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ENVELOPING SKEWFIELDS OF THE NILPOTENT POSITIVE PART AND THE BOREL SUBSUPERALGEBRA OF $\mathfrak{osp}(1, 2n)$

JACQUES ALEV AND FRANÇOIS DUMAS

ABSTRACT. We study an analogue of the Gelfand-Kirillov property for some Lie superalgebras. More precisely, we consider in the classical simple orthosymplectic Lie superalgebra $\mathfrak{osp}(1, 2n)$ of dimension $n^2 + 3n$ the positive nilpotent subsuperalgebra $\mathfrak{n}^+$ of dimension $n^2 + n$ and the solvable Borel subsuperalgebra $\mathfrak{b}^+$ of dimension $n^2 + 2n$. We prove that the enveloping algebras $\mathcal{U}(\mathfrak{n}^+)$ and $\mathcal{U}(\mathfrak{b}^+)$ are rationally equivalent to some super-analogues of Weyl algebras over polynomial superalgebras.

INTRODUCTION

In the classical context of the seminal paper [7], an algebraic finite-dimensional complex Lie algebra $\mathfrak{g}$ is said to satisfy the Gelfand-Kirillov property when its enveloping algebra $\mathcal{U}(\mathfrak{g})$ is rationally equivalent to a Weyl algebra $A_n(K)$ over a commutative purely transcendental field extension K of $\mathbb{C}$. It was proved that this property is satisfied when $\mathfrak{g}$ is nilpotent ([7]), or more generally solvable ([5], [10], [12]). This profound problem gave rise to an abundant literature with many developments within Lie theory itself (see references in [3] or [16]) or for variants concerning quantum groups, invariant algebras or other classes of interesting noncommutative algebras.

The study of an analogue of the Gelfand-Kirillov property for Lie superalgebras was explored in [13] and [2]. In this context, it is relevant to introduce for any integer $n \geq 1$ the noncommutative polynomial superalgebra $\tilde{\mathcal{O}}(\mathbb{C}^n)$ and an analogue $\tilde{A}_n(\mathbb{C})$ of the Weyl algebra. By definition $\tilde{\mathcal{O}}(\mathbb{C}^n)$ is the algebra generated over $\mathbb{C}$ by n indeterminates $y_1, \dots, y_n$ with relations $y_i y_j + y_j y_i = 0$ for all $1 \leq i \neq j \leq n$. Then $\tilde{A}_n(\mathbb{C})$ is the algebra generated over $\tilde{\mathcal{O}}(\mathbb{C}^n)$ by n other indeterminates $x_1, \dots, x_n$ with relations $x_i y_j + y_j x_i = x_i x_j + x_j x_i = 0$ for all $1 \leq i \neq j \leq n$ and $x_i y_i - y_i x_i = 1$ for any $1 \leq i \leq n$. These algebras are noetherian domains and then they admit a skewfield of fractions. Otherwise the only case in the classification of classical simple finite dimensional complex Lie superalgebras $\mathfrak{g}$ where the

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enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a domain is when $\mathfrak{g}$ is an orthosymplectic Lie superalgebra $\mathfrak{osp}(1, 2n)$ (see [4], [9]). This is the case we consider here.

For any integer $n \geq 1$, the superalgebra $\mathfrak{g} = \mathfrak{osp}(1, 2n)$ is of dimension $2n^2 + 3n$. In its $\mathbb{Z}_2$ -graded decomposition $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, the even part $\mathfrak{g}_{\bar{0}}$ is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(2n)$ of dimension $2n^2 + n$, and the odd part $\mathfrak{g}_{\bar{1}}$ is a vector space of dimension $2n$. There is a natural definition of the nilpotent positive part $\mathfrak{n}^+ = \mathfrak{osp}^+(1, 2n)$, which is a nilpotent subsuperalgebra of $\mathfrak{g}$ of dimension $n^2 + n$. In its $\mathbb{Z}_2$ -graded decomposition $\mathfrak{n}^+ = \mathfrak{n}_0^+ \oplus \mathfrak{n}_1^+$, the even part $\mathfrak{n}_0^+ \simeq \mathfrak{sp}^+(2n)$ is just the nilpotent positive part of dimension n^2 in the ordinary triangular decomposition of the Lie algebra $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{sp}(2n)$, and the odd part $\mathfrak{n}_1^+$ is a subspace of dimension n in $\mathfrak{g}_{\bar{1}}$. Denoting by $\mathfrak{h}$ the Cartan Lie subalgebra of $\mathfrak{g}_{\bar{0}}$, the Borel subsuperalgebra $\mathfrak{b}^+$ of $\mathfrak{g}$ is defined as the subsuperalgebra generated by $\mathfrak{n}^+$ and $\mathfrak{h}$. We have $\mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h} = \mathfrak{b}_0^+ \oplus \mathfrak{n}_1^+$, where the even part $\mathfrak{b}_0^+ = \mathfrak{n}_0^+ \oplus \mathfrak{h}$ is the positive Borel Lie subalgebra of $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{sp}(2n)$. In particular, $\mathfrak{b}_0^+$ and $\mathfrak{b}^+$ are of dimensions $n^2 + n$ and $n^2 + 2n$ respectively. The enveloping algebras $\mathcal{U}(\mathfrak{n}^+)$ and $\mathcal{U}(\mathfrak{b}^+)$ can be described as iterated skew polynomial algebras; then they are noetherian domains and therefore we can consider their skewfields of fractions. We prove in the paper:

Main theorem. *For any $n \geq 2$, the enveloping algebras of the subsuperalgebras $\mathfrak{n}^+$ and $\mathfrak{b}^+$ of $\mathfrak{osp}(1, 2n)$ satisfy the following isomorphisms:*

- (i) $\text{Frac}\mathcal{U}(\mathfrak{n}^+) \simeq \text{Frac}\left(A_{n(n-1)/2}(\mathbb{C}) \otimes \tilde{\mathcal{O}}(\mathbb{C}^n)\right)$, and its center is a purely transcendental extension of dimension n over $\mathbb{C}$.
- (ii) $\text{Frac}\mathcal{U}(\mathfrak{b}^+) \simeq \text{Frac}\left(A_{n(n-1)/2}(\mathbb{C}) \otimes \tilde{A}_n(\mathbb{C})\right)$, and its center is $\mathbb{C}$.

The method we use to prove this theorem allows to recover as a corollary the classical Gelfand-Kirillov property for the even parts, with already known descriptions of $\text{Frac}\mathcal{U}(\mathfrak{n}_0^+)$ as a Weyl skewfield $D_{n(n-1)/2}(K)$ over a commutative field K which is a purely transcendental extension of dimension n over $\mathbb{C}$, and of $\text{Frac}\mathcal{U}(\mathfrak{b}_0^+)$ as a Weyl skewfield $D_{n(n+1)/2}(\mathbb{C})$ over $\mathbb{C}$. The particular case $n = 2$ of the main theorem was previously proved in Proposition 3.2 and Theorem 3.4 of [2]. The natural and probably difficult question of an extension of the main theorem to the enveloping algebra of the Lie superalgebra $\mathfrak{osp}(1, 2n)$ itself must be appreciated by recalling that even for the even part the answer is unknown, the case of $\mathfrak{sp}(2n)$ being one of the situations where the original Gelfand-Kirillov conjecture remains open (see [16]).

The paper is in three parts. The first one introduces the analogues of Weyl algebras for polynomial superalgebras and some canonical form of their skewfields of fractions. The second part describes the enveloping algebras $\mathcal{U}(\mathfrak{n}^+)$ and $\mathcal{U}(\mathfrak{b}^+)$ as iterated skew polynomial algebras. The last section contains the proof of the main theorem, based on suitable rational changes of generators reducing step by step the commutation relations.

