks + tw)(a_{i_1}b_{i_2} + a_{i_2}b_{i_1})y] \\ &\equiv 0 \pmod{p^2}. \end{aligned}$$

This implies

$$\phi(C^\perp) \subseteq \phi(C)^\perp,$$

and, since ϕ is a bijection the first statement follows by showing that both sides have the same size.

The second statement for LCD codes is as in [4, Th. 5.2]. The second statement for self-dual codes follows by plugging $C^\perp = C$ in the first statement. \square

Example: If $p = 3$, then $3p^2 = 27 = 16 + 9 + 1 + 1$. The Gray map can be taken to be $a + by \mapsto (4a + 3b, a + b)$.

2.4 Finite fields

If L, K are two finite fields of respective orders p^{rs} and p^r satisfying $K \subseteq L$, we write the *trace* from L down to K as

$$\text{Tr}_{p^r}^{p^{rs}}(z) = z + z^{p^r} + \dots + z^{p^{r(s-1)}},$$

where r, s are positive integers.

2.5 Codes over fields and asymptotics

Let p be an odd prime, and denote by \mathbb{F}_p the finite field of order p . By a **code** of length N over \mathbb{F}_p , we shall mean a proper subset of \mathbb{F}_p^N . This code is **linear** if it is a \mathbb{F}_p -vector subspace of \mathbb{F}_p^N . The **dimension** of a code C , denoted by k , is equal to its dimension as a vector space. Its (minimum) **distance**, denoted by d or $d(C)$, is defined as the minimum Hamming weight of its nonzero elements. The **Hamming weight** of $x = (x_1, x_2, \dots, x_n) \in \mathbb{F}_p^n$, denoted by $w(x)$, is the number of indices i where $x_i \neq 0$. The three parameters of a code are written compactly as $[n, k, d]$. We extend this notation to a possibly nonlinear code $C \subseteq \mathbb{F}_p^n$, by letting then $k = \log_p(|C|)$, and letting d be the minimum pairwise distance between two nonzero codewords. If $C(n)$ is a family of codes of parameters $[n, k_n, d_n]$, the **rate** r and **relative distance** δ are defined as

$$r = \limsup_{n \rightarrow \infty} \frac{k_n}{n},$$

and

$$\delta = \liminf_{n \rightarrow \infty} \frac{d_n}{n}.$$

A family of codes is said to be **good** iff $r\delta > 0$.

Recall that the q -ary **entropy function** $H_q(\cdot)$ is defined for $0 < y < \frac{q-1}{q}$ by

$$H_q(y) = y \log_q(q-1) - y \log_q(y) - (1-y) \log_q(1-y).$$

3 Main results

3.1 Enumeration in a special case

Assume that p is a primitive root modulo n with n an odd prime. Then, $(p^2)^{\frac{n-1}{2}} \equiv 1 \pmod{n}$, and we have $x^n - 1 = (x-1)\tilde{g}_2(x)\tilde{g}_3(x)$ with $\tilde{g}_2(x), \tilde{g}_3(x)$ monic irreducible polynomials over \mathbb{F}_{p^2} . By Hensel lifting, since $(n, p) = 1$, we have $x^n - 1 = \vartheta(x-1)g_2(x)g_3(x)$ over R , with $g_2(x) \equiv \tilde{g}_2(x) \pmod{p}$, $g_3(x) \equiv \tilde{g}_3(x) \pmod{p}$ and $\vartheta \in R^*$. Both $g_2(x)$ and $g_3(x)$ are monic basic irreducible polynomials over R . The following lemma is taken from [13].

Lemma 4. Let $R_1 = GR(p^s, p^{sm})$ and $h(x)$ be a monic basic irreducible polynomial of degree l over R_1 . Then the residue class ring $\frac{R_1[x]}{(h(x))}$ is a Galois ring of characteristic p^s and cardinality $p^{sm l}$ and contains R_1 as a subring. Thus

$$\frac{R_1[x]}{(h(x))} = GR(p^s, p^{sm l}).$$

Thus this lemma shows that the alphabet rings \mathcal{R}_i for $i = 2, 3$ of the constituent codes defined below are also Galois rings.

3.1.1 $n \equiv 1 \pmod{4}$

Firstly, if $n \equiv 1 \pmod{4}$, since p is primitive modulo n , we can deduce $(p^2)^{\frac{n-1}{4}} \equiv -1 \pmod{n}$. This implies that -1 is in the p^2 -cyclotomic coset modulo n . Consequently, we obtain $g_i^*(x) = g_i(x)$ for $i = 2, 3$. Note that $\deg(g_i(x)) = \frac{n-1}{2}$. By the CRT, we have $C = C_1 \oplus C_2 \oplus C_3$, where C_1 is a code of length 2 over R and C_i is a code of length 2 over the ring $\frac{R[x]}{(g_i(x))}$ for $i = 2, 3$. Let $\mathcal{R}_i = \frac{R[x]}{(g_i(x))}$ for $i = 2, 3$. The properties of C_3 being similar to that of C_2 , we only investigate C_2 for simplicity's sake.

