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A SIMPLE PROOF OF A MULTIDIMENSIONAL BORG-LEVINSON TYPE THEOREM

MOURAD CHOULLI

ABSTRACT. We provide a simple and short proof of a multidimensional Borg-Levinson type theorem. Precisely, we prove that the spectral boundary data determine uniquely the corresponding potential appearing in the Schrödinger operator on an admissible Riemannian manifold. We also sketch the proof of the case of incomplete spectral boundary data.

1. INTRODUCTION

Let $\mathcal{M} = (\mathcal{M}, \mathbf{g})$ be a smooth compact Riemannian manifold of dimension $n \geq 3$ and with boundary $\partial\mathcal{M}$. We recall that, in local coordinates, the usual Laplace-Beltrami operator is given by

$$\Delta_{\mathbf{g}} = \frac{1}{\sqrt{|\mathbf{g}|}} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(\sqrt{|\mathbf{g}|} \mathbf{g}^{jk} \frac{\partial}{\partial x_k} \cdot \right).$$

We define, for $q \in L^\infty(\mathcal{M}, \mathbb{R})$, the unbounded operator $L = L(q)$ acting on $L^2(\mathcal{M})$ as follows

$$L = -\Delta_{\mathbf{g}} + q \quad \text{with} \quad D(L) = H_0^1(\mathcal{M}) \cap H^2(\mathcal{M}).$$

As L is self-adjoint operator with compact resolvent, its spectrum is reduced to a sequence of eigenvalues:

$$-\infty < \lambda_1 \leq \dots \leq \lambda_k \leq \dots \quad \text{and} \quad \lambda_k \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Of course $\lambda_k = \lambda_k(q)$, $k \geq 1$.

In addition, there exists (ϕ_k) , $\phi_k = \phi_k(q)$, $k \geq 1$, an orthonormal basis of $L^2(\mathcal{M})$ consisting of eigenfunctions. Each ϕ_k is associated to λ_k , $k \geq 1$.

For simplicity convenience, we use the notation

$$\psi_k = \partial_\nu \phi_k|_{\partial\mathcal{M}}, \quad k \geq 1,$$

where ν is the unit normal vector field on $\partial\mathcal{M}$ with respect to $\mathbf{g}$.

When $q \in L_+^\infty(\mathcal{M}) = \{q \in L^\infty(\mathcal{M}, \mathbb{R}); q \geq 0\}$, 0 is not in the spectrum of L and all the eigenvalues of L are positive. Then, for $f \in H^{1/2}(\partial\mathcal{M})$, the BVP

$$(1.1) \quad (-\Delta_{\mathbf{g}} + q)u = 0 \text{ in } \mathcal{M} \quad \text{and} \quad u = f \text{ on } \partial\mathcal{M}$$

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admits a unique solution $u = u(q, f) \in H^1(\mathcal{M})$ with $\partial_\nu u \in H^{-1/2}(\partial\mathcal{M})$. Moreover

$$\|\partial_\nu u\|_{H^{-1/2}(\partial\mathcal{M})} \leq C\|f\|_{H^{1/2}(\partial\mathcal{M})},$$

where the constant C is independent of f . In other words, the mapping

$$\Lambda(q) : f \in H^{1/2}(\partial\mathcal{M}) \rightarrow \partial_\nu u \in H^{-1/2}(\partial\mathcal{M})$$

defines a bounded operator. The operator $\Lambda(q)$ is usually called the Dirichlet-to-Neumann map associated to q .

For $\tilde{q} \in L_+^\infty(\mathcal{M})$, we set $\tilde{\lambda}_k = \lambda_k(\tilde{q})$, $\tilde{\phi}_k = \phi_k(\tilde{q})$ and $\tilde{\psi}_k = \psi_k(\tilde{q})$. We introduce then the notation

$$\mathfrak{D}(q, \tilde{q}) = \sum_{k \geq 1} |\lambda_k - \tilde{\lambda}_k| + \sum_{k \geq 1} \|\tilde{\psi}_k - \psi_k\|_{L^2(\partial\mathcal{M})}.$$

We aim to prove the following result:

Theorem 1.1. *Let $\aleph > 0$. If $0 \leq q \leq \aleph$, $0 \leq \tilde{q} \leq \aleph$ and $\mathfrak{D}(q, \tilde{q}) < \infty$, then $\Lambda(q) - \Lambda(\tilde{q})$ extends to a bounded operator on $L^2(\partial\mathcal{M})$ and*

$$\|\Lambda(q) - \Lambda(\tilde{q})\|_{\mathcal{B}(L^2(\partial\mathcal{M}))} \leq C\mathfrak{D}(q, \tilde{q}),$$

where the constant C only depends on n , $\mathcal{M}$ and $\aleph$.

