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How many weights can a linear code have?

Minjia Shi^{1,2} · Hongwei Zhu² · Patrick Solé³ ·
Grard D. Cohen⁴

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Abstract We study the combinatorial function $L(k, q)$, the maximum number of nonzero weights a linear code of dimension k over $\mathbb{F}_q$ can have. We determine it completely for $q = 2$, and for $k = 2$, and provide upper and lower bounds in the general case when both k and q are ≥ 3 . A refinement $L(n, k, q)$, as well as nonlinear analogues $N(M, q)$ and $N(n, M, q)$, are also introduced and studied.

Keywords Linear codes · Hamming weight · Perfect difference sets

Mathematics Subject Classification 94B05 · 05B10

1 Introduction

There are several problems in extremal combinatorics on distances in codes. For instance, the famous paper [5] derives an upper bound on the size of a code C over $\mathbb{F}_q$ with exactly s distinct distances:

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✉ Minjia Shi
mjshi@ahu.edu.cn

¹ Key Laboratory of Intelligent Computing & Signal Processing, Ministry of Education, Anhui University, No. 3 Feixi Road, Hefei 230039, Anhui, People’s Republic of China

² School of Mathematical Sciences, Anhui University, Hefei 230601, People’s Republic of China

³ CNRS/LAGA, University of Paris 8, 2 rue de la Libert, 93 526 Saint-Denis, France

⁴ TelecomParisTech, 46 rue Barrault, 75 013 Paris, France

$$|C| \leq \sum_{j=0}^s \binom{n}{j} (q-1)^j. \quad (1)$$

In the same spirit, other authors have given upper bounds on the size of codes with one or several forbidden distances [6].

In this note, we tackle a related but distinctly different problem: how many distinct weights can a linear code of given dimension over a given finite field have? In other words, we study the combinatorial function $L(k, q)$, the maximum number of nonzero weights a code of dimension k over $\mathbb{F}_q$ may have. While an upper bound is easy to prove (Proposition 2), its tightness is nontrivial¹ and we only manage to establish it in some special cases like $k = 2$ or $q = 2$ (Cf. Theorems 1 and 2). Numerical experiments with very long random codes suggest it is tight for all k 's and q 's. We leave the question as an open problem. We can also study the more refined function $L(n, k, q)$, the maximum number of nonzero weights an $[n, k]_q$ code may have. This latter function is related to both $L(k, q)$ and Eq. (1) above. The nonlinear counterpart of $L(k, q)$ denoted by $N(M, q)$, can be determined explicitly (Theorem 6). The nonlinear counterpart of $L(n, k, q)$ denoted by $N(n, M, q)$, can also be studied. The rate of convergence of $N(n, M, q)$ towards $N(M, q)$ requires perfect difference sets [3] and primes in short intervals [2] for its careful study.

The material is organized as follows. Section 2 collects the necessary notations and definitions. Section 3 studies upper bounds in the linear code case. Section 4 derives lower bounds in that situation. Section 5 introduces and investigates the function $L(n, k, q)$. Section 6 tackles the nonlinear analogues of $L(k, q)$ and $L(n, k, q)$, denoted by $N(M, q)$, and $N(n, M, q)$, respectively. Section 7 concludes the article. An appendix collects some numerical values, which comfort the Conjecture that Proposition 2 is tight.

2 Definitions and notation

Let q be a prime power, and $\mathbb{F}_q$ denote the finite field of order q . By a **code** of length n over $\mathbb{F}_q$, we shall mean a proper subset of $\mathbb{F}_q^n$. This code is **linear** if it is a $\mathbb{F}_q$ -vector subspace of $\mathbb{F}_q^n$. The **dimension** of a code, denoted by k , is equal to its dimension as a vector space. The parameters of such a code are written compactly as $[n, k]_q$. The **Hamming weight** of $x \in \mathbb{F}_q^n$, denoted by $w(x)$, is the number of indices i where $x_i \neq 0$. The **Hamming distance** between $x \in \mathbb{F}_q^n$, and $y \in \mathbb{F}_q^n$, denoted by $d(x, y)$, is defined by $d(x, y) = w(x - y)$. For a given prime power q and given values of k , let $L(k, q)$ denote the largest possible number of nonzero weights a q -ary code can have. If $C(n)$ is a family of codes of parameters $[n, k_n]_q$, the **rate** R is defined as

$$R = \limsup_{n \rightarrow \infty} \frac{k_n}{n}.$$

Recall that the q -ary **entropy function** $H_q(\cdot)$ is defined for $0 < y < 1$, by

$$H_q(y) = y \log_q(q-1) - y \log_q(y) - (1-y) \log_q(1-y).$$

3 Upper bounds

The following monotonicity properties of $L(k, q)$ are given without proof.