1. ANALOGUES OF WEYL ALGEBRAS FOR POLYNOMIAL SUPERALGEBRAS

1.1. Polynomial superalgebras.

1.1.1. *Notations.* It is usual in quantum group theory to denote by $\mathcal{O}_\Lambda(\mathbb{C}^n)$ the algebra of polynomials in n indeterminates $y_1, \dots, y_n$ with coefficients in $\mathbb{C}$ and noncommutative multiplication twisted by relations $y_i y_j = \lambda_{ij} y_j y_i$ for any $1 \leq i, j \leq n$, where $\Lambda = (\lambda_{ij})$ is a $n \times n$ multiplicatively skew-symmetric matrix with entries in $\mathbb{C}$. In the particular cases where all $\lambda_{ij} = 1$, or $\lambda_{ij} = -1$ for all $i \neq j$, we use respectively the two following notations:

- (i) $\mathcal{O}(\mathbb{C}^n)$ is the commutative algebra of polynomials in n indeterminates $y_1, y_2, \dots, y_n$ with coefficients in $\mathbb{C}$.
- (ii) $\tilde{\mathcal{O}}(\mathbb{C}^n)$ is the noncommutative algebra of polynomials in n indeterminates $y_1, y_2, \dots, y_n$ with coefficients in $\mathbb{C}$ and product twisted by relations $y_i y_j = -y_j y_i$ for any $1 \leq i \neq j \leq n$.

1.1.2. *Superalgebra structure.* The monomials $y_1^{\beta_1} y_2^{\beta_2} \dots y_n^{\beta_n}$ whose total degree $\beta_1 + \beta_2 + \dots + \beta_n$ is even generate a commutative subalgebra R_0 of $\tilde{\mathcal{O}}(\mathbb{C}^n)$. Denoting by R_1 the R_0 -submodule of $\tilde{\mathcal{O}}(\mathbb{C}^n)$ generated by $y_1, \dots, y_n$, the $\mathbb{C}$ -submodules R_0 and R_1 satisfy $\tilde{\mathcal{O}}(\mathbb{C}^n) = R_0 \oplus R_1$ and $R_i R_j \subseteq R_{i+j}$ (with indices taken modulo 2). This gives rise to a structure of superalgebra on $\tilde{\mathcal{O}}(\mathbb{C}^n)$. The algebras $\tilde{\mathcal{O}}(\mathbb{C}^n)$ being noetherian domains, they admit skew fields of fractions. The following proposition gives an alternative form of $\tilde{\mathcal{O}}(\mathbb{C}^n)$ in terms of planes $\tilde{\mathcal{O}}(\mathbb{C}^2)$ up to rational equivalence.

1.1.3. *Proposition.* For any integer $n \geq 2$, we have:

If $n = 2p$, then $\text{Frac}(\tilde{\mathcal{O}}(\mathbb{C}^n)) \simeq \text{Frac}(\tilde{\mathcal{O}}(\mathbb{C}^2)^{\otimes p})$

If $n = 2p + 1$, then $\text{Frac}(\tilde{\mathcal{O}}(\mathbb{C}^n)) \simeq \text{Frac}(\tilde{\mathcal{O}}(\mathbb{C}^2)^{\otimes p} \otimes \mathcal{O}(\mathbb{C}))$

Proof. Suppose that $n \geq 3$ and define from generators $y_1, \dots, y_n$ of $\tilde{\mathcal{O}}(\mathbb{C}^n)$ with $y_i y_j = -y_j y_i$ for all $1 \leq i \neq j \leq n$ the monomials: $y'_1 = y_1$, $y'_2 = y_2$ and $y'_i = y_1 y_2 y_i$ for any $3 \leq i \leq n$. They satisfy $y'_1 y'_2 = -y'_2 y'_1$ and

$$y'_1 y'_i = y'_i y'_1, \quad y'_2 y'_i = y'_i y'_2, \quad y'_i y'_j = -y'_j y'_i \quad \text{for all } 3 \leq i \neq j \leq n.$$

It is clear that the subfield of $\text{Frac} \tilde{\mathcal{O}}(\mathbb{C}^n)$ generated by $y'_1, y'_2, \dots, y'_n$ is equal to $\text{Frac} \tilde{\mathcal{O}}(\mathbb{C}^n)$. We deduce that $\text{Frac} \tilde{\mathcal{O}}(\mathbb{C}^n) \simeq \text{Frac}(\tilde{\mathcal{O}}(\mathbb{C}^2) \otimes \tilde{\mathcal{O}}(\mathbb{C}^{n-2}))$, and the result follows by iteration. $\square$

1.2. Weyl algebras for polynomial superalgebras.

1.2.1. *Notations.* The definition of quantum differential calculi on various quantum algebras with adapted de Rham complexes gives rise to suitable versions of quantum Weyl algebras. In the case of quantum spaces $\mathcal{O}_\Lambda(\mathbb{C}^n)$, these algebras $A_n^{\bar{q}, \Lambda}$ (introduced in [1] and [19] and studied in many papers) take in consideration both the quantization parameters $\Lambda = (\lambda_{ij})$ of the

space and the quantization parameters $\bar{q} = (q_1, \dots, q_n)$ of the “differential” operators on this space. We are interested here in two particular cases:

(i) For $\lambda_{ij} = q_i = 1$ for all $1 \leq i, j \leq n$, we find the usual Weyl algebra $A_n(\mathbb{C})$, that is the algebra of polynomials in $2n$ indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$ with coefficients in $\mathbb{C}$ and a noncommutative product twisted by relations:

$$\begin{cases} x_i x_j - x_j x_i = y_i y_j - y_j y_i = x_i y_j - y_j x_i = 0 & (1 \leq i \neq j \leq n), \\ x_i y_i - y_i x_i = 1 & (1 \leq i \leq n). \end{cases} \quad (1)$$

We also introduce the subalgebra $U_n(\mathbb{C})$ of $A_n(\mathbb{C})$ generated by $y_1, \dots, y_n$ and $w_1, \dots, w_n$ with $w_i = y_i x_i$ for any $1 \leq i \leq n$. We have:

$$\begin{cases} w_i w_j - w_j w_i = y_i y_j - y_j y_i = w_i y_j - y_j w_i = 0 & (1 \leq i \neq j \leq n), \\ w_i y_i - y_i w_i = y_i & (1 \leq i \leq n). \end{cases} \quad (2)$$

(ii) For $q_i = 1$ and $\lambda_{ij} = -1$ for all $1 \leq i \neq j \leq n$, we denote by $\tilde{A}_n(\mathbb{C})$ the algebra of polynomials in $2n$ indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$ with coefficients in $\mathbb{C}$ and a noncommutative product twisted by:

$$\begin{cases} x_i x_j + x_j x_i = y_i y_j + y_j y_i = x_i y_j + y_j x_i = 0 & (1 \leq i \neq j \leq n), \\ x_i y_i - y_i x_i = 1 & (1 \leq i \leq n). \end{cases} \quad (3)$$

We denote by $\tilde{U}_n(\mathbb{C})$ the subalgebra of $\tilde{A}_n(\mathbb{C})$ generated by $y_1, \dots, y_n$ and $w_1, \dots, w_n$ with $w_i = y_i x_i$ for any $1 \leq i \leq n$. We have:

$$\begin{cases} y_i y_j + y_j y_i = 0 & (1 \leq i \neq j \leq n), \\ w_i w_j - w_j w_i = w_i y_j - y_j w_i = 0 & (1 \leq i \neq j \leq n), \\ w_i y_i - y_i w_i = y_i & (1 \leq i \leq n). \end{cases} \quad (4)$$

1.2.2. Remarks. The algebras $\tilde{A}_n(\mathbb{C})$ are the algebras denoted by $S_{n,n}^\Lambda$ and studied in [17] and [18] when all non diagonal entries of Λ are equal to -1 . Section 11 of [19] gives a slightly different presentation of these algebras as “generalized enveloping algebras”; with the notations of this article, case (i) corresponds to the values $q_{ij} = p_i = q = 1$, and case (ii) is for $q_{ij} = -1$ and $p_i = q = 1$. They also appear in example 2.1 of [8] (taking $q = 1$ and $p_{ij} = -1$). We refer to paragraph 1.3.3 of [17] for more ringtheoretical references and a survey, and only mention here that, for any integer $n \geq 1$, the algebra $\tilde{A}_n(\mathbb{C})$ is simple, has center $\mathbb{C}$, and has the same Hochschild homology and cohomology as the usual Weyl algebra $A_n(\mathbb{C})$.