Let $(\mathcal{M}, g)$ a compact Riemannian manifold with boundary $\partial\mathcal{M}$. We say that $\mathcal{M}$ is admissible if $\mathcal{M} \Subset \mathbb{R} \times \mathcal{M}_0$, for some $(n-1)$ -dimensional simple manifold $(\mathcal{M}_0, g_0)$, and if $g = c(\mathfrak{e} \oplus g_0)$, where $\mathfrak{e}$ is the Euclidean metric on $\mathbb{R}$ and c is a smooth positive function on $\mathcal{M}$. We recall that $(\mathcal{M}_0, g_0)$ is simple if it is a compact Riemannian manifold with boundary and, for any $x \in \mathcal{M}_0$, the exponential map $\exp_x$ with its maximal domain of definition is a diffeomorphism onto $\mathcal{M}_0$, and $\partial\mathcal{M}_0$ is strictly convex (that is, the second fundamental form of $\partial\mathcal{M}_0 \hookrightarrow \mathcal{M}_0$ is positive definite).

Observing that

$$\|\Lambda(q) - \Lambda(\tilde{q})\|_{\mathcal{B}(H^{1/2}(\partial\mathcal{M}), H^{-1/2}(\partial\mathcal{M}))} \leq \|\Lambda(q) - \Lambda(\tilde{q})\|_{\mathcal{B}(L^2(\partial\mathcal{M}))}$$

and $\mathfrak{D}(q, \tilde{q}) = \mathfrak{D}(q - \lambda, \tilde{q} - \lambda)$, for any λ not belonging to the spectrum of $L(q)$ and $L(\tilde{q})$, we deduce as a consequence of Theorem 1.1 and [8, Theorem 1] the following stability inequality:

Corollary 1.1. *Let $\aleph > 0$, $t \in (0, 1/2)$ and assume that $\mathcal{M}$ is admissible with $\mathcal{M} \Subset \mathbb{R} \times \mathcal{M}_0$. Then there exist two constants $C > 0$ and $0 < c < 1$, only depending on n , $\mathcal{M}$ and $\aleph$, so that, for any $q, \tilde{q} \in L^\infty(\mathcal{M}, \mathbb{R}) \cap H^t(\mathcal{M})$ satisfying*

$$\|q\|_{L^\infty(\mathcal{M}) \cap H^t(\mathcal{M})} \leq \aleph, \quad \|\tilde{q}\|_{L^\infty(\mathcal{M}) \cap H^t(\mathcal{M})} \leq \aleph \quad \text{and} \quad \mathfrak{D}(q, \tilde{q}) \leq c,$$

we have

$$\|q - \tilde{q}\|_{L^2(\mathbb{R}, H^{-3}(\mathcal{M}_0))} \leq C \left| \ln \left[\mathfrak{D}(q, \tilde{q}) + |\ln(\mathfrak{D}(q, \tilde{q}))|^{-1} \right] \right|^{-t/4}.$$

In particular, this corollary says that in an admissible manifold the spectral boundary data $(\lambda_k(q), \psi_k(q))_{k \geq 1}$ determine uniquely $q \in L^\infty(\mathcal{M}, \mathbb{R}) \cap H^t(\mathcal{M})$:

Theorem 1.2. *Let $t \in (0, 1/2)$ and assume that $\mathcal{M}$ is admissible. If $q, \tilde{q} \in L^\infty(\mathcal{M}, \mathbb{R}) \cap H^t(\mathcal{M})$ satisfy*

$$\lambda_k(q) = \lambda_k(\tilde{q}) \quad \text{and} \quad \psi_k(q) = \psi_k(\tilde{q}), \quad k \geq 1,$$

then $q = \tilde{q}$.

Results of the same kind as Theorem 1.2 are known in the literature as multidimensional Borg-Levinson type theorems.

