On the notion of mathematical model in educational research: Insights from a new proposal

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This paper introduces a definition of mathematical model which is intended to be applied in educational research. A synthesis of the background literature leads us to a notion of mathematical modelling that summarises contributions from a wide range of authors. This notion can be made operational in research by using the concept of mathematical model introduced in this study. This resulting proposal for defining model allows the establishing of suitable dimensions of analysis for research. The main features of this new conceptualization and its application to investigate mathematical models are also discussed.

Keywords: Mathematical models, problem solving, educational purposes, research tools.

Introduction

In recent years, the discussion about which capabilities are linked to the mathematical competence points towards modelling skills. These capabilities include not only knowledge related to mathematical contents, but also transversal competences such as problem solving, representation and connections (Montoya Delgadillo, Viola, & Vivier, 2017; Niss & Højgaard, 2011). Therefore, modelling is a focus of interest for researchers in mathematics education (Carreira & Baioa, 2017). A great number of authors have addressed this issue from a wide range of perspectives.

Our synthesis of the different approaches reveals five features that describe how educational researchers understand mathematical modelling: (i) Modelling is a student's activity. Indeed, Chevallard, Bosch, and Gascón (1997) stated that mathematical knowledge is a human activity and, as a part of Mathematics, so is modelling. What is more, they emphasised that a great part of all mathematical activities can be understood as modelling activities; (ii) Modelling is connected to a problem originated from a certain system. In this regard, Castro and Castro (1997) identified "modelling" and "problem solving", whereas other researchers claimed that the purpose of modelling is to produce knowledge about the studied system (Chevallard, 1989). A third group distinguished between modelling "for", if the purpose is to solve a problem, and modelling "of", if the purpose is to generalise to produce knowledge (Streefland, 1985); (iii) Modelling entails a process, i.e., modelling follows a sequence of steps that individuals carry out when tackling a problem. These steps can make up an ideal cycle (Blum & Leiß, 2007; Borromeo Ferri, 2006) or just a set of activities (Ärlebäck, 2009); (iv) Modelling drives students to acquire mathematical skills. On this point, Niss and Højgaard (2011) understand modelling as a competence to be developed in school. Likewise, Blomhøj (2004) interprets it as a teaching practice that links real world and mathematics. Further, García, Gascón, Higueras, and Bosch (2006) pointed to a double didactic power of mathematical modelling, namely that it can be used to teach mathematical contents, or it can constitute the objective of learning; (v) Modelling leads to the construction of a model, which is the product of modelling (Sriraman, 2006). In this sense, modelling means a translation between real world and mathematics (Blum & Borromeo Ferri, 2009) or between different mathematical domains (García et al., 2006). The discussed five features above allow us to understand modelling in mathematics education as *the activity that entails the development of some mathematical skills, originating from a question within a system and is carried out during the process of design, application and evaluation of models. These models allow the use of mathematics to produce knowledge about the system.*

Importance of the concept of model and research question

The synthesised notion of modelling given above relies on several ideas that researchers must specify depending on their goals: What kind of *skills* are entailed by modelling? Which activities are related to *the process of design, application and evaluation*? What is a *model*? As commented on above, there is an extensive literature addressing the process and competencies associated with modelling (see Kaiser, 2014 for a review). Nevertheless, there is a lower number of papers addressing the analysis from the notion of model. The importance of having an operational definition of model is crucial for research purposes. Firstly, it allows focusing the research on the actual model whenever the whole modelling process is inaccessible or too complex to analyse. Secondly, it also provides what elements to focus on when analysing a mathematical model, and which in addition is also useful when discussing other dimensions of modelling.

In this context, the research questions addressed in this paper are: Which elements comprise a mathematical model in the framework of modelling outlined above? Can these elements make up a functional definition of mathematical model? With the aim of answering these questions, we now turn to discuss and elaborate on the main features of the preceding discussion.