1.2.3. Superalgebra structure.

(i) Let $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be two $\mathbb{C}$ -vector spaces of dimension n with respective basis $\{z_1, \dots, z_n\}$ and $\{y_1, \dots, y_n\}$. Let $\mathfrak{k}_0$ be a linear extension of $\mathfrak{g}_0$ of dimension $2n$ with basis $\{z_1, \dots, z_n, w_1, \dots, w_n\}$. Then we define a Lie superalgebra structure $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of dimension $2n$ and a Lie superalgebra structure $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{g}_1$ of dimension $3n$ by setting for the values of the super-bracket on these generators:

$$\begin{cases} \{y_i, y_i\} = z_i, [w_i, y_i] = y_i, [w_i, z_i] = 2z_i \text{ for any } 1 \leq i \leq n, \\ \text{the brackets are zero for all other pairs of generators.} \end{cases}$$

By construction $\mathfrak{g}_0$ is a subsuperalgebra of $\mathfrak{k}_0$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a subsuperalgebra of $\mathfrak{k}$. It is clear by PBW theorem (see Theorem 6.1.2 in [14]) that the enveloping algebra $\mathcal{U}(\mathfrak{k})$ is isomorphic to $\tilde{U}_n(\mathbb{C})$ of GK-dim $2n$. Moreover the subalgebra $\mathcal{U}(\mathfrak{k}_0)$ is isomorphic to $U_n(\mathbb{C})$ of GK-dim $2n$, and the subalgebra $\mathcal{U}(\mathfrak{g})$ is isomorphic to $\tilde{\mathcal{O}}(\mathbb{C}^n)$ of GK-dim n .

(ii) We have for the usual Weyl algebra the classical isomorphism $A_n(\mathbb{C}) \simeq A_1(\mathbb{C})^{\otimes n}$. We also have $\tilde{A}_n(\mathbb{C}) \simeq \tilde{A}_1(\mathbb{C})^{\hat{\otimes} n}$ in the tensor category of superalgebras assigning the parity 1 to the generators x_i and y_i .

(iii) All isomorphisms and tensor products considered in the following are in the tensor category of associative algebras.

1.2.4. *Remark.* All algebras introduced in 1.2.1 can be described as iterated skew polynomial algebras over $\mathbb{C}$; then they are noetherian domains, and therefore they admit a skewfield of fractions. The skewfield $\text{Frac } A_n(\mathbb{C}) = \text{Frac } U_n(\mathbb{C})$ is the well known Weyl skewfield $D_n(\mathbb{C})$, with center $\mathbb{C}$. Similarly we deduce from (4) that $\text{Frac } \tilde{A}_n(\mathbb{C}) = \text{Frac } \tilde{U}_n(\mathbb{C})$, and it is proved in Proposition 3.3.1 of [18] that its center is also $\mathbb{C}$. In parallel with Proposition 1.1.3, we have for these skewfields the following canonical form.

1.2.5. *Theorem.* For any integer $n \geq 1$, we have the following isomorphisms:

- (i) If $n = 2p$ is even, then $\text{Frac } \tilde{A}_n(\mathbb{C}) = \text{Frac } (\tilde{A}_2(\mathbb{C})^{\otimes p})$.
- (ii) If $n = 2p + 1$ is odd, then $\text{Frac } \tilde{A}_n(\mathbb{C}) = \text{Frac } (\tilde{A}_2(\mathbb{C})^{\otimes p} \otimes A_1(\mathbb{C}))$.

Proof. We can suppose $n \geq 1$. We consider the subalgebra $\tilde{U}_n(\mathbb{C})$ defined in 1.2.1 with generators $w_1, \dots, w_n, y_1, \dots, y_n$ and relations (4). As in the proof of Proposition 1.1.3, we introduce: $y'_1 = y_1, y'_2 = y_2$ and $y'_i = y_1 y_2 y_i$ for $3 \leq i \leq n$. Then we define: $w'_1 = w_1 - (w_3 + \dots + w_n), w'_2 = w_2 - (w_3 + \dots + w_n)$ and $w'_i = w_i$ for $3 \leq i \leq n$. Obvious calculations give:

$$\begin{cases} y'_1 y'_2 = -y'_2 y'_1, & w'_1 w'_2 = w'_2 w'_1, & y'_1 w'_2 = w'_2 y'_1, & y'_2 w'_1 = w'_1 y'_2, \\ [w'_1, y'_1] = y'_1, & [w'_2, y'_2] = y'_2. \end{cases} \quad (5)$$

$$\begin{cases} y'_1 y'_j = y'_j y'_1, & y'_2 y'_j = y'_j y'_2, & y'_1 w'_j = w'_j y'_1, & \text{for } 3 \leq j \leq n. \\ y'_2 w'_j = w'_j y'_2, & w'_1 w'_j = w'_j w'_1, & w'_2 w'_j = w'_j w'_2, \end{cases} \quad (6)$$

We calculate for $\ell = 1$ or 2 and any $3 \leq j \leq n$ the commutator: $[w'_\ell, y'_j] = [w_\ell, y_1 y_2 y_j] - \sum_{i=3}^n [w_i, y_1 y_2 y_j] = y_1 y_2 y_j - [w_j, y_1 y_2 y_j] = 0$ to conclude :

$$y'_j w'_1 = w'_1 y'_j \text{ and } y'_j w'_2 = w'_2 y'_j \text{ for any } 3 \leq j \leq n. \quad (7)$$

Moreover:

$$[w'_j, y'_j] = y'_j \text{ for any } 3 \leq j \leq n, \quad (8)$$

$$y'_i y'_j = -y'_j y'_i, \quad w'_i w'_j = w'_j w'_i, \quad w'_i y'_j = y'_j w'_i, \quad \text{for } 3 \leq i \neq j \leq n. \quad (9)$$

We denote by V the subalgebra of $\tilde{U}_n(\mathbb{C})$ generated by $y'_1, \dots, y'_n, w'_1, \dots, w'_n$, by W the subalgebra of V generated by y'_1, y'_2, w'_1, w'_2 , and by W' the

subalgebra of V generated by $y'_3, \dots, y'_n, w'_3, \dots, w'_n$. It follows from (5) that $W' \simeq \tilde{U}_2(\mathbb{C})$, from (8) and (9) that $W'' \simeq \tilde{U}_{n-2}(\mathbb{C})$, and from (7) and (9) that $V \simeq W \otimes W'$. Since it is clear that $\text{Frac } V = \text{Frac } \tilde{U}_n(\mathbb{C})$, we deduce that $\text{Frac } \tilde{U}_n(\mathbb{C}) \simeq \text{Frac } (\tilde{U}_2(\mathbb{C}) \otimes \tilde{U}_{n-2}(\mathbb{C}))$. We conclude that $\text{Frac } \tilde{A}_n(\mathbb{C}) \simeq \text{Frac } (\tilde{A}_2(\mathbb{C}) \otimes \tilde{A}_{n-2}(\mathbb{C}))$ and finish by induction on n . $\square$