Write $C_2 = \langle [1, b] \rangle$, where $b \in \mathcal{R}_2$. Note that b can be uniquely decomposed in base p as $b = \alpha + p\beta$, where $\alpha, \beta \in \mathcal{T} = \{x \in \mathcal{R}_2 \mid x^{p^{n-1}} = x\}$. Define a generalized **Frobenius map** F as $F(b) = \alpha^{p^2} + p\beta^{p^2}$, then $F^{\frac{n-1}{4}}(b) = \alpha^{p^{\frac{n-1}{2}}} + p\beta^{p^{\frac{n-1}{2}}}$. The **conjugate** \bar{b} of b is $F^{\frac{n-1}{4}}(b)$. Throughout this paper, we let $u = p^{\frac{n-1}{2}}$. With this notation $\bar{b} = \alpha^u + p\beta^u$. We can then define a **Hermitian scalar product** on \mathcal{R}_2^2 , by the formula $x \cdot \bar{y} = x_1 \bar{y}_1 + x_2 \bar{y}_2$ for $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{R}_2^2$.

From the introduction we know there is $w \in R$ such that $w^2 = -1$. Write $w = w_1 + pw_2$, $w_1, w_2 \in \mathcal{T}_1 = \{x \in R \mid x^{p^2} = x\}$. Then $w^2 = w_1^2 + 2pw_1w_2 = -1$, which is equivalent to $w_1^2 = -1, w_2 = 0$. That is to say, only has two choices for w in R such that $w^2 = -1$. Let us denote one of this two w 's by $\sqrt{-1}$.

Theorem 3. Writing $C_1 = \langle [1, a] \rangle$, where $a \in R$. Then we have C_1 is self-dual over R iff $a = \pm\sqrt{-1}$.

Proof. C_1 is self-dual iff $1 + aa = 0$, which implies $a = \pm\sqrt{-1}$. □

The following lemma generalizes the *Yamada normal form* of [13].

Lemma 5. *Let $A, B \in \mathcal{T}$, then we have $A + B = T_1 + pT_2$ with $T_1, T_2 \in \mathcal{T}$ is uniquely given by*

$$\begin{cases} T_1 = (A^{\frac{1}{p}} + B^{\frac{1}{p}})^p, \\ T_2 \equiv -\mathcal{P}_p(A^{\frac{1}{p}}, B^{\frac{1}{p}}) \pmod{p}, \end{cases}$$

where $p\mathcal{P}_p(A, B) = \sum_{i=1}^{p-1} \binom{p}{i} A^i B^{p-i}$, and $\mathcal{P}_p(A, B)$ is a polynomial in A, B with integral coefficients.

Proof. Note that

$$(A + B)^p = A^p + B^p + p\mathcal{P}_p(A, B).$$

We then claim that $(A + B)^{p^i} = A^{p^i} + B^{p^i} + p\mathcal{P}_p(A^{p^{i-1}}, B^{p^{i-1}})$ for any integer i . This can be proved by induction on n as follows.

$$\begin{aligned} (A + B)^{p^n} &= (A^{p^{n-1}} + B^{p^{n-1}} + p\mathcal{P}_p(A^{p^{n-1}}, B^{p^{n-1}}))^p \\ &= (A^{p^{n-1}} + B^{p^{n-1}})^p \\ &= A^{p^n} + B^{p^n} + p\mathcal{P}_p(A^{p^{n-1}}, B^{p^{n-1}}), \end{aligned}$$

where the first and last equality hold by induction hypothesis. We then obtain

$$A + B = (A^{\frac{1}{p}} + B^{\frac{1}{p}})^p - p\mathcal{P}_p(A^{\frac{1}{p}}, B^{\frac{1}{p}}),$$

which implies

$$\begin{cases} T_1 = (A^{\frac{1}{p}} + B^{\frac{1}{p}})^p, \\ T_2 \equiv -\mathcal{P}_p(A^{\frac{1}{p}}, B^{\frac{1}{p}}) \pmod{p}, \end{cases}$$

since it can be checked by the formulas above that $(A^{\frac{1}{p}} + B^{\frac{1}{p}})^{p^n} = (A^{\frac{1}{p}} + B^{\frac{1}{p}})^p$, showing that $(A^{\frac{1}{p}} + B^{\frac{1}{p}})^p \in \mathcal{T}$. This determines T_1 uniquely. We can only determine $T_2 \pmod{p}$, which is enough for our purpose. Then the result follows. \square

With Lemma 5, we then obtain the following important theorem, which gives a necessary and sufficient condition for C_2 to be a self-dual code over \mathcal{R}_2 .

