PVTOL Exact Linearization Control

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1 Introduction

The term unmanned aerial vehicle (UAV) has a very specific meaning in the context of automatic control research, but it can be substituted by the word "drone" for the general public. These types of autonomous vehicles have seen a rise in popularity in the commercial and industrial fields, and not only as a didactic novelty. There has been a growing interest in many different types of field for their use, like law enforcement or disaster relief to name a few [1]. This in turn makes these kind of vehicles a popular option for the development of innovative control strategies.

The basis of many of these control strategies come from solving a simplified case and extending it to encompass more complex control algorithms. One of the most used models is the planar vertical take-off and landing (PVTOL), which is an underactuated nonlinear vehicle that can be viewed as a simplification of many UAV models, including some types of helicopters, special airplanes and the multirotor platforms.

The main approach used for the stabilization of the PVTOL platform is the separation of the attitude and the translation dynamics. The attitude dynamics are fully actuated and controllable while the translational dynamics depend on the attitude of the vehicle and the thrust of the motors, which is generally considered a control input. Separating the dynamics allows the design of the control for the vehicle a two-part problem. The first part consists of designing a position control, which will be implemented as an outer control loop. The second part consists on designing an attitude controller, where the desired position control is fed and is implemented as an inner control loop. The stability of the whole system depends on the ability of the inner attitude control to successfully track the outer loop output of the position control. The problem with this kind of formulation comes from the fact that the design is separated for both control loops and in practice need constant adjustments to the control parameters in order to make them work.

Some examples of global stabilization of the PVTOL platform using a nonlinear control can be found in [2] and [3]. The problem with these type of controllers is that of parameter adjustment, as it is easier to adjust a linear control than a nonlinear one. Another approach to solve the PVTOL control problem is to linearize the dynamics of the vehicle so that a single control for both attitude and position dynamics may be applied, like the one presented in [4]. This is usually done using approximations with small angles assumptions and/or Taylor series. As these are approximations, the stability can only be ensured locally and the region tends to be small.

The method proposed in this work is called feedback linearization. This is a nonlinear control technique that allows the system to be transformed into a linear system using a feedback function. The advantage of this technique is that it is not based on approximations like the previous case, thus, the possibility exists that global stability can be ensured for this method. Even if the system is locally stable, it usually yields a better attraction zone than the one obtained with linear approximations. However, it is not without any drawbacks, one of them being that the exact knowledge of the state is crucial in order for the feedback function to successfully linearize the system. In the case of PVTOLs, there exists filters that can measure accurately the attitude of the vehicle, which makes it possible to mitigate this drawback. Another drawback that is the main focus of this work is that the transformation of the system often presents singularities in certain vehicle states.

Feedback linearization has been applied to the PVTOL platform before. In the case presented in [5], a feedback linearization control is used to stabilize the inner attitude loop. One of the controllers proposed in [6] goes a step further and shows a feedback linearization controller that uses high-order derivative terms. In the works presented in [7], [8] and [9], it is mentioned that the thrust of the vehicle needs to be different from zero in order for the matrix transformation necessary for feedback linearization to be non-singular. The authors of [10] mention that the thrust of the vehicle is always positive in practice and, thus, do not consider the case when it is zero. The example presented in [11] does limit the range of action of the control inputs to a region.

Some works propose this method for a quadcopter platform instead of a PVTOL. Both [12] and [13] use feedback linearization with Euler angles for the attitude representation and high-order derivative terms. In both cases, the Euler angles present singularities in the attitude dynamics. Another feedback linearization control for a quadcopter can be found in [14] using high-order derivatives. In all of these instances, the authors acknowledge that the feedback linearization transformation employed is defined only if the thrust of the vehicle is different than zero. However, they do not dwell on the implications that this condition has on the system and control design. The main contribution of this work is to determine under what conditions it is possible to determine local asymptotic stability using a control based on feedback linearization of the PVTOL platform that prevents the system from reaching any singularity due to the transformation of the feedback function.

This work is divided in four sections. Section 2 presents the PVTOL model that is going to be used throughout the control design and numeric validations. The control design is presented in section 3, which in turn separates the design in various parts. In section 3.1 the problem to be solved is introduced. Section 3.2 extends the model presented in section 2 in order to allow the use of an exact linearization methodology as presented in [15]. In section 3.3, a state feedback control is proposed to solve the tracking problem for the linearized PVTOL. In section 4 a numeric simulation is presented, along with its results. Finally, section 5 is used to present the conclusions and to outline the possible future work for the proposed design methodology.

2 System Model

The PVTOL is considered to behave like a rigid body. The external forces acting on it are considered to be just the effects of the motors and the gravity. It is considered that the speed is small enough that any aerodynamic effects, such as drag, is negligible in the model. The free body diagram of all the forces considered is shown in figure 1 and explained further in this section.

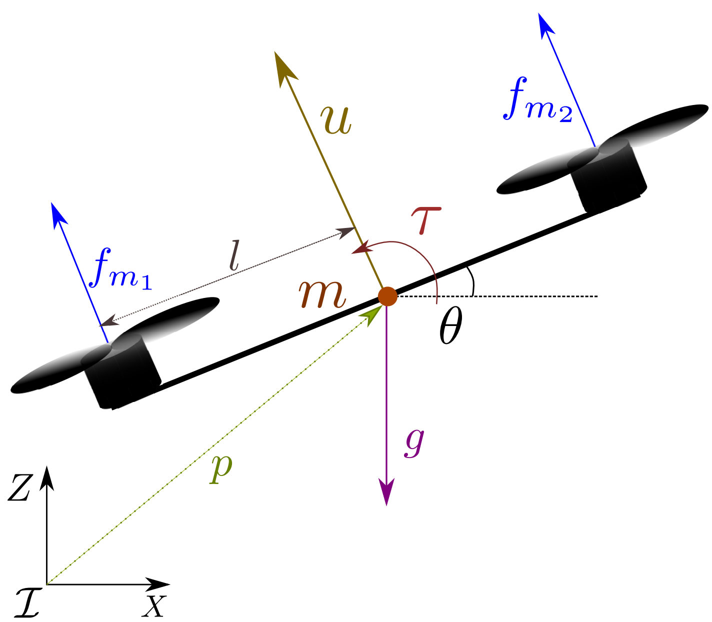

Figure 1: The PVTOL free body diagram.

The position of the vehicle $p \in \mathbb{R}^2$ and its attitude $\theta \in \mathbb{R}$ are referenced w.r.t. the inertial frame $\mathcal{I}$. The position can be further decomposed into

Definition 1.

$$p := \begin{bmatrix} x \\ z \end{bmatrix} \tag{1}$$

The mass of the vehicle m and the effect of the gravity g are assumed to be positive constants.