2. THE LIE SUPERALGEBRA $\mathfrak{osp}(1, 2n)$ AND ITS ENVELOPING ALGEBRA

2.1. Generators and relations. We refer for instance to [6], [15], [2]. The basefield is $\mathbb{C}$. For any $n \geq 1$, the odd part $\mathfrak{g}_{\bar{1}}$ of $\mathfrak{g} = \mathfrak{osp}(1, 2n)$ is a vector space of dimension $2n$ and the even part $\mathfrak{g}_{\bar{0}}$ is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(2n)$ of dimension $2n^2 + n$. As a Lie superalgebra, $\mathfrak{osp}(1, 2n)$ is generated by the elements $b_i^{\pm}$ ($1 \leq i \leq n$) of a basis of $\mathfrak{g}_{\bar{1}}$. Then the elements $\{b_j^{\pm}, b_k^{\pm}\}$ ($1 \leq j \leq k \leq n$) and $\{b_j^+, b_k^-\}$ ($1 \leq j, k \leq n$) form a basis of $\mathfrak{g}_{\bar{0}}$. The brackets in $\mathfrak{osp}(1, 2n)$ are given by the “para-Bose” relations on the generators:

$$[\{b_j^{\xi}, b_k^{\eta}\}, b_{\ell}^{\epsilon}] = (\epsilon - \xi)\delta_{j\ell}b_k^{\eta} + (\epsilon - \eta)\delta_{k\ell}b_j^{\xi} \quad (10)$$

$$\begin{aligned} [\{b_i^{\xi}, b_j^{\eta}\}, \{b_k^{\epsilon}, b_{\ell}^{\varphi}\}] &= (\epsilon - \eta)\delta_{jk}\{b_i^{\xi}, b_{\ell}^{\varphi}\} + (\epsilon - \xi)\delta_{ik}\{b_j^{\eta}, b_{\ell}^{\varphi}\} \\ &\quad + (\varphi - \eta)\delta_{j\ell}\{b_i^{\xi}, b_k^{\epsilon}\} + (\varphi - \xi)\delta_{i\ell}\{b_j^{\eta}, b_k^{\epsilon}\}, \end{aligned} \quad (11)$$

with $1 \leq i, j, k, \ell \leq n$ and $\epsilon, \varphi, \xi, \eta$ are $\pm$. By PBW theorem, the enveloping algebra $\mathcal{U}(\mathfrak{osp}(1, 2n))$ is generated by the $2n^2 + n$ elements:

$$b_i^{\pm} \text{ and } k_i := \frac{1}{2}\{b_i^-, b_i^+\} \text{ for } 1 \leq i \leq n, \quad (12)$$

$$a_{ij}^{\pm} := \frac{1}{2}\{b_i^{\pm}, b_j^{\pm}\} \text{ for } 1 \leq i < j \leq n, \quad (13)$$

$$s_{ij} := \frac{1}{2}\{b_i^-, b_j^+\} \text{ and } t_{ij} := \frac{1}{2}\{b_i^+, b_j^-\} \text{ for } 1 \leq i < j \leq n, \quad (14)$$

of $\mathfrak{osp}(1, 2n)$. In particular, denoting

$$c_i^{\pm} := \frac{1}{2}\{b_i^{\pm}, b_i^{\pm}\} \text{ for } 1 \leq i \leq n, \quad (15)$$

the enveloping algebra $\mathcal{U}(\mathfrak{sp}(2n))$ of the even part $\mathfrak{g}_{\bar{0}}$ is the subalgebra of $\mathcal{U}(\mathfrak{osp}(1, 2n))$ generated by the $2n^2 + n$ elements $(b_i^{\pm})^2 = c_i^{\pm}, k_i$ for $1 \leq i \leq n$, and $a_{ij}^{\pm}, s_{ij}, t_{ij}$ for $1 \leq i < j \leq n$. The commutation relations in the associative algebra $\mathcal{U}(\mathfrak{osp}(1, 2n))$ are deduced from (10) and (11) taking $\{x, y\} = xy + yx$ if $x, y \in \mathfrak{g}_{\bar{1}}$, and $[x, y] = xy - yx$ otherwise. In particular commutation relations between the generators defined by (12), (13), (14), (15) are obtained by injecting them in relations (10) and (11). For instance:

$$b_i^+ b_j^+ + b_j^+ b_i^+ = 2a_{ij}^+ \quad \text{for all } 1 \leq i < j \leq n, \quad (16)$$

$$[c_i^+, b_j^+] = [c_i^+, c_j^+] = 0 \quad \text{for all } 1 \leq i < j \leq n, \quad (17)$$

$$[a_{ij}^+, b_{\ell}^+] = [a_{ij}^+, c_{\ell}^+] = 0 \quad \text{for all } 1 \leq i < j \leq n, 1 \leq \ell \leq n, \quad (18)$$

$$[a_{ij}^+, a_{k\ell}^+] = 0 \quad \text{for all } 1 \leq i < j \leq n, 1 \leq k < \ell \leq n. \quad (19)$$

Similarly with (10) and (14), we deduce for all $1 \leq i < j \leq n$ and $1 \leq k \leq n$:

$$[t_{ij}, b_j^+] = b_i^+ \quad \text{and} \quad [t_{ij}, b_k^+] = 0 \quad \text{if } k \neq j. \quad (20)$$

The relations involving the t_{ij} 's are obtained from (11), (13) and (14), which give for $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$:

$$[t_{ij}, a_{k\ell}^+] = \delta_{jk} \frac{1}{2} \{b_i^+, b_\ell^+\} + \delta_{j\ell} \frac{1}{2} \{b_i^+, b_k^+\}, \quad (21)$$

$$[t_{ij}, t_{k\ell}] = \delta_{jk} \frac{1}{2} \{b_i^+, b_\ell^-\} - \delta_{i\ell} \frac{1}{2} \{b_k^+, b_j^-\}, \quad (22)$$

or in other words:

$$[t_{ij}, a_{k\ell}^+] = 0 \quad \text{if } j \neq k, j \neq \ell \quad (1 \leq i < j \leq n, 1 \leq k < \ell \leq n), \quad (23)$$

$$[t_{ij}, a_{j\ell}^+] = a_{i\ell}^+ \quad (1 \leq i < j < \ell \leq n), \quad (24)$$

$$[t_{ij}, a_{ij}^+] = c_i^+ \quad (1 \leq i < j \leq n), \quad (25)$$

$$[t_{ij}, a_{kj}^+] = a_{ik}^+ \quad \text{if } i < k \quad (1 \leq i < j \leq n, 1 \leq k < j \leq n), \quad (26)$$

$$[t_{ij}, a_{ki}^+] = a_{kj}^+ \quad \text{if } k < i \quad (1 \leq i < j \leq n, 1 \leq k < j \leq n), \quad (27)$$

$$[t_{ij}, t_{j\ell}] = t_{i\ell} \quad (1 \leq i < j < \ell \leq n), \quad (28)$$

$$[t_{ij}, t_{ki}] = -t_{kj} \quad (1 \leq k < i < j \leq n), \quad (29)$$

$$[t_{ij}, t_{k\ell}] = 0 \quad \text{if } j \neq k, i \neq \ell \quad (1 \leq i < j \leq n, 1 \leq k < \ell \leq n). \quad (30)$$

The action of the generators k_i follows from (10) and (12):

$$[k_i, k_j] = 0 \quad (1 \leq i, j \leq n), \quad (31)$$

$$[k_i, b_i^+] = b_i^+, \quad [k_i, b_j^\pm] = 0 \quad (1 \leq i \neq j \leq n), \quad (32)$$

then with (13) and (14), for all $1 \leq i < j \leq n$ and $1 \leq \ell \leq n$:

$$\begin{cases} [k_i, a_{ij}^+] = a_{ij}^+, \\ [k_j, a_{ij}^+] = a_{ij}^+, \\ [k_\ell, a_{ij}^+] = 0 \text{ if } \ell \neq i, j. \end{cases} \quad \begin{cases} [k_i, t_{ij}] = t_{ij}, \\ [k_j, t_{ij}] = -t_{ij}, \\ [k_\ell, t_{ij}] = 0 \text{ if } \ell \neq i, j. \end{cases} \quad (33)$$