Definition of mathematical model for research

Background and relevant items

The importance of lending meaning to the term "mathematical model" has been discussed in the past years. Even though almost all the authors share the intuition of a model as a mapping between a system and a mathematical structure enabling to answer questions or get information about the system, there is no agreement on the nature of mathematical models or what their key components are. An early explicit definition of mathematical model was proposed in the context of engineering (Minsky, 1965). Under this approach, a model is a supporting decision-making tool. This pioneer idea focuses on the usefulness for the system but is far away from being functional for research. However, it contains the first relevant item to define a model up: the analysis of a problem within a system. Connecting back to mathematics education, Lesh and Harel (2003) proposed a notion of model which was based on two components: a mathematical conceptual system together with its accompanying procedures. Both components are expressed through different representations in order to solve a problem. Lesh and Harel complementary perspective is more aligned with the purposes of our paper, which from our point of view stresses two elements that must be considered in educational research: a mathematical description of a problem and the representations that allow working with that description to solve the problem. Niss (2012) defined the term mathematical model without associating it with a problem through the triplet (D, f, M) that connects an extramathematical domain (D) with a mathematical realm (M) via the "mathematisation" mapping (f). Niss' definition included the domain as a part of the model and also objectified the intuition of the mapping that links the system with mathematics. However, f is not always easy to use in analysis. Sometimes it is quite difficult to find the D-M-correspondences because the objects/variables of interest might be implicit (e.g. the best value for money in the second example below). In other cases, the key point of the model does not rely on the connection but in the use of mathematical properties (the extensive nature of area in the first example). Velten (2009) avoided the mathematisation mapping by defining a mathematical model using another triplet, (S, Q, M), but with a different interpretation: S is a system, Q is a question related to S and M is a set of mathematical statements that can be used to answer Q. As with Niss' definition, Velten's approach provides suitable categories of analysis, but it focuses on a problem and does not (explicitly) take representations into account.

The synthesis of the ideas explained above stress the following key features of a mathematical model: (i) that a model is determined by a set of specific components, that lead to define suitable categories of analysis; (ii) that a models is associated with a system (instead of a problem); (iii) that a model includes a mathematical structure that allow working with the model; and (iv) that a model includes representations. Our integration of these features gives rise to the new definition of model we propose below and which we suggest being used for research in mathematics education.

Definition of model

A mathematical model is a triplet (S, M, R), where

- S is the *system* the model is concerned with. It consists of a set of objects, their properties, and the relationships between objects and properties.
- M is the (conceptual) *mathematization* of the system, i.e., the set of mathematical concepts and properties that abstract the relevant information from S, along with the set of relationships applied to produce mathematical knowledge from the system.
- R is the set of mathematical *representations* of the system. It contains the explicit representations that enable to work mathematically with the elements of M and, thus, produce knowledge about the system.

This definition is constructed from three inseparable components that show different levels of abstraction S is real (as well as the context in which the system is framed), but M is purely conceptual, so it is inaccessible and usually not evident. It is necessary, therefore, to use R (observable) to describe and work with the model. Figure 1 represents these levels and two types of connections between the components of a mathematical model, which are discussed below.

Categories of analysis

The main characteristic of this definition is its functionality, since it allows establishing categories for the analysis of mathematical models produced in educational contexts. In order to provide a complete view of what a model creator (the research participant) contributes, the analysis based on the triplet (S, M, R) may be decomposed into two parts. The first part is composed of the elements received by the model creator, that is, the explicit information he/she receives (the enunciation of a task, for example). The second part is constituted by the elements contributed by the creator, which may include identified objects in, or hypothesis about, S, latent concepts or relationships of M, or new representations incorporated in R in order to generate knowledge by working mathematically. It is of interest to specify two subcategories of contributed elements within the mathematization (M) component: (i) premises, which are mathematical statements used without justification (they might be explicit or latent and made visible through R); and (ii) deductions, which are those statements justified by an explicit rationale. These categories are not exhaustive, since one can find contributed elements of mathematization that are neither premises nor deductions (see the examples below).