Theorem 4. *C_2 is self-dual over \mathcal{R}_2 with respect to Hermitian inner product iff*

$$\begin{cases} 1 + \alpha^{\frac{1+u}{p}} \equiv 0 \pmod{p}, \\ \beta\alpha^u + \beta^u\alpha - \mathcal{P}_p(1, \alpha^{\frac{1+u}{p}}) \equiv 0 \pmod{p}. \end{cases}$$

Proof. C_2 is self-dual iff $1 + b\bar{b} = 0$, i.e. $1 + bF^{\frac{n-1}{4}}(b) = 0$, which is equivalent to

$$1 + \alpha^{1+u} + p\beta\alpha^u + p\beta^u\alpha = 0 = T_1 + pT_2, \quad (3)$$

where $T_1 \in \mathcal{T}, T_2 \in \mathcal{T}$ and $T_1 = T_2 = 0$.

By Equation (3) and Lemma 5, we can then have

$$1 + \alpha^{1+u} = (1 + \alpha^{\frac{1+u}{p}})^p - p\mathcal{P}_p(1, \alpha^{\frac{1+u}{p}}).$$

Then Equation (3) is equivalent to

$$(1 + \alpha^{\frac{1+u}{p}})^p + p(\beta\alpha^u + \beta^u\alpha - \mathcal{P}_p(1, \alpha^{\frac{1+u}{p}})) = 0.$$

That implies

$$\begin{cases} 1 + \alpha^{\frac{1+u}{p}} \equiv 0 \pmod{p} \\ \beta\alpha^u + \beta^u\alpha - \mathcal{P}_p(1, \alpha^{\frac{1+u}{p}}) \equiv 0 \pmod{p} \end{cases}$$

Then the result follows. \square

Next, we will enumerate the number of possible choices for C_2 .

Theorem 5. *The number of self-dual codes C_2 over \mathcal{R}_2 is equal to $u(1+u)$.*

Proof. Let $x = \alpha^{\frac{1}{p}}$ and we then consider the equation

$$x^{1+u} \equiv -1 \pmod{p}.$$

It can be obtained that, noting that $2(1+u)$ is a divisor of $u^2 - 1$, the number of choices for x , hence for α , is $1+u$. Then by the equation $\beta\alpha^u + \beta^u\alpha - \mathcal{P}_p(1, \alpha^{\frac{1+u}{p}}) \equiv 0 \pmod{p}$, we get

$$Tr_u^{u^2}(\widehat{\beta}\widehat{\alpha}^u) = f(\widehat{\alpha}),$$

where $f(\widehat{\alpha})$ denotes $\mathcal{P}_p(1, \alpha^{\frac{1+u}{p}}) \equiv 0 \pmod{p}$. That implies that there are u choices for β when fixed α . Thus, the number of C_2 is equal to $u(1+u)$. \square

By the CRT, the following theorem can be derived.

Theorem 6. *The number of self-dual codes C over R is equal to $2u^2(u+1)^2$.*

Proof. By Theorems 3 and 5, we obtain the number of C 's as $2u^2(u+1)^2$. \square

We are now ready to investigate the number of LCD codes. We firstly introduce the following proposition.

Proposition 1. C_2 is LCD over \mathcal{R}_2 iff $1 + b \cdot \bar{b} \in \mathcal{R}_2^*$.

Proof. “ \Leftarrow ”

1. If $b \in \mathcal{R}_2^*$, we then obtain $C_2^{\perp_H} = \langle [1, -\frac{1}{b}] \rangle$. Suppose C_2 is not LCD, then there exist nonzero $t, k \in \mathcal{R}_2$ such that $t(1, b) = k(1, -\frac{1}{b})$. We then obtain $t(1 + b\bar{b}) = 0$, which implies $t = 0$ since $1 + b \cdot \bar{b} \in \mathcal{R}_2^*$. Contradiction !
2. If $b \in \mathcal{R}_2 \setminus \mathcal{R}_2^*$, write $b = pb'$ with $b' \in \mathcal{T}$. Then we have

$$C_2^{\perp_H} = \begin{pmatrix} 2\bar{b}' & 1 \\ 0 & 2b'' \end{pmatrix}$$

with $b'' \in \mathcal{T}$. Suppose C_2 is not LCD, then there exist nonzero $m, n, l \in \mathcal{R}_2$ such that $m(1, 2b') = n(2\bar{b}', 1) + l(0, 2b'')$. That implies

$$\begin{cases} m = 2n\bar{b}', \\ 2mb' = n + 2lb'', \end{cases}$$

then we obtain $m = 0$, a contradiction !

“ \Rightarrow ” Suppose $1 + b \cdot \bar{b} \in \mathcal{R}_2 \setminus \mathcal{R}_2^*$, then we have the following two cases.

1. If $b \in \mathcal{R}_2^*$, we have $p(1 + b \cdot \bar{b}) = 0 = p(1, b) \cdot \overline{(1, b)}$. Note that $C_2^{\perp_H} = \langle [1, -\frac{1}{b}] \rangle$, we then obtain $p(1, b) \in C_2^{\perp_H}$, which implies $p(1, b) \in C_2^{\perp_H} \cap C_2$. Contradiction!
2. If $b \in \mathcal{R}_2 \setminus \mathcal{R}_2^*$, we then get $1 \in \mathcal{R}_2 \setminus \mathcal{R}_2^*$, a contradiction !