2.2. Chevalley generators and Serre relations. Following [15] (up to a renormalization of the n -th generator by $\sqrt{2}$), we introduce:

$$t_i := t_{i,i+1} \quad \text{for any } 1 \leq i \leq n-1, \quad \text{and } t_n := b_n^+, \quad (34)$$

which satisfy:

$$[t_i, t_j] = 0 \quad \text{if } |i-j| > 1 \quad (1 \leq i, j \leq n), \quad (35)$$

$$[t_i, t_{i+1}] = t_{i,i+2} \quad (1 \leq i \leq n-2), \quad (36)$$

$$[t_{n-1}, t_n] = b_{n-1}^+. \quad (37)$$

It also follows from (20) that for any $1 \leq i \leq n-1$ and $1 \leq j \leq n$:

$$[t_i, b_{i+1}^+] = b_i^+ \quad \text{and} \quad [t_i, b_j^+] = 0 \quad \text{if } j \neq i+1. \quad (38)$$

By straightforward calculations using (29), (35), (36), (37), the elements $t_1, \dots, t_n$ defined in (34) satisfy the following Serre identities which already appear as (2.11) in [15]:

$$t_i^2 t_{i+1} - 2t_i t_{i+1} t_i + t_{i+1} t_i^2 = 0 \quad (1 \leq i \leq n-1), \quad (39)$$

$$t_i^2 t_{i-1} - 2t_i t_{i-1} t_i + t_{i-1} t_i^2 = 0 \quad (2 \leq i \leq n-1), \quad (40)$$

$$t_n^3 t_{n-1} - (t_n^2 t_{n-1} t_n + t_n t_{n-1} t_n^2) + t_{n-1} t_n^3 = 0. \quad (41)$$

2.3. Enveloping algebra of the nilpotent subsuperalgebra $\mathfrak{n}^+$.

2.3.1. *Definition of $\mathfrak{n}^+$.* We define $\mathfrak{n}^+ := \mathfrak{osp}^+(1, 2n)$ as the subsuperalgebra of $\mathfrak{osp}(1, 2n)$ generated by Chevalley generators $t_1, t_2, \dots, t_n$. By (34), (37) and (38), $\mathfrak{n}^+$ contains the elements $b_1^+, b_2^+, \dots, b_n^+$. Hence by (13), $\mathfrak{n}^+$ contains the elements a_{ij}^+ for $1 \leq i < j \leq n$. It also follows inductively from (28) and (36) that $\mathfrak{n}^+$ contains the elements t_{ij} for $1 \leq i < j \leq n$. Hence:

$$\mathfrak{n}^+ = \mathfrak{n}_0^+ \oplus \mathfrak{n}_1^+,$$

where $\mathfrak{n}_0^+$ is the $\mathbb{C}$ -vector space with basis $(b_i^+)_{1 \leq i \leq n}$, and $\mathfrak{n}_0^+ \simeq \mathfrak{sp}^+(2n)$ is the Lie subalgebra of $\mathfrak{g}_0 \simeq \mathfrak{sp}(2n)$ with basis $(c_i^+, a_{ij}^+, t_{ij})_{1 \leq i < j \leq n}$. Then:

$$\dim \mathfrak{n}_0^+ = \dim \mathfrak{sp}^+(2n) = n^2 \quad \text{and} \quad \dim \mathfrak{n}_1^+ = n.$$

Let us observe that relations (17), (18) and (19) imply that the Lie subalgebra with basis $(c_i^+, a_{ij}^+)_{1 \leq i < j \leq n}$ in $\mathfrak{n}_0^+$ is abelian of dimension $n(n+1)/2$.

2.3.2. *Description of $\mathcal{U}(\mathfrak{n}^+)$ as an iterated skew polynomial algebra.* We denote by U^+ the enveloping algebra $\mathcal{U}(\mathfrak{n}^+)$ of $\mathfrak{n}^+$. By PBW theorem U^+ is generated as an associative algebra over $\mathbb{C}$ by the n^2 elements b_i^+ , a_{ij}^+ and t_{ij} for $1 \leq i < j \leq n$. The relations detailed in 2.1 allow us to describe U^+ as an iterated skew polynomial extension. We start with $A := \mathbb{C}[a_{ij}^+]_{1 \leq i < j \leq n}$, which is by (19) a commutative polynomial algebra in $n(n-1)/2$ variables. Then we set:

$$B_0 := A, \quad B_1 := A[b_1^+], \quad B_j := B_{j-1}[b_j^+; \sigma_j, \delta_j] \quad \text{for } 2 \leq j \leq n, \quad (42)$$

where σ_j is the automorphism of B_{j-1} and δ_j is the σ_j -derivation of B_{j-1} defined by $\sigma_j(a_{k\ell}^+) = a_{k\ell}^+$ and $\delta_j(a_{k\ell}^+) = 0$ for all $1 \leq k < \ell \leq n$, and $\sigma_j(b_i^+) = -b_i^+$ and $\delta_j(b_i^+) = 2a_{ij}^+$ for any $1 \leq i < j$, in order to translate relations (16) and (18). At the last step we denote:

$$B := B_n = \mathbb{C}[a_{ij}^+][b_1^+][b_2^+; \sigma_2, \delta_2] \cdots [b_n^+; \sigma_n, \delta_n]. \quad (43)$$

Then we add the generators t_{ij} for $1 \leq i < j \leq n$ inductively:

$$\begin{aligned} R_1 &:= B[t_{1,2}; d_{1,2}][t_{1,3}; d_{1,3}] \cdots [t_{1,n}; d_{1,n}], \\ R_i &:= R_{i-1}[t_{i,i+1}; d_{i,i+1}][t_{i,i+2}; d_{i,i+2}] \cdots [t_{i,n}; d_{i,n}], \quad (2 \leq i \leq n-2) \\ R_{n-1} &:= R_{n-2}[t_{n-1,n}; d_{n-1,n}] = U^+, \end{aligned}$$

where the derivations d_{ij} are suitably defined to account for relations (20) and (23) to (30). To sum up we conclude:

$$A = \mathbb{C}[a_{ij}^+]_{1 \leq i < j \leq n}, \quad (44)$$

$$B = A[b_i^+; \sigma_i, \delta_i]_{1 \leq i \leq n}, \quad (45)$$

$$U^+ = B[t_{ij}; d_{ij}]_{1 \leq i < j \leq n}. \quad (46)$$

Let us recall that in this presentation, A is commutative and the b_i^+ 's commute with the elements of A .

2.3.3. Restriction to the even part. Because of relations (17) and (18), we can consider in B the commutative subalgebra C generated over A by the elements $c_k^+ = (b_k^+)^2$ for $1 \leq k \leq n$. It follows from previous results that:

$$C = \mathbb{C}[a_{ij}^+, c_k^+]_{1 \leq i < j \leq n, 1 \leq k \leq n}, \quad (47)$$

$$\mathcal{U}(\mathfrak{n}_0^+) = C[t_{ij}; d_{ij}]_{1 \leq i < j \leq n}. \quad (48)$$

2.4. Enveloping algebra of the solvable Borel subsuperalgebra $\mathfrak{b}^+$.

2.4.1. Definition of $\mathfrak{b}^+$. The Cartan Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$ is the Lie subalgebra with basis $(k_i)_{1 \leq i \leq n}$, which is abelian by relation (31). The positive Borel subsuperalgebra $\mathfrak{b}^+$ is defined as the subsuperalgebra of $\mathfrak{osp}(1, 2n)$ generated by $\mathfrak{n}^+$ and $\mathfrak{h}$. We have clearly:

$$\mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h} = \mathfrak{b}_0^+ \oplus \mathfrak{n}_1^+, \quad \text{with } \mathfrak{b}_0^+ := \mathfrak{n}_0^+ \oplus \mathfrak{h},$$

where the even part $\mathfrak{b}_0^+$ is the positive Borel Lie subalgebra of $\mathfrak{g}_0 \simeq \mathfrak{sp}(2n)$.