The paper by Nachman, Sylvester and Uhlmann [18] was the starting point of new developments and results on multidimensional Borg-Levinson type theorems. We quote here the following short list of references: [1, 2, 3, 4, 6, 7, 9, 10, 12, 13, 14, 15, 16, 17]. Of course many other works on spectral inverse problems exist in the literature. A short survey on multidimensional Borg-Levinson type theorems can be found in [4]. The case of unbounded potentials was considered by Pohjola in [19]. We also mention that a new stability inequality for the two dimensional case was recently established by Imanuvilov and Yamamoto in [11].

2. PROOF OF THE MAIN RESULT

Henceforward, the usual scalar products on $L^2(\mathcal{M})$ and $L^2(\partial\mathcal{M})$ are denoted respectively by $(\cdot|\cdot)$ and $\langle\cdot|\cdot\rangle$.

From [4, Lemma 3.1], the solution of the BVP 1.1 is given by

$$u = - \sum_{k \geq 1} \frac{\langle \psi_k | f \rangle}{\lambda_k} \phi_k.$$

This identity yields

$$\sum_{k \geq 1} \frac{|\langle \psi_k | f \rangle|^2}{\lambda_k^2} = \|u\|_{L^2(\mathcal{M})}^2.$$

Pick (ρ_k) an orthonormal basis of $L^2(\mathcal{M})$, $s \geq 0$ and define

$$\mathcal{H}^s = \left\{ w \in H^2(\mathcal{M}), \sum_{k \geq 1} k^{4s/n} \left(k^{4/n} |(w|\rho_k)|^2 + |(\Delta_{\mathfrak{g}} w |\rho_k)|^2 \right) < \infty \right\}.$$

We have by Parseval's identity

$$\begin{aligned} & \sum_{k \geq 1} k^{4s/n} \left(k^{4/n} |(w|\rho_k)|^2 + |(\Delta_{\mathfrak{g}} w |\rho_k)|^2 \right) \\ &= \left\| \sum_{k \geq 1} k^{2(s+1)/n} (w|\rho_k) \rho_k \right\|_{L^2(\mathcal{M})}^2 + \left\| \sum_{k \geq 1} k^{2s/n} (\Delta_{\mathfrak{g}} w |\rho_k) \rho_k \right\|_{L^2(\mathcal{M})}^2. \end{aligned}$$

In consequence, the left hand side of this identity does not depend on the choice of the orthonormal basis (ρ_k) . We endow $\mathcal{H}^s$ with the norm

$$\|w\|_{\mathcal{H}^s} = \|w\|_{H^2(\mathcal{M})} + \left(\sum_{k \geq 1} k^{4s/n} \left(k^{4/n} |(w|\rho_k)|^2 + |(\Delta_{\mathfrak{g}} w |\rho_k)|^2 \right) \right)^{1/2}.$$

It is not hard check that $\mathcal{H}^s$ contains all the eigenfunctions (φ_k) of the operator $-\Delta_{\mathfrak{g}}$, under Neumann boundary condition, and therefore it contains also the closure, with respect to $\|\cdot\|_{\mathcal{H}^s}$, of the vector space spanned by these eigenfunctions. This can be easily seen just by taking $\rho_k = \varphi_k$, $k \geq 1$, in the definition of $\mathcal{H}^s$.

We associate to $\mathcal{H}^s$ the trace space

$$\mathfrak{H}^s = \{g \in H^{3/2}(\partial\mathcal{M}); g = w|_{\partial\mathcal{M}} \text{ for some } w \in \mathcal{H}^s\}$$

that we equip with its natural quotient norm

$$\|g\|_{\mathfrak{H}^s} = \inf \{ \|w\|_{\mathcal{H}^s}; w|_{\partial\mathcal{M}} = g \}.$$

Lemma 2.1. *We fix $s > n/4$. Then, for any $g \in \mathfrak{H}^s$, the series $\sum_{k \geq 1} \frac{\langle \psi_k | g \rangle}{\lambda_k} \phi_k$ converges in $H^2(\mathcal{M})$ with*

$$(2.1) \quad \sum_{k \geq 1} \frac{|\langle \psi_k | g \rangle|}{\lambda_k} \|\phi_k\|_{H^2(\mathcal{M})} \leq C \|g\|_{\mathfrak{H}^s},$$

where the constant C is independent of g .