It is assumed that the velocity of each motor $\omega_i \forall \{1, 2\}$ is real and controllable. The thrust that each motor generates f_{m_i} is considered to be proportional to the square of the propellers' speed, that is

$$f_{m_i} = k_{m_i}\, \omega_i^2 \qquad \forall i \in \{1, 2\} \tag{2}$$

for some real constant $k_{m_i} > 0$.

In order to simplify the design and analysis of the control law, the control inputs that will be considered can obtained as

$$\begin{aligned} u &:= \quad f_{m_1} + f_{m_2} \\ \tau &= \quad l\left(f_{m_2} - f_{m_1}\right) \end{aligned} \tag{3}$$

which are the PVTOL thrust force and torque, respectively.

The PVTOL model can the be described by applying a Newton-Euler method to obtain

$$m\,\ddot{x} = \quad -u\,\sin\theta \tag{4}$$

$$m\,\ddot{z} = \quad u\,\cos\theta - m\,g \tag{5}$$

$$J\,\ddot{\theta} = \quad \tau \tag{6}$$

where $J \in \mathbb{R}$ is a positive constant that represents the second inertia moment.

More information about this model can be obtained from [16].

3 Control Design

In this section, the control for the PVTOL is designed in a four part procedure. The first part explains the objective that the designed control should reach and which will guide the rest of the design process. What follows is the extension of the PVTOL model presented in section 2 in order to transform the higher derivatives of equations (4) and (5) into an affine system. The altitude dynamics are then extracted from this system and will be used in the stability proof. The next part proposes a feedback linearization control for the new affine system and gives the conditions in which the control strategy is valid. The last part presents a state feedback control in order to stabilize the vehicle in a trajectory tracking scheme and it will be shown that the control algorithm locally stabilizes the PVTOL system.

3.1 Problem Statement

The problem addressed in this work is known as the tracking problem. In order to present a better understanding of it from a theoretical point of view the following definition is needed

Definition 2. *The state of the PVTOL vehicle is usually defined as*

$$\hat{\xi} = \left[\begin{array}{cccccc} x & \dot{x} & z & \dot{z} & \theta & \dot{\theta} \end{array}\right]^T \tag{7}$$

with this definition, the control objective is presented as

Control Objective 1. *Design the control inputs u and τ in equations* (4) - (6) *such that*

$$\lim_{t\to\infty} \left|\left|\hat{\xi} - \hat{\xi}_d\left(t\right)\right|\right| = 0 \tag{8}$$

where $\hat{\xi}_d\left(t\right) : \mathbb{R} \to \mathbb{R}^6$ is a function that represents the desired trajectory of the vehicle.

If the objective 1 is accomplished, this means that the vehicle is capable of tracking the trajectory $\hat{\xi}_d\left(t\right)$.

3.2 System Model Extension

The control input u in (4) and (5) can be extended to include higher order derivatives. In order to do this, the following remark is used.

Remark 1. *The control input u is considered to be a C^2 function, that is, its derivatives up to second order $\dot{u}, \ddot{u}$ are considered to exist and be continuous.*

Taking into consideration remark 1, the differentiation of equations (4) and (5) w.r.t. time results in

$$m\,x^{(3)} = m\,x^{(3)}(u,\theta,\dot{u},\dot{\theta}) = \quad -\dot{u}\,\sin\theta - u\,\dot{\theta}\,\cos\theta \tag{9}$$

$$m\,z^{(3)} = m\,x^{(3)}(u,\theta,\dot{u},\dot{\theta}) = \quad \dot{u}\,\cos\theta - u\,\dot{\theta}\,\sin\theta \tag{10}$$

which can be differentiated once again, which results in

$$m\,x^{(4)} = -\ddot{u}\,\sin\theta - 2\,\dot{u}\,\dot{\theta}\,\cos\theta - u\,\ddot{\theta}\,\cos\theta + u\,\dot{\theta}^2\,\sin\theta \tag{11}$$
$$m\,z^{(4)} = \ddot{u}\,\cos\theta - 2\,\dot{u}\,\dot{\theta}\,\sin\theta - u\,\ddot{\theta}\,\sin\theta - u\,\dot{\theta}^2\,\cos\theta \tag{12}$$

Substituting $\ddot{\theta}$ using equation (6) results in

$$m\,x^{(4)} = -\ddot{u}\,\sin\theta - 2\,\dot{u}\,\dot{\theta}\,\cos\theta - \frac{u\,\tau\,\cos\theta}{J} + u\,\dot{\theta}^2\,\sin\theta \tag{13}$$
$$m\,z^{(4)} = \ddot{u}\,\cos\theta - 2\,\dot{u}\,\dot{\theta}\,\sin\theta - \frac{u\,\tau\,\sin\theta}{J} - u\,\dot{\theta}^2\,\cos\theta \tag{14}$$

which can be organized in matrix form as

$$m\begin{bmatrix} x^{(4)} \\ z^{(4)} \end{bmatrix} = \begin{bmatrix} -\sin\theta & -\dfrac{u\,\cos\theta}{J} \\ \cos\theta & -\dfrac{u\,\sin\theta}{J} \end{bmatrix}\begin{bmatrix} \ddot{u} \\ \tau \end{bmatrix} + \begin{bmatrix} -2\,\dot{\theta}\,\cos\theta & u\,\dot{\theta}\,\sin\theta \\ -2\,\dot{\theta}\,\sin\theta & -u\,\dot{\theta}\,\cos\theta \end{bmatrix}\begin{bmatrix} \dot{u} \\ \dot{\theta} \end{bmatrix} \tag{15}$$

Definition 3. *The following definitions can be obtained from the form of equation* (15)

$$E\left(u,\theta\right) := \begin{bmatrix} -\sin\theta & -\dfrac{u\,\cos\theta}{J} \\ \cos\theta & -\dfrac{u\,\sin\theta}{J} \end{bmatrix} \tag{16}$$

$$F\left(u,\dot{u},\theta,\dot{\theta}\right) := \begin{bmatrix} -2\,\dot{\theta}\,\cos\theta & u\,\dot{\theta}\,\sin\theta \\ -2\,\dot{\theta}\,\sin\theta & -u\,\dot{\theta}\,\cos\theta \end{bmatrix}\begin{bmatrix} \dot{u} \\ \dot{\theta} \end{bmatrix} \tag{17}$$

Substituting equation (16) into (15) results in

$$m\begin{bmatrix} x^{(4)} \\ z^{(4)} \end{bmatrix} = E\left(u,\theta\right)\begin{bmatrix} \ddot{u} \\ \tau \end{bmatrix} + F\left(u,\dot{u},\theta,\dot{\theta}\right) \tag{18}$$

which is in an affine control form.