This completes the proof. □

Similarly, we have Proposition 2 about C_1 , which is a constituent code over R .

Proposition 2. The code C_1 is LCD over R iff $1 + a \cdot a \in R^*$. In particular, the number of possible choices for a is $p^4 - 2p^2$.

Proof. The proof of the first statement is similar to that of Proposition 1, with the Euclidean inner product replacing the Hermitian inner product. We omit it here. Write $a = a_1 + pa_2$ with $a_1, a_2 \in \mathcal{T}_1$. By this criterion the number of possible choices for a is equal to

$$p^4 - |\{a \mid a^2 + 1 = pa', a' \in \mathcal{T}_1\}| = p^4 - |\{(a_1, a_2) \mid a_1^2 + 1 = 0\}|,$$

where $\mathcal{T}_1 = \{x \in R \mid x^{p^2} = x\}$. Then the result follows. \square

To determine the number of LCD codes C_2 , we will reason by complementation.

Theorem 7. C_2 is not LCD code over \mathcal{R}_2 with respect to Hermitian inner product iff

$$1 + \alpha^{\frac{1+u}{p}} \equiv 0 \pmod{p}.$$

Proof. C_2 is not LCD iff $1 + b\bar{b} \in p\mathcal{R}_2$, i.e.,

$$1 + \alpha^{1+u} + p\beta\alpha^u + p\beta^u\alpha = T_1 + pT_2, \quad (4)$$

where $T_1 = 0, T_2 \in \mathcal{T}$. By Equation (4) and Lemma 5, we can then have

$$1 + \alpha^{1+u} = (1 + \alpha^{\frac{1+u}{p}})^p - p\mathcal{P}_p(1, \alpha^{\frac{1+u}{p}}).$$

Then Equation (4) is equivalent to

$$(1 + \alpha^{\frac{1+u}{p}})^p + p(\beta\alpha^u + \beta^u\alpha - \mathcal{P}_p(1, \alpha^{\frac{1+u}{p}})) = pT_2.$$

That implies

$$1 + \alpha^{\frac{1+u}{p}} \equiv 0 \pmod{p},$$

with β is arbitrary. \square

Combining Proposition 2 and Theorem 7, we have the following important theorem.

Theorem 8. The number of LCD codes C over R is equal to $(p^4 - 2p^2)(p^{2(n-1)} - u^3 - u^2)^2$.

Proof. We know that the total number of codes over R and \mathcal{R}_2 are p^4 and $p^{2(n-1)}$, respectively. Based on Theorems 3 and 5, the choice for non LCD codes is $(u+1)u^2$, then the result follows by subtraction. \square

3.1.2 $n \equiv 3 \pmod{4}$

If $n \equiv 3 \pmod{4}$, we have $g_3^*(x) = g_2(x)$, $g_2^*(x) = g_3(x)$. Then by the CRT, we obtain $C = C_1 \oplus C'_2 \oplus C'_3$, where C'_2 is a code of length 2 over the ring $\frac{R[x]}{(g_2(x))}$ and C'_3 is a code of length 2 over the ring $\frac{R[x]}{(g_2^*(x))}$. Writing $C'_2 = \langle [1, b'] \rangle$. Let $\mathcal{R}'_i = \frac{R[x]}{(g_i(x))}$ for $i = 2, 3$. Similarly, b' can be uniquely decomposed in base p as $b' = \alpha' + p\beta'$, where $\alpha', \beta' \in \mathcal{T}' = \{x \in \mathcal{R}'_2 \mid x^{p^{n-1}} = x\}$.

Note that, if C is self-dual we then have $C'_3 = C'^{\perp}_2$. The following result is needed to count the number of self-dual C 's.

Theorem 9. *The number of dual pairs $(C'_2, C'_3 = C'^{\perp}_2)$ over \mathcal{R}'_2 is equal to $p^{2(n-1)} - p^{n-1}$.*

Proof. Assume that $C'_2 = \langle [1, b'] \rangle$, let $C'_3 = C'^{\perp}_2$, then we have the following discussion.

- (i) If b' is a unit, we then obtain $C'_3 = \langle [1, -\frac{1}{b'}] \rangle$.
- (ii) If b' is not a unit, let $b' = p\beta'$, then we get $C'_3 = \langle [-p\beta', 1] \rangle$.

Thus, from the form of the generator matrix of C'_3 , it is clear that the number of dual pairs $(C'_2, C'_3 = C'^{\perp}_2)$ is exactly the size of \mathcal{R}'_2 , i.e., is equal to $p^{2(n-1)} - p^{n-1}$. \square

Theorem 10. *The number of self-dual codes C over R is equal to $2 \cdot (p^{2(n-1)} - p^{n-1})$.*

Proof. Based on Theorems 3 and 9, we can obtain the desired results. \square

Lemma 6. *Writing $C'_2 = \langle [1, b'] \rangle$ and $C'_3 = \langle [1, c'] \rangle$. Then*

$$\begin{cases} C'_2 \cap C'^{\perp}_3 = \{0\}, \\ C'^{\perp}_2 \cap C'_3 = \{0\} \end{cases}$$

iff $1 + b'c' \notin p\mathcal{R}'_2$.