The argument above give rise to six categories of analysis: system received (S_r), mathematization received (M_r), representation received (R_r), system contributed (S_c), mathematization contributed (with premises $M^p{}_c$ and deductions $M^d{}_c$) and representation contributed (R_c). These are the elements of the mathematical model to be studied. The categories induce a first stage of analysis (the description of what model creator has received) and a second stage of analysis of what he/she has contributed. In addition, within the analysis of a model, an S-R-M natural strategy emerges, so that the researcher can obtain complete information about the model by (i) paying attention to the involved elements of the system S, then (ii) focusing on the representations R and finally (iii) analyzing the conceptual mathematization of the system M. This analysis sequence deviates from the order of the components (S, M, R) in our definition. The sequence S-M-R was chosen for the definition since it evokes the ideal mental process of building a model. Components of the model may be described following this procedure (Figure 1).

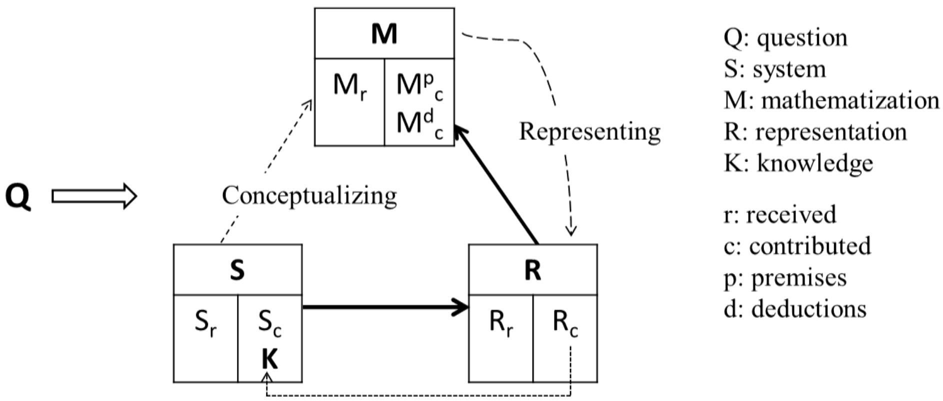

Figure 1: Scheme of the proposed model concept. Note that the analysis strategy (continuous line) and the mental process that generates the model (dashed line) follow different trajectories

Application of the proposed definition: The analysis of school models

Two practical examples

The examples below show the models that two different pre-service teachers at the University of Cordoba (Spain) proposed for solving two released items from the PISA 2003 framework (OECD, 2006). The analyses were carried out following the S-R-M strategy explained above, by first focusing on the elements received by the pre-service teachers and secondly on what the pre-service teachers contributed, with special attention paid to the premises and deductions in the analysis of the used mathematization (M). The results of the analyses are summarized in Figures 2 and 3.

Example 1: Antarctica (Figure 2). The student does not contribute any elements to the system (S). She mostly bases her answer on the picture and provides some verbal reasoning. There are no explicit mathematical concepts, although she implicitly uses the extensive nature of the area. She estimates the area of Antarctica as a portion of an imaginary squared surrounding the island. This approximation is a premise, which shows no clue about how she would proceed for an island with different shape.