3.3 Feedback Linearization

The control inputs of the system presented in (18) are $\ddot{u}$ and τ. The nonlinearity in equation (18) is due to matrix $E\left(u,\theta\right)$ and the term $F\left(u,\dot{u},\theta,\dot{\theta}\right)$. The control inputs $\ddot{u}$ and τ can compensate for the matrix $E\left(u,\theta\right)$ by using its inverse, but its domain will be restricted by the conditions in which the matrix is not singular. The following corollary is used to obtain this domain.

Corollary 1. *Matrix $E\left(u,\theta\right)$ is not singular $\iff$ $u \neq 0$. This can be deduced from the fact that*

$$det\left[E\left(u,\theta\right)\right] = \frac{u}{J} \tag{19}$$

The nonlinear term $F\left(u,\dot{u},\theta,\dot{\theta}\right)$ can be canceled by subtraction. The mass of the vehicle m is also present in equation (18), but can be canceled by multiplying the inputs by it.

Definition 4. *The control inputs $\ddot{u}$ and τ are defined as*

$$\begin{bmatrix} \ddot{u} \\ \tau \end{bmatrix} := E^{-1}\left(u,\theta\right)\left(m\,u_l - \begin{bmatrix} -2\,\dot{\theta}\,\cos\theta & u\,\dot{\theta}\,\sin\theta \\ -2\,\dot{\theta}\,\sin\theta & -u\,\dot{\theta}\,\cos\theta \end{bmatrix}\begin{bmatrix} \dot{u} \\ \dot{\theta} \end{bmatrix}\right) \tag{20}$$

where $u_l \in \mathbb{R}^2$ is the desired control input for the resulting linear system and $E^{-1}\left(u,\theta\right)$ has the form

$$E^{-1}\left(u,\theta\right) = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\dfrac{J\,\cos\theta}{u} & -\dfrac{J\,\sin\theta}{u} \end{bmatrix} \tag{21}$$

Substituting definition 4 in equation (18) and simplifying it results in

$$\begin{bmatrix} x^{(4)} \\ z^{(4)} \end{bmatrix} = u_l \tag{22}$$

In order to design a state feedback control law, the state of the system must first be defined.

Definition 5. *The state vector of the system is defined as*

$$\xi \coloneqq \begin{bmatrix} x & z & \dot{x} & \dot{z} & \ddot{x} & \ddot{z} & x^{(3)} & z^{(3)} \end{bmatrix}^T \tag{23}$$

Corollary 2. *Note that most control design techniques for the PVTOL define the state as shown in* (7)*, which are the variables that could be deduced by the sensors and most state estimators. However, according to equations* (4)*,* (5)*,* (9) *and* (10)*, the derivatives* $\ddot{x}, \ddot{z}, x^{(3)}$ *and* $z^{(3)}$ *can be obtained from the vehicle's variables* $m, \theta, \dot{\theta}, u$ *and* $\dot{u}$*. This implies that there exists a transformation* $T(\cdot)$ *that changes the state presented in* (7) *into the one in definition 5 like*

$$\xi = T\left(\hat{\xi}, u, \dot{u}\right) = \begin{bmatrix} x \\ z \\ \dot{x} \\ \dot{z} \\ -\dfrac{u \sin\theta}{m} \\ \dfrac{u \cos\theta}{m} - g \\ \dfrac{-\dot{u} \sin\theta - u\,\dot{\theta} \cos\theta}{m} \\ \dfrac{\dot{u} \cos\theta - u\,\dot{\theta} \sin\theta}{m} \end{bmatrix} \tag{24}$$

Using the state vector (23) presented in definition 5 and equation (22), the PVTOL system can be changed into the following linear system

$$\dot{\xi} = A\,\xi + B\,u_l + \gamma \tag{25}$$

with the matrices A, B and the vector γ defined as

Definition 6.

$$A \coloneqq \begin{bmatrix} \bar{0}_{2\times2} & I_2 & \bar{0}_{2\times2} & \bar{0}_{2\times2} \\ \bar{0}_{2\times2} & \bar{0}_{2\times2} & I_2 & \bar{0}_{2\times2} \\ \bar{0}_{2\times2} & \bar{0}_{2\times2} & \bar{0}_{2\times2} & I_2 \\ \bar{0}_{2\times2} & \bar{0}_{2\times2} & \bar{0}_{2\times2} & \bar{0}_{2\times2} \end{bmatrix} \tag{26}$$

$$B \coloneqq \begin{bmatrix} \bar{0}_{2\times2} \\ \bar{0}_{2\times2} \\ \bar{0}_{2\times2} \\ I_2 \end{bmatrix} \tag{27}$$

The constant vector γ *is the effect of that the gravity has on system* (25)*, with the following definition*

$$\gamma \coloneqq \begin{bmatrix} \bar{0}_{3\times1} \\ -g \\ \bar{0}_{4\times1} \end{bmatrix} \tag{28}$$

Equation (25) shows that the PVTOL system can be changed to an almost linear representation. The effect of the gravity on the system was not canceled with the proposed approach. However, the desired control u_l can account for this term and compensate it, which will be shown in the following section.

3.3.1 Lateral Dynamical System

In this section, the translation dynamics are separated from system (25). The system presented here will be used in the following sections to help prove the local stability of the system.

Definition 7. *The state vector of the lateral translation dynamic system is defined as*

$$\xi_{tra} \coloneqq \begin{bmatrix} x & \dot{x} & \ddot{x} & x^{(3)} \end{bmatrix}^T \tag{29}$$

Taking into account system (25), the differentiation of ξ_{alt} w.r.t. time is

$$\dot{\xi}_{tra} = A_{tra}\,\xi_{tra} + B_{tra}\,u_{l,tra} \tag{30}$$

with the matrices A_{alt}, B_{alt} and the vector γ_{alt} defined as

Definition 8.

$$A_{tra} \coloneqq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

$$B_{tra} \coloneqq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{32}$$

3.3.2 Altitude Dynamical System

In this section, the altitude dynamics are separated from system (25) so that it may later be used for the stability analysis and the calculation of an attraction region for it.