2.4.2. Description of $\mathcal{U}(\mathfrak{b}^+)$ as an iterated polynomial algebra. We denote by U_b^+ the enveloping algebra $\mathcal{U}(\mathfrak{b}^+)$ of $\mathfrak{b}^+$. The commutation relations of the k_i 's with the generators of U^+ given by formulas (31) to (33) allow an easy description of U_b^+ as the following iterated skew polynomial algebra:

$$U_b^+ = U^+[k_1, \Delta_1][k_2, \Delta_2] \cdots [k_n, \Delta_n], \quad (49)$$

where the derivations Δ_i vanish on all k_j 's and satisfy $\Delta_i(b_i^+) = b_i^+$ and $\Delta_i(b_j^+) = 0$ for any $1 \leq j \neq i \leq n$, $\Delta_i(a_{ij}^+) = a_{ij}^+$ and $\Delta_i(t_{ij}) = t_{ij}$ for $1 \leq i < j \leq n$, $\Delta_i(a_{ji}^+) = a_{ji}^+$ and $\Delta_i(t_{ji}) = -t_{ji}$ for $1 \leq j < i \leq n$, $\Delta_i(a_{k\ell}^+) = 0$ and $\Delta_i(t_{k\ell}) = 0$ for $k \neq i, \ell \neq i$.

2.4.3. Restriction to the even part. It follows from previous results that, with notations (49) and (48):

$$\mathcal{U}(\mathfrak{b}_0^+) = \mathcal{U}(\mathfrak{n}_0^+)[k_i; \Delta_i]_{1 \leq i \leq n} = C[t_{ij}; d_{ij}]_{1 \leq i < j \leq n}[k_i; \Delta_i]_{1 \leq i \leq n}. \quad (50)$$

3. PROOF OF THE MAIN THEOREM

3.1. Some general technical results. We will use the following easy lemmas.

3.1.1. *Lemma.* Let A be a commutative $\mathbb{C}$ -algebra, a an element of A , and R the iterated skew polynomial algebra generated over A by two generators x_1 and x_2 commuting with the elements of A and with relation $x_1x_2 + x_2x_1 = 2a$. Then the element $y = x_1x_2 - a$ satisfies $x_1y = -yx_1$ and $x_2y = -yx_2$.

Proof. Obvious calculation. $\square$

3.1.2. *Lemma.* Let A be a commutative $\mathbb{C}$ -algebra. Let $(a_{ij})_{1 \leq i < j \leq n}$ be a family of elements of A , and R the iterated skew polynomial algebra generated over A by n generators $x_1, x_2, \dots, x_n$ commuting with the elements of A and with relations $x_ix_j + x_jx_i = 2a_{ij}$ for all $1 \leq i < j \leq n$. Then the $n-1$ elements $y_i = x_1x_i - a_{1i}$ for any $2 \leq i \leq n$ satisfy:

$$x_1y_i = -y_ix_1 \quad \text{and} \quad y_iy_j + y_jy_i = 2(a_{1i}a_{1j} - a_{ij}x_1^2).$$

Proof. The first identity follows from previous lemma. For the second one, we observe firstly that:

$$\begin{aligned} x_jy_i &= x_jx_1x_i - x_ja_{1i} = (-x_1x_j + 2a_{1j})x_i - a_{1i}x_j \\ &= -x_1x_jx_i + 2a_{1j}x_i - a_{1i}x_j = x_1x_ix_j - 2a_{ij}x_1 + 2a_{1j}x_i - a_{1i}x_j \\ &= (x_1x_i - a_{1i})x_j - 2a_{ij}x_1 + 2a_{1j}x_i = y_ix_j - 2a_{ij}x_1 + 2a_{1j}x_i. \end{aligned}$$

We use this identity and Lemma 3.1.1 to calculate:

$$\begin{aligned} y_jy_i &= (x_1x_j - a_{1j})y_i = x_1(y_ix_j - 2a_{ij}x_1 + 2a_{1j}x_i) - a_{1j}y_i \\ &= -y_ix_1x_j - 2a_{ij}x_1^2 + 2a_{1j}x_1x_i - a_{1j}y_i \\ &= -y_i(x_1x_j - a_{1j}) - 2a_{1j}y_i - 2a_{ij}x_1^2 + 2a_{1j}x_1x_i \\ &= -y_iy_j - 2a_{1j}y_i - 2a_{ij}x_1^2 + 2a_{1j}x_1x_i \\ &= -y_iy_j + 2a_{1j}(x_1x_i - y_i) - 2a_{ij}x_1^2. \end{aligned}$$

We conclude that $y_jy_i = -y_iy_j + 2a_{1j}a_{1i} - 2a_{ij}x_1^2$. $\square$

3.2. Enveloping skewfield of the Lie subsuperalgebra $\mathfrak{n}^+$.

3.2.1. *Notations.* Recalling the description (44), (45), (46) of U^+ , we introduce the skewfields of rational functions:

$$K := \text{Frac } A = \mathbb{C}(a_{ij}^+)_{1 \leq i < j \leq n}, \quad (51)$$

$$L := \text{Frac } B = K(b_i^+; \sigma_i, \delta_i)_{1 \leq i \leq n}, \quad (52)$$

$$F := \text{Frac } U^+ = L(t_{ij}; d_{ij})_{1 \leq i < j \leq n}. \quad (53)$$

3.2.2. *Notations.* We define the following families of elements of B :

$$y_1 := b_1^+, \quad x_{1,i} := b_i^+, \quad u_{1,i,j} := a_{i,j}^+ \quad \text{for all } 1 \leq i < j \leq n, \quad (54)$$

and more generally for any $1 \leq k \leq n$:

$$\begin{cases} u_{k,i,j} := u_{k-1,k-1,i} u_{k-1,k-1,j} - u_{k-1,i,j} y_{k-1}^2 & \text{for all } k \leq i < j \leq n, \\ x_{k,i} := x_{k-1,k-1} x_{k-1,i} - u_{k-1,k-1,i} & \text{for any } k \leq i \leq n, \\ y_k := x_{k,k}. \end{cases} \quad (55)$$

It is clear that we have with notation (42):

$$y_k \in B_k, \quad x_{k,i} \in B_i, \quad u_{k,i,j} \in B_{k-1} \quad \text{for all } 1 \leq k \leq n, \quad k \leq i < j \leq n. \quad (56)$$

3.2.3. *Lemma.*

(i) The elements $y_1, y_2, \dots, y_n$ of B satisfy the relations:

$$\begin{cases} y_{k-1}y_k = -y_ky_{k-1} & \text{for any } 2 \leq k \leq n, \\ y_ky_\ell = y_\ell y_k & \text{for all } 1 \leq k, \ell \leq n, \quad \ell \neq k \pm 1, \\ y_ka_{ij}^+ = a_{ij}^+y_k & \text{for all } 1 \leq k \leq n, 1 \leq i < j \leq n. \end{cases}$$

(ii) For any $1 \leq k \leq n$, the elements $u_{k,i,j}$ with $k \leq i < j \leq n$ are in the commutative subalgebra $A[y_1^2, y_2^2, \dots, y_{k-1}^2]$ of B ,

Proof. By iterative application of Lemma 3.1.2, we have for any $1 \leq k \leq n$:

$$x_{k-1,k-1}x_{k,i} = -x_{k,i}x_{k-1,k-1} \quad \text{for any } 2 \leq k \leq i \leq n, \quad (57)$$

$$x_{k,i}x_{k,j} + x_{k,j}x_{k,i} = 2u_{k,i,j} \quad \text{for all } k \leq i < j \leq n. \quad (58)$$

We deduce by induction on ℓ that, for any $1 \leq \ell \leq n$:

$$\begin{cases} y_\ell x_{\ell+1,i} = -x_{\ell+1,i}y_\ell & \text{for any } \ell+1 \leq i \leq n, \\ y_\ell x_{k,i} = x_{k,i}y_\ell & \text{for any } \ell+2 \leq k \leq i \leq n, \\ y_\ell u_{k,i,j} = u_{k,i,j}y_\ell & \text{for any } 1 \leq k \leq i < j \leq n. \end{cases} \quad (59)$$

Point (i) of the lemma follows obviously from these relations and point (ii) follows by iterative definition (55) of the $u_{k,i,j}$'s. $\square$

3.2.4. *Proposition.*

(i) The subalgebra of B generated over $\mathbb{C}$ by $y_1, \dots, y_n$ is isomorphic to $\mathcal{O}_\Lambda(\mathbb{C}^n)$, where all entries of the symmetric matrix $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq n}$ are equal to 1 except $\lambda_{i,i+1} = -1 = \lambda_{i+1,i}$ for any $1 \leq i \leq n-1$.