Proof. Let $g \in \mathfrak{H}^s$ and $w \in \mathcal{H}^s$ arbitrary so that $w|_{\partial\mathcal{M}} = g$. From Green's Formula, we have

$$\begin{aligned} \langle \psi_k | g \rangle &= \int_{\partial\mathcal{M}} \partial_\nu \phi_k \bar{g} d\sigma = \int_{\mathcal{M}} \Delta_{\mathfrak{g}} \phi_k \bar{w} dv - \int_{\mathcal{M}} \phi_k \overline{\Delta_{\mathfrak{g}} w} dv. \\ &= \int_{\mathcal{M}} (-\lambda_k + q) \phi_k \bar{w} dv - \int_{\mathcal{M}} \phi_k \overline{\Delta_{\mathfrak{g}} w} dv. \end{aligned}$$

Therefore, taking into account that $\lambda_k \sim k^{2/n}$ as $k \rightarrow +\infty$ (by Weyl's asymptotic formula), we get

$$|\langle \psi_k | g \rangle| \leq C \left(k^{2/n} |(\phi_k | w)| + |(\phi_k | \Delta_{\mathfrak{g}} w)| \right), \quad k \geq 1,$$

where the constant C is independent of w .

But $\|\phi_k\|_{H^2(\mathcal{M})} \leq C \lambda_k$ for any $k \geq 1$. Hence

$$\frac{|\langle \psi_k | g \rangle|}{\lambda_k} \|\phi_k\|_{H^2(\mathcal{M})} \leq C k^{-2s/n} \left[k^{2s/n} \left(k^{2/n} |(\phi_k | w)| + |(\phi_k | \Delta_{\mathfrak{g}} w)| \right) \right], \quad k \geq 1,$$

Whence, the series $\sum_{k \geq 1} \frac{\langle \psi_k | g \rangle}{\lambda_k} \phi_k$ converges in $H^2(\mathcal{M})$ and using Cauchy-Schwarz's inequality we find

$$\sum_{k \geq 1} \frac{|\langle \psi_k | g \rangle|}{\lambda_k} \|\phi_k\|_{H^2(\mathcal{M})} \leq C \|w\|_{\mathcal{H}^s}.$$

As $w \in \mathcal{H}^s$ is chosen arbitrary so that $w|_{\partial\mathcal{M}} = g$, inequality (2.1) follows. $\square$

Proof of Theorem 1.1. In this proof C is a generic constant depending only on n , $\mathcal{M}$ and $\mathfrak{N}$.

In light of Lemma 2.1, we have

$$\Lambda(q)(f) = - \sum_{k \geq 1} \frac{\langle \psi_k | f \rangle}{\lambda_k} \psi_k, \quad f \in \mathfrak{H}^s$$

and

$$\Lambda(\tilde{q})(f) = - \sum_{k \geq 1} \frac{\langle \tilde{\psi}_k | f \rangle}{\tilde{\lambda}_k} \tilde{\psi}_k, \quad f \in \mathfrak{H}^s.$$

We split $\Lambda_q(f) - \Lambda_{\tilde{q}}(f)$, $f \in \mathfrak{H}^s$, into three terms:

$$\Lambda_q(f) - \Lambda_{\tilde{q}}(f) = A_1 + A_2 + A_3,$$

with

$$\begin{aligned} A_1 &= \sum_{k \geq 1} \left(\frac{1}{\tilde{\lambda}_k} - \frac{1}{\lambda_k} \right) \langle \tilde{\psi}_k | f \rangle \tilde{\psi}_k, \\ A_2 &= \sum_{k \geq 1} \frac{\langle \tilde{\psi}_k | f \rangle - \langle \psi_k | f \rangle}{\lambda_k} \tilde{\psi}_k, \\ A_3 &= \sum_{k \geq 1} \frac{\langle \psi_k | f \rangle}{\lambda_k} (\tilde{\psi}_k - \psi_k). \end{aligned}$$

Then it is straightforward to check that

$$\begin{aligned} \|A_1\|_{L^2(\partial\mathcal{M})} &\leq C \sum_{k \geq 1} |\lambda_k - \tilde{\lambda}_k| \|f\|_{L^2(\partial\mathcal{M})}, \\ \|A_2\|_{L^2(\partial\mathcal{M})} + \|A_3\|_{L^2(\partial\mathcal{M})} &\leq C \sum_{k \geq 1} \|\tilde{\psi}_k - \psi_k\|_{L^2(\partial\mathcal{M})} \|f\|_{L^2(\partial\mathcal{M})}. \end{aligned}$$