Definition 9. *The state vector of the altitude system is defined as*

$$\xi_{alt} \coloneqq \begin{bmatrix} z & \dot{z} & \ddot{z} & z^{(3)} \end{bmatrix}^T \tag{33}$$

Taking into account system (25), the differentiation of ξ_{alt} w.r.t. time is

$$\dot{\xi}_{alt} = A_{alt}\,\xi_{alt} + B_{alt}\,u_{l,alt} + \gamma_{alt} \tag{34}$$

with the matrices A_{alt}, B_{alt} and the vector γ_{alt} defined as

Definition 10.

$$A_{alt} \coloneqq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_{tra} \tag{35}$$

$$B_{alt} \coloneqq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = B_{tra} \tag{36}$$

The constant vector γ_{alt} is the effect of that the gravity has on system (34)*, with the following definition*

$$\gamma_{alt} \coloneqq \begin{bmatrix} 0 \\ -g \\ 0 \\ 0 \end{bmatrix} \tag{37}$$

3.4 State Feedback Control Design

The proposed control in this work for system (25) is a state feedback because of its simplicity. Its description and the stability analysis are shown in the following definition and theorem.

Definition 11. *Let us define the gain matrix $K \in \mathbb{R}^{2\times 8}$ such that*

$$K := \begin{bmatrix} K_{1,1} & 0 & K_{2,1} & 0 & K_{3,1} & 0 & K_{4,1} & 0 \\ 0 & K_{1,2} & 0 & K_{2,2} & 0 & K_{3,2} & 0 & K_{4,2} \end{bmatrix} \tag{38}$$

where $K_{i,j} > 0, \quad K_{i,j} \in \mathbb{R} \quad \forall i \in \{1,2,3,4\}\, \forall j \in \{1,2\}$.

The gain scalars $K_{i,j}$ are chosen in order to comply with the Lyapunov equations

$$P_{alt}\ (A_{alt} - B_{alt}\, K_2) + (A_{alt} - B_{alt}\, K_2)^T\ P_{alt} = -Q_{alt} \tag{39}$$

$$P_{tra}\ (A_{tra} - B_{tra}\, K_1) + (A_{tra} - B_{tra}\, K_1)^T\ P_{tra} = -Q_{tra} \tag{40}$$

where the matrices $Q_., P_. \in \mathbb{R}^{4\times 4}$ are positive definite symmetric and the feedback gain matrices $K_i \in \mathbb{R}^{1\times 4} \quad \forall i \in \{1,2\}$ are defined as

$$K_i := \begin{bmatrix} K_{1,i} & K_{2,i} & K_{3,i} & K_{4,i} \end{bmatrix} \tag{41}$$

Theorem 1. *The proposed control law is defined as*

$$u_l = -K\,[\xi - \gamma_u] \tag{42}$$

where

$$\gamma_u := \begin{bmatrix} \bar{0}_{1\times 3} & g & \bar{0}_{1\times 4} \end{bmatrix}^T \tag{43}$$

The control law (42) *makes system* (25) *be locally, asymptotically stable to the state*

$$\xi^* = \begin{bmatrix} \bar{0}_{5\times 1} \\ g \\ \bar{0}_{2\times 1} \end{bmatrix} \tag{44}$$

Proof. First let us calculate the equilibrium point of the system by substituting u_l defined in (42) into (25) as

$$\begin{aligned} \dot{\xi} =&\ A\,\xi - B\,K\,[\xi - \gamma_u] + \gamma \\ =&\ (A - B\,K)\ \xi + B\,K\,\gamma_u + \gamma \\ =&\ (A - B\,K)\ \xi + \begin{bmatrix} \bar{0}_{3\times 1} \\ -g \\ \bar{0}_{3\times 1} \\ K_{3,2}\,g \end{bmatrix} = \bar{0} \\ \implies \quad \xi =&\ (A - B\,K)^{-1} \begin{bmatrix} \bar{0}_{3\times 1} \\ g \\ \bar{0}_{3\times 1} \\ -K_{3,2}\,g \end{bmatrix} \end{aligned} \tag{45}$$

solving equation (45) for ξ results in the state presented in equation (44)

The control law (42), just like it was done with the system, can be divided into a part that affects the translation system and one that guides the altitude system as

$$u_{l,tra} := -K_1 \xi_{tra} \tag{46}$$

and

$$u_{l,alt} := -K_2 \left(\xi_{alt} - \begin{bmatrix} \bar{0}_{2\times 1} \\ g \\ 0 \end{bmatrix} \right) \tag{47}$$

Now, let us define the Lyapunov candidate function $V(\xi, t)$ as

$$V(\xi,t) \coloneqq \Delta\xi_{alt}^T P_{alt}\, \Delta\xi_{alt} + \xi_{tra}^T P_{tra}\, \xi_{tra} > 0 \tag{48}$$

where the altitude error $\Delta\xi_{alt}$ is defined as

$$\Delta\xi_{alt} \coloneqq \; \xi_{alt} - \begin{bmatrix} \bar{0}_{2\times 1} \\ g \\ 0 \end{bmatrix} \tag{49}$$

and its differentiation w.r.t. time, substituting from equations (30) and (47), is

$$\begin{aligned}
\Delta\dot{\xi}_{alt} =& \quad \dot{\xi}_{alt} \\
=& \quad A_{alt}\,\xi_{alt} + B_{alt}\, u_{l,alt} + \gamma_{alt} \\
=& \quad A_{alt}\,\xi_{alt} - B_{alt}\,K_2 \left(\xi_{alt} - \begin{bmatrix} \bar{0}_{2\times 1} \\ g \\ 0 \end{bmatrix} \right) + \gamma_{alt} \\
=& \quad (A_{alt} - B_{alt}\,K_2)\;\xi_{alt} + \begin{bmatrix} 0 \\ -g \\ 0 \\ k_{3,2}\,g \end{bmatrix} \\
=& \quad (A_{alt} - B_{alt}\,K_2)\;\Delta\xi_{alt}
\end{aligned} \tag{50}$$

It can be seen that substituting equation (44) into (48) results in $V(\xi^*,t) = 0$.