(ii) Its skewfield of fractions Q satisfies:

$$L = Q(a_{ij}^+)_{1 \leq i < j \leq n}, \quad (60)$$

$$Q \simeq \text{Frac}(\tilde{\mathcal{O}}(\mathbb{C}^n)), \quad (61)$$

where each a_{ij}^+ commutes with the elements of Q .

Proof. By induction from relations (55), we prove that for any $1 \leq k \leq n$,

$$y_k = (y_{k-1}y_{k-2} \cdots y_2y_1)b_k^+ + r_{k-1}, \quad (62)$$

where the term r_{k-1} lies in the subalgebra of B_{k-1} generated by $y_1, \dots, y_{k-1}$ and the a_{ij}^+ 's for $1 \leq i < j \leq k$. It follows that the skewfield L defined by (52) is also the skewfield of rational functions generated over K by $y_1, \dots, y_n$. Then the commutation relations of Lemma 3.2.3 imply point (i) and assertion (60) of point (ii). In order to prove assertion (61), we suppose that $n \geq 3$ and define: $y'_1 = y_1$, $y'_2 = y_2$, $y'_3 = y_1y_3$ and $y'_i = y_i$ for any $4 \leq i \leq n$. It is clear that the subfield of Q generated by the elements $(y'_i)_{1 \leq i \leq n}$ is equal to Q . Moreover by straightforward calculations, we have:

$$\begin{cases} y'_1 y'_2 = -y'_2 y'_1, & y'_1 y'_i = y'_i y'_1, & y'_2 y'_i = y'_i y'_2, & \text{for any } 3 \leq i \leq n, \\ y'_3 y'_4 = -y'_4 y'_3, & y'_3 y'_i = y'_i y'_3, & & \text{for any } 5 \leq i \leq n, \end{cases}$$

We deduce that $Q \simeq \text{Frac}(\tilde{\mathcal{O}}(\mathbb{C}^2) \otimes T_{n-2})$ where T_{n-2} is the subalgebra of Q generated by the elements $(y'_i)_{3 \leq i \leq n}$. Applying the same reduction on the generators of T_{n-2} , we conclude by iteration and using Proposition 1.1.3 that $Q \simeq \text{Frac}(\tilde{\mathcal{O}}(\mathbb{C}^n))$. $\square$

3.2.5. *Lemma.* For any $1 \leq i \leq n-1$ and any $1 \leq m \leq n$, we have:

$$[t_i, u_{m,i,i+1}] = x_{m,i}^2 \quad (63)$$

$$[t_i, u_{m,k,i+1}] = u_{m,k,i} \quad \text{for } m \leq k < i \leq n, \quad (64)$$

$$[t_i, u_{m,i+1,\ell}] = u_{m,i,\ell} \quad \text{for } m < i+1 < \ell \leq n, \quad (65)$$

$$[t_i, u_{m,i+1,\ell}] = 0 \quad \text{for } m = i+1 < \ell \leq n, \quad (66)$$

$$[t_i, u_{m,k,\ell}] = 0 \quad \text{for } k \neq i+1 \text{ and } \ell \neq i+1, \quad (67)$$

$$[t_i, x_{m,i+1}] = x_{m,i} \quad \text{for } i+1 > m, \quad (68)$$

$$[t_i, x_{m,i+1}] = 0 \quad \text{for } i+1 = m, \quad (69)$$

$$[t_i, x_{m,k}] = 0 \quad \text{for } k \neq i+1. \quad (70)$$

Proof. We study the action on B of the Chevalley generators $t_1, t_2, \dots, t_{n-1}$ defined in 2.2. Let us recall that, by (38) we have:

$$[t_i, b_{i+1}^+] = b_i^+ \quad \text{and} \quad [t_i, b_k^+] = 0 \quad \text{if } k \neq i+1. \quad (71)$$

Similarly by (23) to (27), we obtain:

$$[t_i, a_{i,i+1}^+] = (b_i^+)^2, \quad (72)$$

$$[t_i, a_{k,i+1}^+] = a_{k,i}^+ \quad \text{for } 1 \leq k < i \leq n, \quad (73)$$

$$[t_i, a_{i+1,\ell}^+] = a_{i,\ell}^+ \quad \text{for } i+1 < \ell \leq n, \quad (74)$$

$$[t_i, a_{k,\ell}^+] = 0 \quad \text{for } k \neq i+1 \text{ and } \ell \neq i+1. \quad (75)$$

Then for fixed i , we prove the assertions of the lemma by induction on m . By (54), assertion (71) is the case $m = 1$ of assertions (68) to (70), and (72) to (75) are the case $m = 1$ of assertions (63) to (67). Suppose that the eight assertions are satisfied to some rank $m \geq 1$. By (55), we have:

$$\begin{aligned} [t_i, u_{m+1,k,\ell}] &= [t_i, u_{m,m,k}] u_{m,m,\ell} + u_{m,m,k} [t_i, u_{m,m,\ell}] \\ &\quad - [t_i, u_{m,k,\ell}] x_{m,m}^2 - u_{m,k,\ell} [t_i, x_{m,m}^2]. \end{aligned}$$

It follows from (69) and (70) that the fourth term is zero. Then using induction assumptions (63) to (67), tedious but straightforward calculations prove that these five assertions remain valid at the rank $m+1$. Similarly by (55) we have:

$$[t_i, x_{m+1,k}] = [t_i, x_{m,m}] x_{m,k} + x_{m,m} [t_i, x_{m,k}] - [t_i, u_{m,m,k}].$$

It follows from (69) and (70) that the first term is zero. Then using induction assumptions (63) to (70), it is easy to check that: $[t_i, x_{m+1,i+1}]$ equals 0 if

$m + 1 = i + 1$ and equals $x_{m+1,i}$ if $i + 1 > m + 1$, and $[t_i, x_{m+1,k}] = 0$ for any $k \neq i + 1$. This finishes the proof. $\square$

3.2.6. Corollary. *Each generator t_{ij} for $1 \leq i < j \leq n$ commutes with each generator y_m for $1 \leq m \leq n$.*

Proof. By (69) and (70), the Chevalley generators $t_1, t_2, \dots, t_{n-1}$ commute with $y_m = x_{m,m}$ for any $1 \leq m \leq n$. By (28) and (36), each generator t_{ij} for $1 \leq i < j \leq n$ is an iterated bracket of $t_1, t_2, \dots, t_{n-1}$. $\square$

3.2.7. Proposition. *The skewfield of fractions of U^+ satisfies:*

$$F = Q(a_{ij}^+)_{1 \leq i < j \leq n} (t_{ij}; d_{ij})_{1 \leq i < j \leq n}, \quad (76)$$

where the generators t_{ij} commute with all elements of Q , and the derivations d_{ij} translate the commutation relations between the t_{ij} 's and the a_{ij}^+ 's [see relations (23) to (27)] or the t_{ij} 's with each other [see relations (28) to (30)].

Proof. This is just a summary of assertions (53) and (60), Proposition 3.2.4 and Corollary 3.2.6. $\square$

3.2.8. Lemma. *Denote by $(q_{i,j})_{1 \leq i < j \leq n}$ the elements of F defined by:*

$$q_{i,j} := y_i^{-2} u_{i,i,j} \quad \text{for any } 1 \leq i < j \leq n. \quad (77)$$

Then the $q_{i,j}$'s commute with each other and with all elements of Q , and the subfield of L generated over Q by the $(q_{i,j})_{1 \leq i < j \leq n}$ is equal to L .