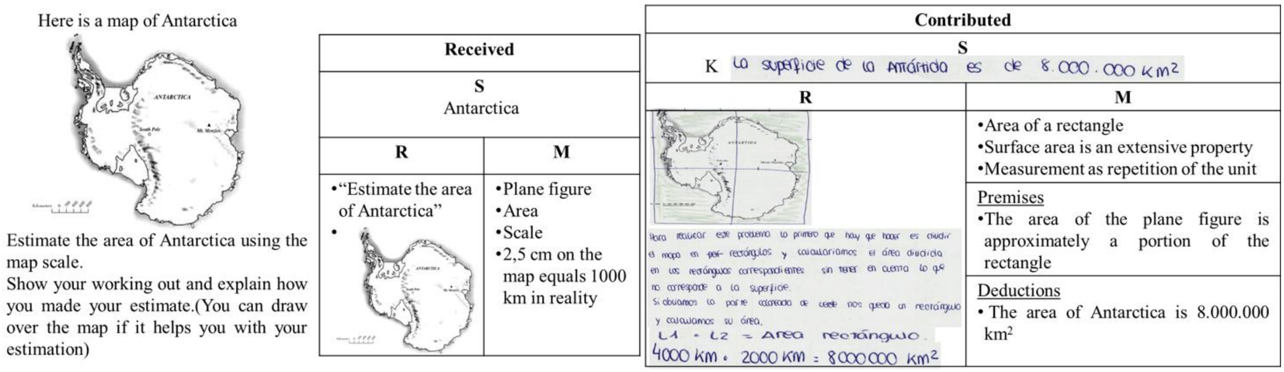

Figure 2: Proposed analysis for a given answer to the task "Antarctica"

Example 2: Pizzas (Figure 3). The student does not contribute any explicit element to the system. However, he tacitly introduces the value for money notion when he states "with one zed, I am buying a portion of 31,416 cm^2". He mostly uses symbolic representation. The premises are relevant because he (i) identifies that the relation diameter-area is not linear and (ii) recognizes the area as the correct variable for estimating the best value for money. This is evidence of a possible general procedure to apply for other kind of sizes of pizzas.

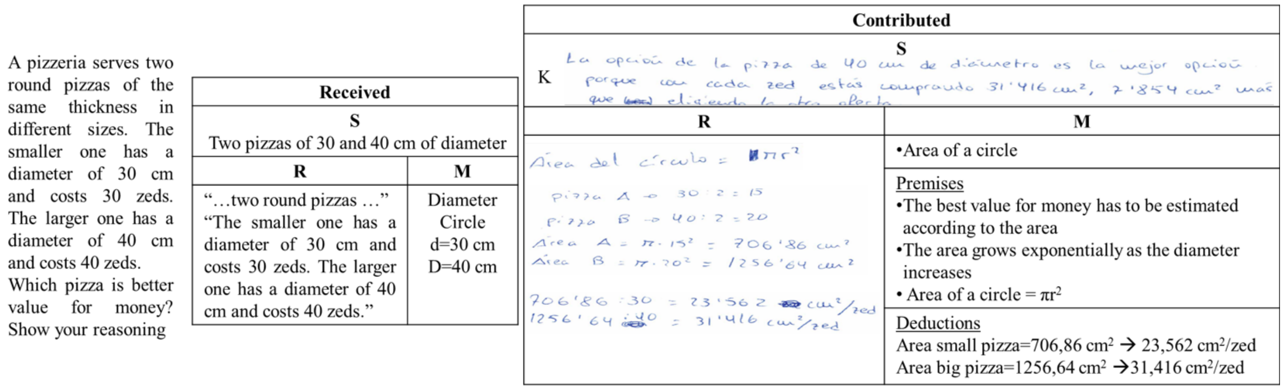

Figure 3: Proposed analysis for a given answer to the task "Pizzas"

Discussion

We now turn to discuss some of the aspects and features of the provided definition of the notion of mathematical model in this paper. Firstly, readers should note that the components of a model are not specified univocally in the formal nature of the definition (Velten, 2009). This is intentionally done in order to let each researcher specify the meanings of each component according to his/her various needs.

Secondly, when dealing with the components S, M, and R of the definition, these should be understood as inseparable elements. In this way, the definition is applicable from different perspectives and at different levels of abstraction. In Mathematics Education, S very often depends on the task context (which may be mathematical or extra-mathematical). Therefore, S is susceptible to empirical or conceptual experimentation. In turn, R contains the evidences of the application of the model in specific cases. Particularly, R captures the representations of the knowledge of the system obtained by the model creator and represents the source of information for the analysis of the models linked to tasks. This is the reason why R should be of special interest for research in mathematics education. Finally, M contains information about the level of abstraction of the model. M is not limited to a conceptual structure or a set of affirmations. As Niss (2012) pointed out, M contains a large number of objects, relationships, properties, results, hypotheses and ways of reasoning that enable understanding of the evidence collected in R. Subcategories within M (premises and deductions) are specified to provide a better understanding of the model. Premises offer information about how the modeller conceives the system and what mathematics can be applied to get knowledge of relevance to S. On the other hand, deductions show the way of reasoning and the conclusions obtained. By combining the analysis of both elements, the researcher is able to identify whether there is an underlying general model. It should be pointed out that premises are sometimes not evident to identify and that they have to be conjectured based on the collected information. For instance, in the example of Antarctica, it is difficult to identify the premises used by the student to create her model. Nevertheless, the student in example 2 provides evidence that he would be able to tackle the task given pizzas of different prices, shapes and sizes.