Differentiating (48) w.r.t. time results in

$$\dot{V}(\xi,t) = \quad \Delta\xi_{alt}^T P_{alt}\, \Delta\dot{\xi}_{alt} + \Delta\dot{\xi}_{alt}^T P_{alt}\, \Delta\xi_{alt} + \xi_{tra}^T P_{tra}\, \dot{\xi}_{tra} + \dot{\xi}_{tra}^T P_{tra}\, \xi_{tra} \tag{51}$$

The term $\dot{\xi}_{alt}$ can be substituted from equation (50), and the term $\dot{\xi}_{tra}$ can be substituted from equations (30) and (46). This results in

$$\begin{aligned}
\dot{V}(\xi,t) =& \quad \Delta\xi_{alt}^T P_{alt}\,(A_{alt} - B_{alt}\,K_2)\;\Delta\,\xi_{alt} + \Delta\xi_{alt}^T\,(A_{alt} - B_{alt}\,K_2)^T\; P_{alt}\,\Delta\xi_{alt} + \\
& \quad \xi_{tra}^T P_{tra}\,(A_{tra} - B_{tra}\,K_1)\,\xi_{tra} + \xi_{tra}^T\,(A_{tra} - B_{tra}\,K_1)^T\; P_{tra}\,\xi_{tra} \\
=& \quad \Delta\xi_{alt}^T \left[P_{alt}\,(A_{alt} - B_{alt}\,K_2) + (A_{alt} - B_{alt}\,K_2)^T\; P_{alt} \right] \Delta\xi_{alt} + \\
& \quad \xi_{tra}^T \left[P_{tra}\,(A_{tra} - B_{tra}\,K_1) + (A_{tra} - B_{tra}\,K_1)^T\; P_{tra} \right] \xi_{tra}
\end{aligned} \tag{52}$$

which can be further simplified using equations (39) and (40), that results in

$$\dot{V}(\xi,t) = \quad -\Delta\xi_{alt}^T Q_{alt}\, \Delta\xi_{alt} - \xi_{tra}^T Q_{tra}\, \xi_{tra} < 0 \tag{53}$$

Therefore, system (25) is proven to be asymptotically, exponentially stable to the state (44) using control law (42). □

Theorem 1 provides the basis to achieve the control objective 1. In order to solve it the desired trajectory needs to be transformed into the new state coordinates.

Corollary 3. *The desired trajectory presented in* (8) *can be transformed into a desired trajectory in the new state space as*

$$\xi_d(t) \coloneqq \xi_d \coloneqq T\left(\hat{\xi}_d(t), u, \dot{u}\right) \tag{54}$$

It is assumed that this trajectory is continuous and smooth and that $u \neq 0$*. Also, its differentiation w.r.t. time is also assumed to have the form*

$$\dot{\xi}_d = A\,\xi_d \tag{55}$$

Corollary 4. *It follows from corollary 3 and definitions 7 and 9 that from the desired trajectory presented in* (54) *the trajectories for the translation and altitude systems can be obtained as*

$$\xi_{tra,d} \coloneqq \quad \begin{bmatrix} x_d & \dot{x}_d & \ddot{x}_d & x_d^{(3)} \end{bmatrix}^T \tag{56}$$

$$\xi_{alt,d} \coloneqq \quad \begin{bmatrix} z_d & \dot{z}_d & \ddot{z}_d & z_d^{(3)} \end{bmatrix}^T \tag{57}$$

Also from corollary 3, the trajectories from each subsystem have the form

$$\dot{\xi}_{tra,d} = A_{tra}\,\xi_{tra,d} \tag{58}$$

$$\dot{\xi}_{alt,d} = A_{alt}\,\xi_{alt,d} \tag{59}$$

The following theorem is the result of using corollary 3 and the control law presented in theorem 1.

Theorem 2. *The control law*

$$u_l = -K\left[\xi - \xi_d - \gamma_u\right] \tag{60}$$

makes system (25) *be locally, asymptotically stable to the state*

$$\xi^* = \xi_d + \begin{bmatrix} \bar{0}_{5\times 1} \\ g \\ \bar{0}_{2\times 1} \end{bmatrix} \tag{61}$$

Proof. Let us define the tracking error $e(t) \coloneqq e$ as

$$e(t) \coloneqq e \coloneqq \xi - \xi_d \tag{62}$$

which allows the control law (60) to be rewritten as

$$u_l = -K\left[e - \gamma_u\right] \tag{63}$$

and the translation and altitude tracking errors as

$$e_{tra} \coloneqq \xi_{tra} - \xi_{tra,d} \tag{64}$$

$$e_{alt} \coloneqq \xi_{alt} - \xi_{alt,d} \tag{65}$$

The differentiation of the tracking error w.r.t. time, using the definition of the control law (63) and from systems (25) and (55), is

$$\begin{aligned} \dot{e} &= \dot{\xi} - \dot{\xi}_d \\ &= A\,\xi - B\,K\left[e - \gamma_u\right] + \gamma - A\,\xi_d \\ &= A\,e - B\,K\left[e - \gamma_u\right] + \gamma \\ &= \left(A - B\,K\right)\,e + B\,K\,\gamma_u + \gamma \\ &= \left(A - B\,K\right)\,e + \begin{bmatrix} \bar{0}_{3\times 1} \\ -g \\ \bar{0}_{3\times 1} \\ K_{3,2}\,g \end{bmatrix} \end{aligned} \tag{66}$$

The differentiation of (64) and (65) w.r.t. time can be obtained from (66) as

$$\dot{e}_{tra} = \dot{\xi}_{tra} - \dot{\xi}_{tra,d} = \left(A_{tra} - B_{tra}\,K_1\right)e_{tra} \tag{67}$$

$$\dot{e}_{alt} = \dot{\xi}_{alt} - \dot{\xi}_{alt,d} = \left(A_{alt} - B_{alt}\,K_2\right)\,e_{alt} + \begin{bmatrix} 0 \\ -g \\ 0 \\ K_{3,2}\,g \end{bmatrix} \tag{68}$$

Let us now define an altitude difference as

$$\Delta e_{alt} \coloneqq e_{alt} - \begin{bmatrix} \bar{0}_{2\times 1} \\ g \\ 0 \end{bmatrix} \tag{69}$$

differentiating this altitude difference w.r.t. time using equation (68) results in

$$
\begin{aligned}
\Delta\dot{e}_{alt} = &\ \dot{e}_{alt} \\
= &\ (A_{alt} - B_{alt}\, K_2)\ e_{alt} + \begin{bmatrix} 0 \\ -g \\ 0 \\ K_{3,2}\, g \end{bmatrix} \\
= &\ (A_{alt} - B_{alt}\, K_2)\ \Delta e_{alt}
\end{aligned} \tag{70}
$$

The equilibrium point for the tracking error system can be obtained by solving the equation $\dot{e} = \bar{0}$ for the tracking error e, which was already solved in (45). Using the definition of the tracking error in (62), this results in

$$
\begin{aligned}
e = \begin{bmatrix} \bar{0}_{5\times 1} \\ g \\ \bar{0}_{2\times 1} \end{bmatrix} = \xi - \xi_d \\
\implies \quad \xi = \xi_d + \begin{bmatrix} \bar{0}_{5\times 1} \\ g \\ \bar{0}_{2\times 1} \end{bmatrix}
\end{aligned} \tag{71}
$$

which is the same state as the one defined in (61).