Proof. “ \Rightarrow ” If $1 + b'c' \in p\mathcal{R}'_2$, we then obtain $p(1 + b'c') = 0$, which implies $p(1, b') \cdot (1, c') = 0$. That is equivalent to $p(1, b') \in C'^{\perp}_3$, noting that $p(1, b') \in C'_2$, which is a contradiction with $C'_2 \cap C'^{\perp}_3 = \{0\}$.

“ \Leftarrow ” If $C'_2 \cap C'^{\perp}_3 \neq \{0\}$, then we must have $\lambda \in GR(p^2, p^{2(n-1)})$ such that $\lambda(1, b')(1, c') = \lambda(1 + b'c') = 0$.

- (1) If λ is a unit, we get $1 + b'c' = 0$, which is a contradiction with $1 + b'c' \notin p\mathcal{R}'_2$.

- (2) If λ is not a unit, writing $\lambda = p\lambda_1$ with $\lambda_1 \in \mathcal{T}_{e_j}$, we can obtain $1 + b'c' \in p\mathcal{R}'_2$,
Contradiction !

Then the result follows. \square

Theorem 11. *Suppose $u' = p^{n-1}$, the number of LCD codes over R is equal to $(p^4 - 2p^2) \cdot (u'^4 - u'^2 + u')$.*

Proof. Just keep the same notations as in Lemma 6. In the following, we aim to count the possible choices for C'_2 and C'_3 . Then we have

- (1) If b' is a unit, then c' is arbitrary except the case when $c' \in \frac{-1}{b'} + p\mathcal{R}'_2$, which implies the number of pairs (b', c') is $(u'^2 - u')^2$.
- (2) If b' is not a unit, i.e., $b' = pb'_1$ with $b'_1 \in \mathcal{T}'$, then c' is arbitrary. In detail, $c' = c'_1 + pc'_2$ with $c'_1, c'_2 \in \mathcal{T}'$, we have $1 + b'c' = 1 + pc'_1b'_1 \notin p\mathcal{R}'_2$. Thus, the number of pairs (b', c') is $u' \cdot u'^2 = u'^3$.

Thus, the total number of C'_j, C''_j is $(u'^2 - u')^2 + u'^3 = u'^4 - u'^3 + u'^2$. Therefore, the total number of LCD double circulant codes is equal to, based on Proposition 2,

$$(p^4 - 2p^2) \cdot (u'^4 - u'^3 + u'^2).$$

This completes the proof. \square

3.2 Enumeration in the general case

The following result, while not needed for the asymptotics, is of interest in its own right.

Theorem 12. *Let n be an odd integer, assume that the factorization of $x^n - 1$ into irreducible polynomials over R is of the form*

$$x^n - 1 = \varsigma(x - 1) \prod_{i=2}^s g_i(x) \prod_{j=1}^t h_j(x) h_j^*(x),$$

with $\varsigma \in R^*$, and $g_i(x)$ is a self-reciprocal polynomial of degree d_i with d_i a even integer, the polynomial $h_j(x)$ is of degree e_j and $*$ denotes reciprocation. The number of self-dual double circulant codes over R is

$$2 \prod_{i=2}^s (u_i^2 + u_i) \prod_{j=1}^t (u_j'^2 - u_j'),$$

The number of LCD double circulant codes over R is

$$(p^4 - 2p^2) \prod_{i=2}^s (u_i^4 - u_i^3 - u_i^2) \prod_{j=1}^t (u_j'^4 - u_j'^3 + u_j'^2),$$

where $u_i = p^{d_i}$, $u_j' = p^{2e_j}$.

Proof. Let $\mathcal{R} = \frac{R[x]}{(x^n-1)}$. We know that

$$\mathcal{R} \simeq \frac{R[x]}{(x-1)} \oplus \left(\bigoplus_{i=2}^s \frac{R[x]}{(g_i(x))} \right) \oplus \left(\bigoplus_{j=1}^t \left(\frac{R[x]}{(h_j(x))} \oplus \frac{R[x]}{(h_j^*(x))} \right) \right)$$

by the CRT. Denote by G_i the ring $\frac{R[x]}{(g_i(x))}$, by H_j' the ring $\frac{R[x]}{(h_j(x))}$ and by H_j'' the ring $\frac{R[x]}{(h_j^*(x))}$. This decomposition naturally extends to \mathcal{R}^2 as

$$\mathcal{R}^2 \simeq R^2 \oplus \left(\bigoplus_{i=2}^s G_i^2 \right) \oplus \left(\bigoplus_{j=1}^t (H_j'^2 \oplus H_j''^2) \right).$$