Thirdly, we provide a comment on the role played by the question or problem posed. Velten (2009), in engineering, emphasized the importance of including the problem to be solved (Q) within the model. From Velten's perspective, without including a question about the system, the model would not emerge. However, we decided not to include this element (the question) in our model definition for simplicity reasons and also because we consider a model as an instrument used to generate knowledge, and hence to go beyond specific problems (Chevallard, 1989). Furthermore, the same model is able to answer different questions in a system (for instance, Cuisenaire's strips are useful to answer different arithmetic questions). On the other hand, the knowledge generated (K), which includes possible answers to Q, can be seen as belonging to the model, namely as part of S since K establishes relationships within the system which were previously unknown (see Figure 1 for different relationships between Q, K, S, M and R within the framework of the proposed definition).

Fourthly, the potential of the proposed definition to be a flexible analysis tool should also be highlighted. As indicated above, the system S and its conceptual mathematization M may contain very diverse elements and relationships. Hence, it is natural to establish subcategories for the exploration of the elements of S, R and M in order to obtain a greater depth in the analysis. However, the generality of the definition prevents establishing univocal criteria *a priori* to discriminate between different types of entities in the analysis. These reasons led to the establishment of broad categories. However, as discussed above, subcategories were only included in the study of M, but in any case, when required, the scheme (S, M, R) does admit that particular subcategories are developed to specify a finer categorization or a hierarchy of properties associated with the system under study.

Finally, it is important to draw attention to the specific character of the proposed definition for research in mathematical education. Especially two properties may be highlighted: (1) The inclusion of R as a component of the mathematical model (Lesh & Harel, 2003). This element, not considered in other areas, is essential in education, since most of modelling performed in school mathematics consists of choosing a suitable representation (a manipulative material, the number line, a tree diagram in probability); (2) The application of the model with the focus on the student's activity. When considering the part received and the part contributed by the student, the definition seeks to make the contribution explicit, differentiating the information of S and representations provided (the statement of a task, for example). This distinction also has didactic applications enabling to control the task variables and generate situations of different difficulty within the same system.

Conclusions

This paper provides a definition that includes all the elements a mathematical model should possess according to our literature review. The novelties of this approach are the focus on the model, instead of on the process or the competencies (like in most of the background literature) and the functional nature of the definition provided, which has been exemplified and discussed throughout the paper.

Regarding the elements that comprise a mathematical model and how they can make up a functional definition, this proposal combines three inseparable elements which, as a whole, account for a mathematical model from different levels of abstraction. A system, S, that includes elements of the context; its mathematical conceptualization, M; and the set of representations, R, that a student uses to describe S, M and to work mathematically. Each of these components provides flexible focus of attention for research and an internal logic that offers an analytical strategy suitable for the analysis of modelling in schools. The main advantage of this definition lies in its applicability for the analysis of scholar models. Its disadvantages are also related to its operational nature: on one hand, at least 6 categories must be distinguished to define a model; on the other hand, the desirable idea of a model as a mapping between reality and mathematics (Niss, 2012) then becomes diluted.

We expect that our proposed definition not only is functional, but also general (applicable to a wide range of modelling scenarios). Future research will need to focus on this and of analyzing the effectiveness and usefulness of the given proposal, as well as its applicability to different contents and educational levels. In addition, also the potential didactic applications need to be explored in future work.

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