Now, let us define the Lyapunov candidate function $V_e(\xi, \xi_d, t)$ as

$$
V_e(\xi, \xi_d, t) \coloneqq \Delta e_{alt}^T\, P_{alt}\, \Delta e_{alt} + e_{tra}^T\, P_{tra}\, e_{tra} > 0 \tag{72}
$$

Substituting equation (61) into (72) results in $V_e(\xi^*, \xi_d, t) = 0$. Differentiating (72) w.r.t. time along the trajectories of the system results in

$$
\dot{V}_e(\xi, \xi_d, t) = \quad \Delta\dot{e}_{alt}^T\, P_{alt}\, \Delta e_{alt} + \Delta e_{alt}^T\, P_{alt}\, \Delta\dot{e}_{alt} + \dot{e}_{tra}^T\, P_{tra}\, e_{tra} + e_{tra}^T\, P_{tra}\, \dot{e}_{tra} \tag{73}
$$

The term $\Delta\dot{e}_{alt}$ can be substituted from equation (70) and the term $\dot{e}_{tra}$ from equation (67). This results in

$$
\begin{aligned}
\dot{V}_e(\xi, \xi_d, t) = &\ \Delta e_{alt}^T\, P_{alt}\ (A_{alt} - B_{alt}\, K_2)\ \Delta e_{alt} + \Delta e_{alt}^T\ (A_{alt} - B_{alt}\, K_2)^T\ P_{alt}\, \Delta e_{alt} + \\
&\ e_{tra}^T\, P_{tra}\ (A_{tra} - B_{tra}\, K_1)\ e_{tra} + e_{tra}^T\ (A_{tra} - B_{tra}\, K_1)^T\ P_{tra}\, e_{tra} \\
= &\ \Delta e_{alt}^T \left[P_{alt}\ (A_{alt} - B_{alt}\, K_2) + (A_{alt} - B_{alt}\, K_2)^T\ P_{alt} \right] \Delta e_{alt} + \\
&\ e_{tra}^T \left[P_{tra}\ (A_{tra} - B_{tra}\, K_1) + (A_{tra} - B_{tra}\, K_1)^T\ P_{tra} \right] e_{tra}
\end{aligned} \tag{74}
$$

which can be further simplified using equations (39) and (40), that results in

$$
\dot{V}_e(\xi, \xi_d, t) = \quad -\Delta e_{alt}^T\, Q_{alt}\, \Delta e_{alt} - e_{tra}^T\, Q_{tra}\, e_{tra} < 0 \tag{75}
$$

Therefore, system (25) is proven to be asymptotically, exponentially stable to the state (61) using control law (60). □

Even though the linear system (25) has been proven to be asymptotically, exponentially stable to the desired state ξ_d using theorem 2, the fact that the inverse (21) was used means that the stability is local according to corollary 1. The approach taken in this work is to calculate an attraction region for which the stability analysis of the proposed control law is still valid. As the system was separated into an altitude and a translation system, and this singularity only affects the altitude dynamics, the translation dynamics are still globally asymptotically stable.

Theorem 3. *The region of attraction of system* (25) *with control* (42) *can be defined as*

$$
\Omega \coloneqq \left\{ \xi \in \mathbb{R}^8 \middle|\ ||\xi_{alt}|| < g \right\} \tag{76}
$$

Proof. First, the maximum value of the region is acquired by substituting $u = 0$ in equation (5) and solving for $\ddot{z}$

$$
\ddot{z} = \frac{u\, \cos\theta}{m} - g = -g \tag{77}
$$

The stability proof of theorem 1 uses a Lyapunov candidate function made up of two different positive definite functions, one for the altitude dynamical system and another for the traslational dynamical system. A new positive definite function can then be defined exclusively for the altitude system, which is independent from the translation system and even have different control inputs.

$$V(\xi,t) \geq V_{alt}\left(\xi_{alt},t\right) := \xi_{alt}^{T}\, P_{alt}\, \xi_{alt} \tag{78}$$

Its differentiation w.r.t. time results in

$$\dot{V}_{alt}\left(\xi_{alt},t\right) = -\xi_{alt}^{T}\, Q_{alt}\, \xi_{alt} < 0 \tag{79}$$

Which implies that the following inequality holds true

$$V_{alt}\left(\xi_{alt},t\right) \leq \lambda_{max}\left(P_{alt}\right) \left|\left|\xi_{alt,0}\right|\right|^{2} \leq V\left(\xi,t\right) \tag{80}$$

where $\xi_{alt,0} \in \mathbb{R}^4$ are the initial conditions of the altitude system.

In order to ensure that the control input $u > 0$, the initial conditions can be chosen such that

$$\begin{aligned} & \lambda_{max}\left(P_{alt}\right) \left|\left|\xi_{alt,0}\right|\right|^{2} < \quad \lambda_{max}\left(P_{alt}\right)\, g^{2} \\ \Longrightarrow \quad & \left|\left|\xi_{alt,0}\right|\right| < \quad g \end{aligned} \tag{81}$$

which is the region presented in (76) and thus concluding the proof. □

Theorem 4. *The region of attraction of system* (25) *with control* (60) *can be defined as*

$$\Omega := \left\{ \xi \in \mathbb{R}^{8} \middle| \left|\left|\xi_{alt}\right|\right| < \left|\frac{u_d\, \cos\theta_d}{m} + g\right| \right\} \tag{82}$$

Proof. First, the altitude error is defined as

$$\Delta z = z - z_d \tag{83}$$

where the desired altitude is represented by z_d. Differentiating Δz w.r.t. time and using equations (5) and (55) results in

$$\Delta\ddot{z} = \ddot{z} - \ddot{z}_d = \frac{u\, \cos\theta}{m} - g - \frac{u_d\, \cos\theta_d}{m} \tag{84}$$

where $u_d, \theta_d \in \mathbb{R}$ are the desired thrust force and attitude angle, respectively. It is assumed that $u_d > 0$ is positive and constant.