¹ After submission of this article, a proof was found in [1].

$$\begin{aligned} L(k, q) &\leq L(k+1, q), \\ L(k, q) &< L(k, q^m). \end{aligned}$$

Proposition 2 *For all prime powers q , and all integers $k \geq 1$, we have*

$$L(k, q) \leq \frac{q^k - 1}{q - 1}.$$

Proof The total numbers of nonzero codewords of a code of dimension k over $\mathbb{F}_q$ is $q^k - 1$, and all the nonzero multiples of a given codeword share the same weight. $\square$

This bound is met with equality if $q = 2$.

$$L(k, 2) = 2^k - 1.$$

Proof Denote by G_k the generator matrix of an $[n, k]_q$ with $L(k, 2)$ weights $w_1 < w_2 < \dots < w_{L(k, 2)}$. Define H_{k+1} a matrix obtained from G_k by adding a k by t block of zeros, and by G_{k+1} the matrix obtained by H_{k+1} by adding a row with first n coordinates zero and last t coordinates $= 1$. The code spanned by the rows of G_{k+1} has all these weights plus the $L(k, 2) + 1$ new weights $t < t + w_1 < \dots < t + w_{L(k, 2)}$. The two sets of weights will have void intersection if $w_{L(k, 2)} < t$. This makes $2L(k, 2) + 1$ weights altogether. Note that the rank of G_{k+1} is $k + 1$. Thus we have proved that $L(k + 1, 2) \geq 2L(k, 2) + 1$, which implies by induction, starting from $L(1, 2) = 1$, the lower bound $L(k, 2) \geq 2^k - 1$. The result follows. $\square$

Remark We are now ready to give an alternative proof of Theorem 1. we can exhibit a linear code C with dimension k over $\mathbb{F}_2$ with $2^k - 1$ nonzero weights. Let the generator matrix of C be

[illegible]

where $a_1 = 1, a_2 = 2, a_3 = 2^2, a_4 = 2^3, \dots, a_k = 2^{k-1}$. Since $a_{j_1} + a_{j_2} + \dots + a_{j_t} = (\underbrace{\dots 010 \dots 010 \dots 010 \dots}_k)_2$ in base 2, and the coordinates of 1 's are $j_1, j_2, \dots, j_t$,

respectively. Thus, we obtain all integers of k bits as possible weights that is the set $\{1, 2, 3, \dots, 2^k - 1\}$ of cardinality $2^k - 1$ in all.

The bound in Proposition 2 is also tight when $k = 2$.

Theorem 2 For all prime powers q , we have $L(2, q) = q + 1$.

Proof Let $\{u, v\}$ be a basis of a code C candidate to have $q + 1$ weights. Denote by S, T the supports of u, v respectively. Let $|S \setminus T| = a, |T \setminus S| = b$. On the intersection $S \cap T$ assume v is the all-one vector. Denote by ω a primitive root of $\mathbb{F}_q$. Assume $|S \cap T| = \binom{q}{2}$ and that u restricted to $S \cap T$ is

$$(1, \omega, \omega, \omega^2, \omega^2, \omega^2, \dots, \omega^{q-2}, \dots, \omega^{q-2}),$$

where ω^i occurs $i + 1$ times. With these conventions, we see that the weights of C are

- $w(u) = a + \binom{q}{2}$,
- $w(v) = b + \binom{q}{2}$,
- $w(u - xv) = a + b + \binom{q}{2} - i$ if $x = \omega^{i-1}$ for $i = 1, 2, \dots, q - 1$.

Assume $a < b$. The above weights will be pairwise different if $a + b + \binom{q}{2} - (q - 1) > b + \binom{q}{2}$, that is if $a \geq q$. Thus, under these conditions, C counts $2 + q - 1 = q + 1$ nonzero weights. $\square$

Remark The shortest $[n, 2]_q$ code with $L(2, q)$ nonzero weights obtained by this construction has $n = \binom{q}{2} + 2q + 1$.