In particular, each \mathcal{R} -linear code of length 2 can be decomposed as the ‘‘CRT sum’’

$$C \simeq C_1 \oplus \left(\bigoplus_{i=2}^s C_i \right) \oplus \left(\bigoplus_{j=1}^t (C_j' \oplus C_j'') \right).$$

By Theorem 3, and the analogues of Theorems 5 and 9, the number of self-dual codes is

$$2 \prod_{i=2}^s u_i(1 + u_i) \prod_{j=1}^t (u_j'^2 - u_j').$$

Based on Theorems 8 and 11, the number of LCD codes is equal to

$$(p^4 - 2p^2) \prod_{i=2}^s (p^{4d_i} - u_i^2(u_i + 1)) \prod_{j=1}^t (u_j'^4 - u_j'^3 + u_j'^2),$$

where $u_i = p^{d_i}$, $u_j' = p^{2e_j}$. Then the result follows. \square

4 Relative distance bound

This section only uses enumeration for the special case of the factorization of $x^n - 1$ into two irreducibles. We assume $p \equiv 3 \pmod{4}$ to use the duality-preserving Gray

map given by Section 2.3. If C is an R -code, we call its \mathbb{F}_p image the image of $\phi(C)$ by the Gray map of [9] with $k = 1$. Note that if C is a code of length $2n$ over R then $\phi(C)$ is of length $4n$ over \mathbb{Z}_{p^2} and the \mathbb{F}_p -image of $\phi(C)$ has length $4pn$ over \mathbb{F}_p . We prepare for the proof of the main result by the following result.

Theorem 13. *If $e, f \in R^n$, and $(0, 0) \neq (e, f)$ has Hamming weight $< n$, then the vector (e, f) is in at most λ double circulant codes of length $2n$ with $\lambda = p^{3n+1}$.*

Proof. Write $(e, f) = (e_1, f_1) \oplus (e_2, f_2) \oplus (e_3, f_3)$ for the CRT decomposition of (e, f) . Consider $C_1 = \langle [1, a] \rangle$, $a = \alpha_1 + p\beta_1$, $\alpha_1, \beta_1 \in \mathcal{T}_1 = \{x \in R \mid x^{p^2} = x\}$. Let $(e_1, f_1) \in C_1$, we have $f_1 = e_1 a$.

- (i) If e_1 is a unit, $a = \frac{f_1}{e_1}$.
- (ii) If $e_1 (\neq 0)$ is not a unit, write $e_1 = pe'_1$ with $e'_1 \in \mathcal{T}_1^*$. We then obtain $f_1 = pae'_1$.
 - (ii-1) $\alpha_1 \neq 0$, we have $f_1 = pf'_1 = p\alpha_1 e'_1$, which implies $\alpha_1 = \frac{f'_1}{e'_1}$, β_1 is arbitrary.
 - (ii-2) $\alpha_1 = 0$, we have $a = p\beta_1$, which implies $f_1 = 0$, β_1 is arbitrary.
- (iii) If $e_1 = 0$, we then get $f_1 = 0$. That implies a is arbitrary. We can easily get there are at most p^4 double circulant codes that containing (e_1, f_1) .

Consider $C_2 = \langle [1, b] \rangle$, $b = \alpha_2 + p\beta_2$, $\alpha_2, \beta_2 \in \mathcal{T}_2$, where the case is similar to that of C_1 with \mathcal{T}_2 playing the role of \mathcal{T}_1 . Note that $\mathcal{T}_2 = \mathcal{T}$ or \mathcal{T}' . Thus, the number of constituent codes C_2 is at most $|\mathcal{T}|^2 = |\mathcal{T}'|^2$.

The case is the same as that of C_3 . However, these two analyses cannot be run independently. The detail is as follows.

1. If both e_2 and e_3 are 0, then we obtain $e \in (g_2(x)g_3(x))$, the repetition code of length n . So either $e = 0$ yielding $f = 0$, or $w_H(e) = n$, contradicting the hypothesis. This argument shows that the case (iii) cannot happen simultaneously for C_2 and C_3 .
2. If both e_2 and e_3 are not units, and $e_2, e_3 \neq 0$, we can easily get $\widehat{e} \equiv \widehat{e_2 e_3} \equiv 0 \pmod{g_2(x)g_3(x)}$.
 - (a) If $\widehat{e} \neq 0$, we then obtain $w_H(e) \geq w_H(\widehat{e}) = n$, a contradiction !

- (b) If $\widehat{e} = 0$, assume that $0 \neq e = pe'$, then we have $f = pf'$. Write $C = \langle [1, d] \rangle$, where $d = d_1 + pd_2$ with $d \in \frac{R[x]}{(x^n-1)}$, $d_1, d_2 \in \{x \in \frac{R[x]}{(x^n-1)} \mid x^{p^{2n}} = x\}$. We can get $d_1 = \frac{f'}{e'}$ and d_2 is arbitrary. In this case, there are at most p^{2n} double circulant codes containing (e, f) .