The maximum value of the attraction region can be acquired by obtaining the value of $||\Delta\ddot{z}||$ when $u = 0$. It is calculated from equation (84) as

$$\left|\left|\Delta\ddot{z}\right|\right| = \left|\frac{u_d\, \cos\theta_d}{m} + g\right| \tag{85}$$

Because of the error dynamics presented in (66), the proof of the attraction region is the same as in theorem 3. In order to ensure that the control input $u > 0$, the initial conditions and the desired state can be chosen such that

$$\begin{aligned} & \lambda_{max}\left(P_{alt}\right) \left|\left|\xi_{alt,0}\right|\right|^{2} < \quad \lambda_{max}\left(P_{alt}\right) \left|\frac{u_d\, \cos\theta_d}{m} + g\right|^{2} \\ \Longrightarrow \quad & \left|\left|\xi_{alt,0}\right|\right| < \quad \left|\frac{u_d\, \cos\theta_d}{m} + g\right| \end{aligned} \tag{86}$$

which is the region presented in (82) and thus concluding the proof. □

From equations (60), (28), (20) and (21), the control inputs $\ddot{u}$ and τ are completely defined as

$$\begin{bmatrix} \ddot{u} \\ \tau \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ -\dfrac{J\, \cos\theta}{u} & -\dfrac{J\, \sin\theta}{u} \end{bmatrix} \left[-m\, K \left(\xi - \xi_d - \begin{bmatrix} \bar{0}_{5\times1} \\ g \\ \bar{0}_{2\times1} \end{bmatrix} \right) - \begin{bmatrix} -2\,\dot{\theta}\, \cos\theta & u\,\dot{\theta}\, \sin\theta \\ -2\,\dot{\theta}\, \sin\theta & -u\,\dot{\theta}\, \cos\theta \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{\theta} \end{bmatrix} \right] \tag{87}$$

4 Results

The control law presented in (87) was simulated numerically with the following state feedback gain

$$K = \begin{bmatrix} I_2 & 3.58776721\, I_2 & 3.29779822\, I_2 & 2.75601096\, I_2 \end{bmatrix} \tag{88}$$

The simulation parameters used were

m	$1.8 \quad kg$	J	$0.79 \quad kg \cdot m^2$
g	$9.81 \quad \frac{m}{s^2}$	t_0	$0 \quad s$
Δt	$0.25 \quad s$	t_f	$25 \quad s$
x_d	$-1 \quad m$	y_d	$1.5 \quad m$
$\dot{x}_d$	$0 \quad \frac{m}{s}$	$\dot{y}_d$	$0 \quad \frac{m}{s}$
θ_d	$0 \quad rad$	$\dot{\theta}_d$	$0 \quad \frac{rad}{s}$
u_{des}	$m\,g \quad N$	$\dot{u}_{des}$	$0 \quad \frac{N}{s}$
x_0	$0 \quad m$	y_0	$0 \quad m$
$\dot{x}_0$	$0 \quad \frac{m}{s}$	$\dot{y}_0$	$0 \quad \frac{m}{s}$
θ_0	$0 \quad rad$	$\dot{\theta}_0$	$0 \quad \frac{rad}{s}$
u_0	$m\,g \quad N$	$\dot{u}_0$	$0 \quad \frac{N}{s}$

The results of the simulation can be seen in figures 2 through 9.

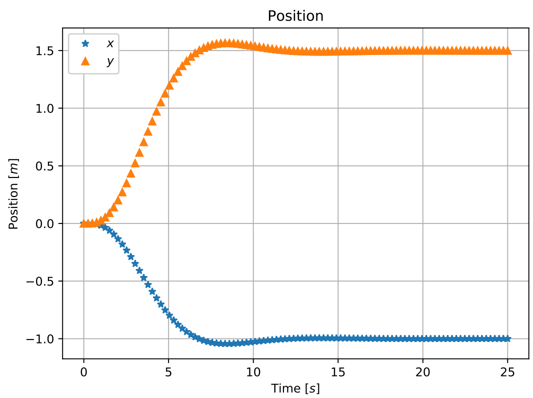

Figure 2: PVTOL Position

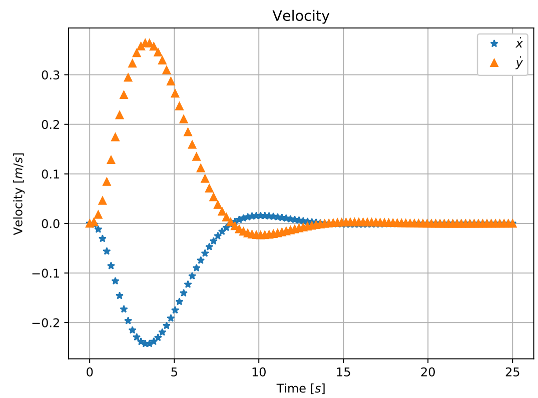

Figure 3: PVTOL Velocity

Figure 2 shows the x and y position of the PVTOL. Because of the initial conditions, the altitude does not drops significantly in the first seconds of the simulation due to the effect of the gravity. Two spikes can be seen in figure 3 that show how the effect of the thrust PVTOLS's thrust force u and attitude θ manage to take the vehicle to a desired state.

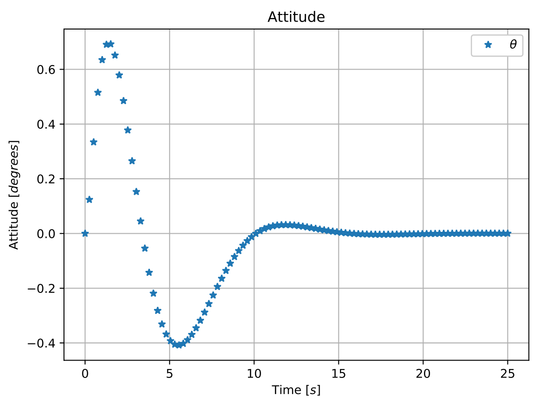

Figure 4: PVTOL Attitude

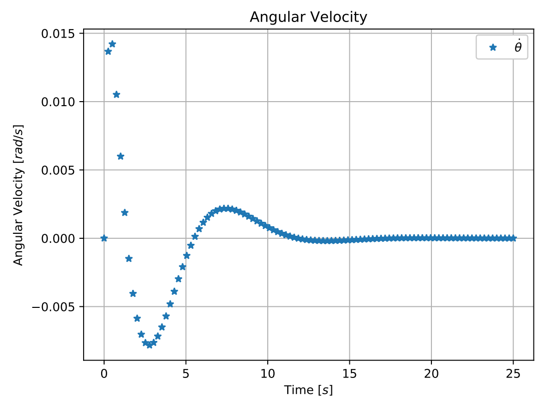

Figure 5: PVTOL Angular Velocity

The change of attitude of the vehicle is reflected in figure 4. It can be seen that it has a jump at the start due to the initial conditions and the desired position. The angular velocity is shown in figure 5. Both of this plots have small angles changes, but is enough to have an effect on the translational dynamics.