4 Lower bounds

The easiest lower bound is

Proposition 3 For all prime powers q , and all integers $k \geq 1$, we have $L(k, q) \geq k$.

Proof Consider the code $\mathbb{F}_q^k$, of length and dimension k . $\square$

This can be improved to a bound that is exponential in k .

Proposition 4 For all prime powers q , and all integers $k \geq 1$, we have

$$L(k + 1, q) \geq 2L(k, q) + 1.$$

In particular, for all integers $k \geq 2$, we have

$$L(k, q) \geq 2^{k-2}q + 2^{k-2} + 1.$$

Proof Same argument as in the first proof of Theorem 1. The second assertion follows by iterating this bound starting from $L(2, q) = q + 1$. $\square$

An asymptotic version of the preceding results is as follows. Define

$$\lambda(q) = \limsup_{n \rightarrow \infty} \frac{1}{k} \log_q(L(k, q)).$$

Theorem 3 For all prime powers q we have

$$\log_q 2 \leq \lambda(q) \leq 1.$$

In particular $\lambda(2) = 1$.

Proof The first inequality comes from Proposition 4. The second one comes Proposition 2. $\square$

Remark Since we conjecture that the bound of Proposition 2 is tight, it is natural to conjecture that $\lambda(q) = 1$ for all prime powers q .

5 Refinements and asymptotics

A more complex function is $L(n, k, q)$ the largest number of nonzero weights an $[n, k]_q$ -code can have. This function is related to $L(k, q)$ in several ways. The following monotonicity properties of $L(n, k, q)$ are given without proof.

Proposition 5 *For all nonnegative integers k, m and all prime powers q we have:*

$$\begin{aligned} L(n, k, q) &\leq L(n, k+1, q), \\ L(n, k, q) &\leq L(n, k, q^m). \end{aligned}$$

The three following lemmas are useful for the proof of Theorem 4.

Lemma 1 *For all prime powers q , and all nonnegative integers n, k we have $L(n, k, q) \leq L(k, q)$.*

Proof Immediate from the definitions. $\square$

The new function is also monotone in n .

Lemma 2 *For all prime powers q , and all nonnegative integers n, k we have $L(n, k, q) \leq L(n+1, k, q)$.*

Proof If C is an $[n, k]_q$ code with $L(n, k, q)$ nonzero weights, then C extended by a constant zero coordinate is an $[n+1, k]_q$ -code with the same number of nonzero weights. $\square$

Lemma 3 *For all prime powers q , and all nonnegative integers n, k we have $L(n, k, q) \leq n$.*

Proof Note that, by definition of the Hamming weight, a code of length n can have at most n distinct weights. $\square$

We now connect the new function $L(n, k, q)$ with $L(k, q)$.

Theorem 4 *For all prime powers q , and all nonnegative integers k we have*

$$\lim_{n \rightarrow \infty} L(n, k, q) = L(k, q).$$

More precisely, there is an integer $n_0 \geq L(k, q)$, such that for all $n \geq n_0$ we have $L(n, k, q) = L(k, q)$.

Proof By Lemmas 1 and 2, the sequence $n \mapsto L(n, k, q)$ is increasing and bounded. Hence, being integral, it converges stably to a limit which can be no other than $L(k, q)$. Let n_0 be such that $L(n_0, k, q) = L(k, q)$. By Lemma 3, we see that $n_0 \geq L(k, q)$. $\square$

Remark The computations of the Appendix suggest that such an n_0 can be very large. If Proposition 2 is tight then, by Theorem 4 $n_0 \geq \frac{q^k-1}{q-1}$. In the special case $q = 2$, the second proof of Theorem 1 shows that $n_0 = 2^k - 1$.

There is a link to Delsarte's bound (Eq. (1)) quoted in the Introduction.