Thus, we obtain $\lambda = p^4 |\mathcal{T}|^2 |\mathcal{T}| = p^{3n+1}$ when $e_1 = e_2 = 0$ and e_3 is not unit. \square

Theorem 14. *If $e, f \in R^n$, and $(0, 0) \neq (e, f)$ has Hamming weight $< n$, then the vector (e, f) is in at most λ self-dual double circulant codes of length $2n$ with $\lambda = 2u^2(u+1)$.*

Proof. Write $(e, f) = (e_1, f_1) \oplus (e_2, f_2) \oplus (e_3, f_3)$ for the CRT decomposition of (e, f) . Let $(e_1, f_1) \in C_1 = \langle [1, a] \rangle$, $a = \alpha_1 + p\beta_1$, $\alpha_1, \beta_1 \in \mathcal{T}_1 = \{x \in R \mid x^{p^2} = x\}$. We can easily get there at most exist 2 self-dual codes which contain (e_1, f_1) . Consider $(e_2, f_2) \in C_2 = \langle [1, b] \rangle$, $b = \alpha_2 + p\beta_2$, with $\alpha_2, \beta_2 \in \mathcal{T}$ or \mathcal{T}' , we then obtain $f_2 = be_2$.

- (i) If e_2 is a unit, then $b = \frac{f_2}{e_2}$.
- (ii) If $e_2 (\neq 0)$ is not a unit, write $e_2 = pe'_2$ with $e'_2 \in \mathcal{T}_2^* = \{x \in R \mid x^{p^2} = x\}$. We then obtain $f_2 = pae'_2$.
 - (ii-1) $\alpha_2 \neq 0$, we have $f_2 = pf'_2 = p\alpha_2 e'_2$, which implies $\alpha_2 = \frac{f'_2}{e'_2}$, β_2 is arbitrary.
 - (ii-2) $\alpha_2 = 0$, we have $b = p\beta_2$, which implies $f_2 = 0$, β_2 is arbitrary.
- (iii) If $e_2 = 0$, we then get $f_2 = 0$. That implies b is arbitrary. We can easily get there are at most $u(1+u)$ self-dual codes that containing (e_2, f_2) .

By Theorem 13, then $e_2, e_3 \neq 0$. If e_2, e_3 are not units with $e_2, e_3 \neq 0$, based on 2 in Theorem 13, for the subcase 2(b), we obtain $1 + d^2 = 0$. We then get $1 + d_1^2 + 2pd_1d_2 = 0$, where $d = d_1 + pd_2$. Then we have $1 + d_1^2 \equiv 0 \pmod{p}$, which implies two coices for d_1 . By Lemma 5, we can get $2\widehat{d}_1\widehat{d}_2 - \mathcal{P}_p(1, \widehat{d}_1^{\frac{2}{p}}) = 0$, which implies $\widehat{d}_2 = \frac{\mathcal{P}_p(1, \widehat{d}_1^{\frac{2}{p}})}{2\widehat{d}_1}$. Then d_2 is uniquely determined by d_1 . That is to say, there are at most 2 self-dual codes containing (e, f) when e_2, e_3 are not units.

Thus, by Theorem 5, we obtain $\lambda = 2u^2(u+1)$ with at most u self-dual codes containing (e_3, f_3) if $n \equiv 1 \pmod{4}$. When $n \equiv 3 \pmod{4}$, C_3 is determined by C_2 .

Then, the result follows. \square

We can now state and prove the main result of this paper.

Theorem 15. *Assume the Artin conjecture for primes in arithmetic progression [11] holds. There is an infinite family of double circulant self-dual (resp. LCD) R -codes with rate $\frac{1}{2}$ and relative Hamming distance of the \mathbb{F}_p image $\delta \geq H_p^{-1}(1/8p)$ (resp. $\delta \geq H_p^{-1}(1/4p)$).*

Proof. Artin conjecture for primes in arithmetic progression shows in particular that p and $\epsilon \in \{\pm 1\}$ being given, there are infinitely many primes $n \equiv \epsilon \pmod{4}$ such that p is primitive modulo n . It should be noted that the size Ω_n of the family of codes we consider is asymptotically equivalent to $2u^4$ for self-dual codes and to p^{4n} for LCD codes. This holds for the case $\epsilon = 1$ by Subsection 3.1.1 like for the case $\epsilon = -1$ by Subsection 3.1.2. Assume we can prove that for n large enough $\Omega_n > \lambda B(d_n)$, where $\lambda = p^{3n+1}$ for LCD codes and $\lambda = 2u^2(u+1)$ for self-dual codes and $B(r)$ denotes the number of vectors in R^{2n} with Hamming weight of their \mathbb{F}_p image $< r$. This would imply by Theorem 13 that there are codes of length $2n$ in the family with \mathbb{F}_p image distance $\geq d_n$. Denote by δ the relative distance of this family of p -ary codes. If we take d_n the largest number satisfying the said inequality, and assume a growth of the form $d_n \sim 4p\delta_0 n$, then, using an entropic estimate for $B(d_n) \sim p^{4pnH_p(\delta_0)}$ (cf. [6, Lemma 2.10.3]) yields, with the said values of Ω_n and λ the estimate $H_p(\delta_0) = \frac{1}{8p}$ for self-dual codes and $H_p(\delta_0) = \frac{1}{4p}$ for LCD codes. The result follows by observing that, by definition of the family of codes so constructed, $\delta \geq \delta_0$. \square