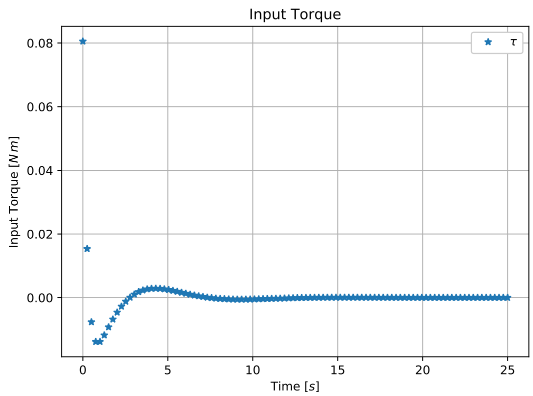

Figure 6: PVTOL Input Torque

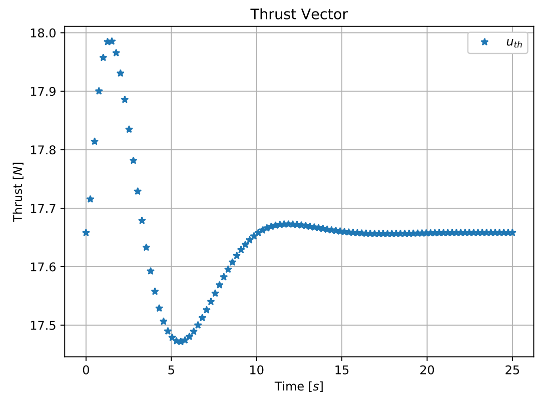

Figure 7: PVTOL Thrust Vector

Figure 6 shows the input torque and figure 7 shows the the thrust vector magnitude. It can be seen in figure 7 that the thrust vector u does not pass through zero, which avoids the singularity in the control algorithm presented.

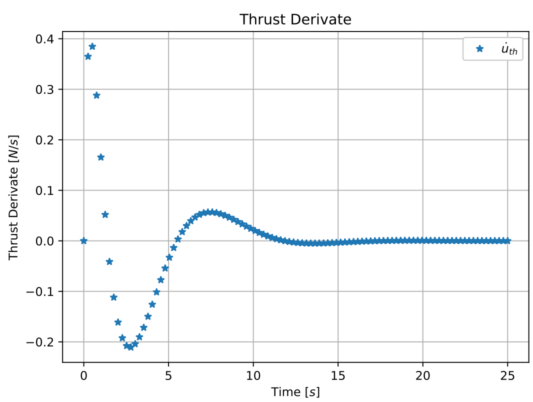

Figure 8: PVTOL Thrust Derivate

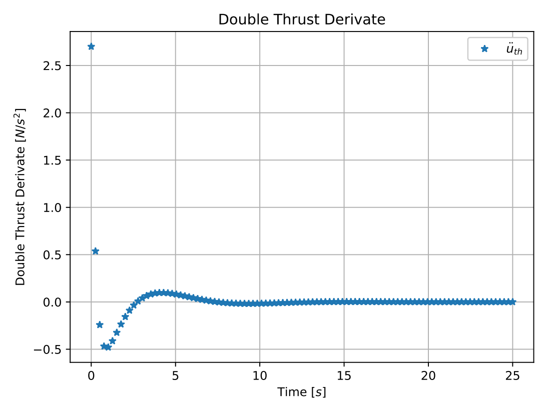

Figure 9: PVTOL Double Thrust Derivate

Figures 7-9 show the dynamics of the thrust vector system. Because the thrust vector u is a control input, the initial conditions of this system can be initialized to any arbitrary value. In this case, the initial conditions where chosen to compensate for the gravity from the start. The thrust does not come close to zero in this case, but this could change depending on the initial conditions chosen.

5 Conclusions

Even though the PVTOL platform model presented in equations (4)-(6) is well known, the control input u can be extended in order to make the platform behave like a linear system. The advantage of this approach is that the control design and stability analysis is easier for the linear system than for the original model. In this work, a state feedback control was used and proven to be able to solve the tracking problem presented. However, this is just one of the possible controls that can be used with this linearized system.

Feedback linearization has been applied to the PVTOL platform before, mostly by extending the state of the system's thrust force u. Some examples include [6], [7], [8], [10] and [9] for the PVTOL, and [12], [13] and [14] use it for the quadcopter platform. However, the feedback linearization in all these controls can only be local, as the feedback function has a singularity when $u = 0$, which is further explained in corollary 1. The singularity for this model is unavoidable, but most of these works only acknowledge it and [10] even mentions that the thrust of the vehicle is always positive in practice. Even if the physical system does have the limitation that the thrust force should always remain positive, that is not a guarantee that the control scheme implemented may not at some point pass the singularity. In [11], the range of action of the control is limited to a region, which could be used to prevent the state f the vehicle from reaching the singularity, but that is not the main objective of these authors.

The main contribution of this work is to determine under what conditions it is possible to determine local asymptotic stability using a control based on feedback linearization for the PVTOL. The problem is approached first by separating and designing a control scheme to both the altitude dynamic system and the translation system. The separation allows the translation system to be globally asymptotically stable while the state of the altitude system is locally asymptotically stable if it is within the region defined in (82). Knowing this region, and because the vehicle's thrust force is a control input, one can modify the vehicle's initial state in order to be located inside it and thus guarantee that the system will not pass through the singularity. It is worth mentioning that the size of the region in (82) also depends on the desired state ξ_d, which may also be chosen as to make the region even bigger.

The gravity compensation term presented throughout this work was assumed to be known. In many cases, this is unknown or can even vary with time. In this cases, it is recommended to add an adaptive control strategy to compensate for it, as this may cause the control to pass through the singularity.

It is also important to remark that the control input u was assumed to be double differentiable in order for the linearization procedure to be as straightforward as possible. However, there may be some other types of functions that can be used that do not necessarily rely on this assumption. As an example, it can even be possible to use a partially differentiable input control instead in order to add robustness or even prevent the system from crossing the singularity from corollary 1.

Finally, one of the main disadvantages that the literature provides for the feedback linearization scheme is that it is highly sensitive to noise. Apart from the fact that the robustness of attitude filters has increased in recent years, another way to avoid these perturbations is to use another control algorithm instead of the state feedback proposed here. This control should also need to be able to avoid the singularity and the region of attraction here is one way t prove it does.

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