Proposition 6 *For all prime powers q , and all integers $n \geq k \geq 1$, we have*

$$q^k \leq \sum_{i=0}^{L(n,k,q)} \binom{n}{i} (q-1)^i.$$

Fig. 1 Definition of $t(q)$

Further

$$L(k, q) \leq \frac{\sum_{i=0}^{L(n,k,q)} \binom{n}{i} (q-1)^i - 1}{q-1}.$$

Proof The first assertion is a direct application of Eq. (1) in Introduction ([5, Theorem 4.1]) with $|C| = q^k$, and $s = L(n, k, q)$. Combining this result with Proposition 2 gives the second assertion. $\square$

We give an asymptotic version of the preceding results. Let

$$\mathcal{L}(R) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log_q(L(n, \lfloor Rn \rfloor, q)).$$

Theorem 5 If C_n is a family of codes of rate R then

$$\mathcal{L}(R) \leq R\lambda(q) \leq H_q(\mathcal{L}(R)).$$

In particular $\mathcal{L}(R) \leq t(q)$, where $t(q)$ is the unique solution in the range $(0, \frac{q-1}{q})$ of $H_q(x) = x$. See Fig. 1.

Proof The first inequality follows by Lemma 1, upon observing that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} (L(k, q)) = R\lambda(q).$$

The second inequality comes from the second assertion of Proposition 6, after using standard entropic estimates [7]. The second assertion is obtained by combining the first and second inequality. $\square$

Define the domain $\mathcal{D}$ as the set of points in the plane $(R, \mathcal{L})$ that are realized by a family of codes. By the preceding result, this domain is contained in the domain of boundaries given by, counterclockwise, in Fig. 2 by

1. the straight line $\mathcal{L} = R$ from $R = 0$ till $R = t(q)$,
2. the horizontal line $\mathcal{L} = t(q)$ from $R = t(q)$ till $R = 1$,
3. the vertical line $R = 1$ from $\mathcal{L} = t(q)$ till $\mathcal{L} = \frac{q-1}{q}$,
4. the curve $\mathcal{L} = H_q^{-1}(R)$, from $R = 1$ till $R = 0$.

Determining the domain $\mathcal{D}$ explicitly, in the same way as the domain of packing and covering codes in [4] is a challenging open problem.

Fig. 2 Boundaries of domain $\mathcal{D}$

6 Nonlinear codes

Warning In this section only q is an arbitrary integer > 1 .

The nonlinear analogue of the function $L(k, q)$ is the function $N(M, q)$ which is the largest number of distances between two codewords of an unrestricted code of size M over some finite alphabet A_q of size q . This function is completely determined in the following Theorem.

Theorem 6 For all integers $M \geq 2$, we have

$$N(M, q) = \binom{M}{2}.$$

Proof By definition we have immediately $N(M, q) \leq \binom{M}{2}$. By an inductive process, we construct a code C_M with $\binom{M}{2}$ distances. To simplify matters take $q = 2$. We search for codes in a special form where nonzero codewords are of the form $(1, 1, \dots, 1, 0, \dots, 0)$, that is a run of ones followed by a run of zeros. Thus the distance between two such codewords is equal to the difference of their weights. For $M = 2$, we may take the length 1 code $\{0, 1\}$. Assume C_M is constructed with codewords of successive weights $w_0 = 0 < w_1 < \dots < w_{M-1}$. We construct a code C_{M+1} by adding a tail of zeros to C_M on the right, of length to be specified later, and by adding a new codeword of weight w_M . The new distances are M in number, given by $w_M, w_M - w_1, \dots, w_M - w_{M-1}$. These distances are pairwise distinct because $(w_M - w_i) - (w_M - w_j) = w_j - w_i$. To make sure they are distinct from the distances in C_M , we must check that

$$(w_M - w_i) \neq w_j - w_k,$$

with i, j, k distinct nonnegative integers $\leq M - 1$. This is enforced if we take w_M large enough. This condition on w_M , in turn, will determine how long the tail must be. Since $\binom{M+1}{2} - \binom{M}{2} = M$, we are done. $\square$

The nonlinear analogue of the function $L(n, k, q)$ is the function $N(n, M, q)$ which is the largest number of distances between two codewords of an unrestricted code of size M and length n over some alphabet A_q , of size q .

The analogue of Theorem 4 in this context is as follows. The proof is similar and omitted.

Theorem 7 For all integers $q > 1$, and all nonnegative integers M we have

$$\lim_{n \rightarrow \infty} N(n, M, q) = N(M, q).$$

More precisely, there is an integer $n_0 \geq N(M, q)$, such that for all $n \geq n_0$ we have $N(n, M, q) = N(M, q)$.