5 Numerical examples

Assume $p = 3$ and $C = \langle [I, A_0 + yA_1] \rangle$, where A_0, A_1 are circulant matrices of size n and I is the identity matrix of the same size. The generator matrix of $\phi(C)$ can be computed as

$$\begin{pmatrix} 4I & I & 4A_0 + 3A_1 & A_0 + A_1 \\ 3I & I & 3A_0 - 4A_1 & A_0 - A_1 \end{pmatrix}.$$

Define the base 3 decomposition of $x \in \mathbb{Z}_9$ as

$$x = r_0(x) + 3r_1(x),$$

where $r_i(x) \in \{0, 1, 2\}$. Then we can define the Gray map of [9] as

$$\Phi : \mathbb{Z}_9 \rightarrow \mathbb{Z}_3^3$$

Table 1: Double circulant LCD codes over $\mathbb{Z}_9 + u\mathbb{Z}_9$ and their Gray images ϕ, Φ

n	$a_1(x)$	$a_0(x)$	$(4n, 9^{2n}, d_{\phi(C)})_{\mathbb{Z}_9}$	$[12n, 4n, d_{\Phi(C)}]_{\mathbb{Z}_3}$	Distance of BKLC over \mathbb{Z}_3
2	41	51	$(8, 9^4, 4)$	$[24, 8, 6]$	11
3	811	081	$(12, 9^6, 6)$	$[36, 12, 12]$	15
4	3651	6505	$(16, 9^8, 5)$	$[48, 16, 14]$	18
5	10856	57664	$(20, 9^{10}, 6)$	$[60, 20, 16]$	21

$$a \mapsto (a_0, a_1, a_2),$$

where $a_i = r_1(a) + ir_0(a)$. The explicit map is tabulated below.

x	$\Phi(x)$	$w_H(\Phi(x))$
0	(0,0,0)	0
1	(0,1,2)	2
2	(0,2,1)	2
3	(1,1,1)	3
4	(1,2,0)	2
5	(1,0,2)	2
6	(2,2,2)	3
7	(2,0,1)	2
8	(2,1,0)	2

In Table 1 and Table 2 we have collected some examples of double circulant LCD and self-dual codes obtained by random search in Magma. The coefficients of degree n polynomial $a(x)$ are written in decreasing powers of x , for example for $n = 3$, the entry 811 means $8x^2 + x + 1$. The parameters over \mathbb{Z}_9 and \mathbb{Z}_3 are given in the form $(4n, 9^{2n}, d_{\phi(C)})$ and $[12n, 4n, d_{\Phi(C)}]$ respectively, where $d_{\phi(C)}$ and $d_{\Phi(C)}$ are the Hamming minimum distances of their Gray images ϕ and Φ respectively. The entry in the rightmost column is the best known distance of an $[12n, 4n]$ ternary linear code, obtained by looking up at the tables in www.codetables.de.

Table 2: Double circulant self-dual codes over $\mathbb{Z}_9 + u\mathbb{Z}_9$ and their Gray images ϕ, Φ

n	$a_1(x)$	$a_0(x)$	$(4n, 9^{2n}, d_{\phi(C)})_{\mathbb{Z}_9}$	$[12n, 4n, d_{\Phi(C)}]_{\mathbb{Z}_3}$	Distance of BKLC over \mathbb{Z}_3
2	10	00	$(8, 9^4, 3)$	$[24, 8, 10]$	11
3	811	081	$(12, 9^6, 6)$	$[36, 12, 12]$	15
4	6731	4752	$(16, 9^8, 6)$	$[48, 16, 15]$	18
5	26758	62532	$(20, 9^{10}, 6)$	$[60, 20, 18]$	21

6 Conclusion

In this article we have studied double circulant codes either self-dual or LCD over Galois rings of characteristic p^2 and size p^4 . Extending the study to $GR(p^s, p^{ms})$, with $s > 2$ would result in more terms in the base p expansion of a ring element and would make the computations of Section 3 more difficult. A similar remark can be made about using $m > 2$. More tractable could be to study quasi-cyclic codes of higher index, like four-circulant codes, for instance.

We have used the composition of two Gray maps to derive codes over \mathbb{F}_p . While the choice of the Hamming metric over \mathbb{F}_p is the most natural one, the study of the Lee minimum distance of the \mathbb{F}_p -image could also be worthwhile.

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