Therefore, under the assumption $\mathfrak{D}(q, \tilde{q}) < \infty$, the operator $\Lambda_q - \Lambda_{\tilde{q}}$ extends to a bounded operator on $L^2(\partial\mathcal{M})$ with

$$\|\Lambda_q - \Lambda_{\tilde{q}}\|_{\mathcal{B}(L^2(\partial\mathcal{M}))} \leq C \mathfrak{D}(q, \tilde{q}).$$

The proof is then complete. $\square$

3. THE CASE OF INCOMPLETE SPECTRAL BOUNDARY DATA

We explain briefly in this short section how we can get a Borg-Levinson type theorem for incomplete spectral boundary data. For this purpose, we assume that $\mathcal{M}$ is simple. In that case it is known that there exists another simple manifold $\mathcal{M}_1$ so that $\mathcal{M}_1 \ni \mathcal{M}$.

Let $q, \tilde{q} \in L^\infty(\mathcal{M}, \mathbb{R})$ satisfying $p = (q - \tilde{q})\chi_{\mathcal{M}} \in H^1(\mathcal{M}_1)$,

$$\|q\|_{L^\infty(\mathcal{M})} \leq \aleph, \quad \|\tilde{q}\|_{L^\infty(\mathcal{M})} \leq \aleph \quad \text{and} \quad \|p\|_{H^1(\mathcal{M}_1)} \leq \aleph.$$

From [5], we have

$$(3.1) \quad \|p\|_{L^2(\mathcal{M})}^4 \leq C \left(\frac{1}{|\Im \lambda|} + |\Im \lambda| \|\Lambda_{q-\lambda} - \Lambda_{\tilde{q}-\lambda}\|_{\mathcal{B}(L^2(\partial\mathcal{M}))} \right), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where the constant C only depends on n , $\mathcal{M}$ and $\aleph$.

Now if, for some fixed integer $\ell \geq 1$, $(\lambda_k, \psi_k) = (\tilde{\lambda}_k, \tilde{\psi}_k)$, $k \geq \ell$, then the calculations in the preceding section yield

$$(3.2) \quad \|\Lambda_{q-\lambda} - \Lambda_{\tilde{q}-\lambda}\|_{\mathcal{B}(L^2(\partial\mathcal{M}))} \leq C \left(\sum_{k=1}^{\ell} \frac{\lambda_k^2}{|\lambda_k - \lambda|} + \sum_{k=1}^{\ell} \frac{\tilde{\lambda}_k^2}{|\tilde{\lambda}_k - \lambda|} \right).$$

Again, the constant C only depends on n , $\mathcal{M}$ and $\aleph$.

We deduce by combining together (3.1) and (3.2) that there exist two constants $C > 0$ and $\kappa > 0$, only depending on n , $\mathcal{M}$, $\aleph$ and ℓ , so that

$$(3.3) \quad \|p\|_{L^2(\mathcal{M})}^4 \leq C \left(\frac{1}{|\Im \lambda|} + \frac{|\Im \lambda|}{|\lambda|} \right), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad |\lambda| \geq \kappa.$$

We find by taking in this inequality $\lambda = (\tau + i)^2 = (\tau^2 - 1) + 2i\tau$, with $\tau \geq 1$ sufficiently large,

$$\|p\|_{L^2(\mathcal{M})}^4 \leq C \left(\frac{1}{\tau} + \frac{\tau}{\tau^2 - 1} \right).$$

Taking the limit, as $\tau \rightarrow \infty$, in this inequality, we obtain $p = 0$. In particular, we proved that the spectral boundary data $(\lambda_k, \psi_k)_{k \geq \ell}$ determine uniquely $q \in L^\infty(\mathcal{M}, \mathbb{R}) \cap H_0^1(\mathcal{M})$, for any arbitrary fixed integer $\ell \geq 1$:

Theorem 3.1. *Let $\ell \geq 1$ be an integer and assume that $\mathcal{M}$ is simple. If $q, \tilde{q} \in L^\infty(\mathcal{M}, \mathbb{R}) \cap H_0^1(\mathcal{M})$ satisfy*

$$\lambda_k(q) = \lambda_k(\tilde{q}) \quad \text{and} \quad \psi_k(q) = \psi_k(\tilde{q}), \quad k \geq \ell,$$

then $q = \tilde{q}$.

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