Denote by $N_0(M, q)$ the smallest integer n such that $N(n, M, q) = N(M, q)$.

Proposition 7 *If $M - 1$ is a power of a prime, then $N_0(M, q) \leq 2N(M, q) + 1$.*

Proof Assume $M = s + 1$, where s is a power of a prime. We know there is a Singer difference set [9] $S = \{v_0, v_1, \dots, v_{s+1}\}$, with parameters $(s^2 + s + 1, s + 1, 1)$. Consider the $s + 1$ by $s^2 + s + 1$ matrix with rows g_i , when g_i contains v_i consecutive ones to the left and zeros elsewhere. The Hamming distance from g_i to g_j is $|v_i - v_j|$. The code formed by the M rows of this matrix has length $s^2 + s + 1 = M^2 - M + 1 = 2\binom{M}{2} + 1$ and $\binom{M}{2}$ distances, by the design property. Hence, in this case, $n_0 \leq 2\binom{M}{2} + 1$. For instance, if $s = 2$, we have $S = \{1, 2, 4\}$, and the code is $\{1000000, 1100000, 1111000\}$. See [3, p. 264] for details on, and examples of Singer difference sets. $\square$

Denote, for any integer t , by $pp(t)$ the smallest prime power $\geq t$.

Corollary 1 *For all integers $M > 1$, we have*

$$N_0(M, q) \leq 2N(pp(M - 1) + 1, q) + 1 \leq 2N(2M, q) \sim 8\binom{M}{2}.$$

Proof We claim that $N_0(M, q)$ is a nondecreasing function of M . The first inequality will follow by the previous theorem, since $M \leq pp(M - 1) + 1$. To prove the claim note that, if we have a set of $M + 1$ vectors of length $N_0(M + 1, q)$, with $\binom{M+1}{2}$ distances, removing any vector will result into a set of M vectors with $\binom{M+1}{2} - M = \binom{M}{2}$ distances. Hence $N_0(M, q) \leq N_0(M + 1, q)$. The second inequality follows by the crude bound $pp(x) \leq 2x$, valid for any positive integer x . $\square$

Remark It is possible to reduce the upper bound on $pp(x)$ to $pp(x) \leq x + x^a$, with $a = 0.525$, building on recent estimates on the existence of primes in short intervals [2]. This sharpens the upper bound on $N_0(M, q)$ to $2N(M + O(M^a), q) + 1 \sim 2\binom{M}{2}$, for $M \rightarrow \infty$.

7 Conclusion and open problems

In this note, we have studied a problem of extremal combinatorics: maximizing the number of distinct nonzero weights a linear code can have. We conjecture, based on extensive numerical calculations on very long codes, that the bound of Proposition 2 is tight but cannot prove it. A proof was found later in [1]. A recursive approach in the manner of the proof of Theorem 6 would require to produce q^k new weights to go from $L(k, q)$ to $L(k + 1, q)$. But a code achieving $L(k, q)$ has only $\frac{q^k - 1}{q - 1} < q^k$ distinct weights. Thus establishing the tightness of Proposition 1 is the main open problem of this note. Sharpening the upper bound on $N_0(M, q)$ of Corollary 1 is also a challenging question. Determining explicitly the domain $\mathcal{D}$ of Sect. 5 seems to require better lower bounds on $L(n, kq)$ that those at our disposal.

Appendix: numerical examples

We provide lower bounds on $L(k, q)$ by computing the number of weights in long random codes produced by the computer package Magma [8]. We give some numerical examples in Table 1 about the lower bound of Proposition 4.

When n is in the millions, we can find linear $[n, k]_q$ -codes that meet the upper bound in Proposition 2: see Table 2.

Table 1 Proposition 4

k	3	4	4	6	6	10	10	12	12	12
q	3	5	8	9	13	16	25	29	49	121
$L(k, q) \geq$	11	29	41	177	241	4609	6913	31,745	52,225	125,953

Table 2 $n = 6,000,000$

k	3	3	3	3	3	3	3	4	4	4	5	5
q	3	4	5	7	8	9	11	3	4	5	3	4
$L(k, q) =$	13	21	31	57	73	91	133	40	85	156	121	341

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