Proof. The first assertion follows from Corollary 3.2.3 and third relation in (59). To prove the second assertion, we observe directly from definition (55) that for any $1 \leq i < j \leq n$, we have:

$$u_{i,i,j} = e_{i-1,j} + (-1)^{i-1} a_{i,j}^+ y_1^2 y_2^2 \cdots y_{i-1}^2,$$

where the element $e_{i-1,j}$ lies in the subalgebra E_{i-1} generated by $y_1^2, y_2^2, \dots, y_{i-1}^2$ and the $a_{k,\ell}^+$'s for $1 \leq k \leq i-1$. Then we can write:

$$a_{i,j}^+ = (-1)^{i-1} y_1^{-2} y_2^{-2} \cdots y_{i-1}^{-2} q_{i,j} + e'_{i-1,j}$$

with $e'_{i-1,j} := (-1)^i y_1^{-2} y_2^{-2} \cdots y_{i-1}^{-2} e_{i-1,j} \in E_{i-1}$. This proves the rest of the lemma by induction on i . $\square$

3.2.9. Lemma. *For any $1 \leq i < j \leq n$ and any $1 \leq k < \ell \leq n$, we have:*

$$[t_{ij}, q_{i,j}] = 1, \quad (78)$$

$$[t_{ij}, q_{k,\ell}] = 0 \quad \text{if } j \neq \ell, \quad (79)$$

$$[t_{ij}, q_{k,j}] = 0 \quad \text{if } i < k, \quad (80)$$

$$[t_{ij}, q_{k,j}] = q_{k,i} \quad \text{if } i > k. \quad (81)$$

Proof. We proceed by induction on $j - i$. For $j = i + 1$, the identities directly follows from Lemma 3.2.5. For the general case, we deduced from (28) and (34) that $t_{i,j+1} = [t_{i,j}, t_{j,j+1}] = [t_{i,j}, t_j]$. Then we have $[t_{i,j+1}, q_{k,\ell}] = [[t_{i,j}, t_j], q_{k,\ell}] = [t_{i,j}, [t_j, q_{k,\ell}]] - [t_j, [t_{i,j}, q_{k,\ell}]]$ for any $1 \leq k < \ell \leq n$, and we

finish the proof by induction considering successively the different cases for the indices (i, j) and (k, ℓ) . $\square$

3.2.10. *Corollary.* The elements $(q_{i,j})_{1 \leq i < j \leq n}$ defined in Lemma 3.2.8 and the elements $(p_{i,j})_{1 \leq i < j \leq n}$ defined inductively by:

$$p_{1,j} := t_{1,j} \quad \text{and} \quad p_{i,j} := t_{i,j} - \sum_{m=1}^{i-1} q_{m,i} p_{m,j} \quad (82)$$

satisfy $[p_{ij}, q_{ij}] = 1$ and $[p_{ij}, p_{k\ell}] = [q_{ij}, q_{k\ell}] = [p_{ij}, q_{k\ell}] = 0$ for $(i, j) \neq (k, \ell)$.

Proof. Follows obviously from Lemma 3.2.9 by induction on i . $\square$

3.2.11. *Proposition.* We have: $F = Q(q_{ij})_{1 \leq i < j \leq n} (p_{ij}; \partial_{q_{ij}})_{1 \leq i < j \leq n}$.

Proof. It's clear by Lemma 3.2.8 that $L = Q(q_{ij})_{1 \leq i < j \leq n}$. By definition (82) and Proposition 3.2.7, the subfield of F generated over L by the elements $(p_{i,j})_{1 \leq i < j \leq n}$ is equal to F . Hence the proposition just summaries the results of Lemma 3.2.8 and Corollary 3.2.10. $\square$

3.2.12. *Corollary.* The skewfield F is isomorphic to $\text{Frac}(A_{n(n-1)/2} \otimes \tilde{\mathcal{O}}(\mathbb{C}^n))$, and its center is $\mathcal{Z}(F) = \mathcal{Z}(Q) = \mathbb{C}(y_1^2, \dots, y_n^2)$.

Proof. By Corollary 3.2.10, the generators p_{ij} and q_{ij} commute with all elements of Q and the subalgebra W of F they generate is isomorphic to the usual Weyl algebra $A_{n(n-1)/2}$. Then the theorem follows from Proposition 3.2.11 and assertion (61) of Proposition 3.2.4. $\square$

3.2.13. *Conclusion.* We have proved the first assertion of the main theorem.

3.3. Enveloping skewfield of the Lie subalgebra $\mathfrak{n}_0^+$.

3.3.1. *Notations.* In order to determine the skewfield of fractions of the enveloping algebra of the even part $\mathfrak{n}_0^+$ of $\mathfrak{n}^+$ described in (48), we complete the notations (51), (52), (53) and (61) by defining:

$$Q_0 := \mathbb{C}(y_1^2, y_2^2, \dots, y_n^2) \subset Q, \quad (83)$$

$$L_0 := K((b_1^+)^2, (b_2^+)^2, \dots, (b_n^+)^2) \subset L, \quad (84)$$

$$F_0 := \text{Frac} \mathcal{U}(\mathfrak{n}_0^+) = L_0(t_{ij}; d_{ij})_{1 \leq i < j \leq n} \subset F. \quad (85)$$

3.3.2. *Lemma.* We have the equality $L_0 = K(y_1^2, y_2^2, \dots, y_n^2)$.

Proof. Returning to notations (55) and using (59) and (58), we check by straightforward calculations that: $x_{k,i}^2 = -x_{k-1,k-1}^2 x_{k-1,i}^2 + u_{k-1,k-1,i}^2$ for all $2 \leq k \leq i \leq n$. Using this identity and point (ii) of Lemma 3.2.3, we prove that $y_1^2 = (b_1^+)^2$, $y_2^2 = -(b_1^+)^2 (b_2^+)^2 + a_{1,2}^2$, and :

$$y_i^2 = (-1)^{i-1} y_1^2 y_2^2 \cdots y_{i-1}^2 (b_i^+)^2 + v_{i-1} \quad \text{for any } 2 \leq i \leq n,$$

where the term v_{i-1} lies in the subalgebra generated over A by $y_1^2, y_2^2, \dots, y_{i-1}^2$. Then the result follows by induction on i . $\square$

3.3.3. Proposition. $F_0 = Q_0 (q_{ij})_{1 \leq i < j \leq n} (p_{ij} ; \partial_{q_{ij}})_{1 \leq i < j \leq n}$, with center Q_0 .

Proof. It is clear by previous lemma that $L_0 = Q_0(a_{ij})_{1 \leq i < j \leq n}$. Then the calculations in the proof of Lemma 3.2.8 show that the $q_{i,j}$'s are elements of L_0 , and moreover that the subfield of L_0 generated over Q_0 by $(q_{ij})_{1 \leq i < j \leq n}$ is equal to L_0 . Finally we deduce from (82) that the skew subfield of F_0 generated over L_0 by $(p_{ij})_{1 \leq i < j \leq n}$ is equal to F_0 . $\square$

3.3.4. Conclusion. The previous proposition showed that:

$$\text{Frac} \mathcal{U}(\mathfrak{n}_0^+) \simeq \text{Frac} (A_{n(n-1)/2}(\mathbb{C}) \otimes \mathcal{O}(\mathbb{C}^n)).$$

Hence we retrieve in the context of the classical Gelfand-Kirillov property the usual description of the enveloping skewfield of the nilpotent Lie algebra $\mathfrak{sp}^+(2n)$ as a Weyl skewfield $D_{n(n-1)/2}$ over a purely transcendental extension of dimension n over $\mathbb{C}$.

3.4. Enveloping skewfield of the Lie subsuperalgebra $\mathfrak{b}^+$. Resuming all notations of the previous section and using (49) and (53), we set:

$$H := \text{Frac} U_b^+ = F(k_1, \Delta_1)(k_2, \Delta_2) \cdots (k_n, \Delta_n). \quad (86)$$

Our first step is to determine the action of the generators $k_1, \dots, k_n$ of the Cartan subalgebra $\mathfrak{h}$ on the generators y_i, p_{ij}, q_{ij} ($1 \leq i < j \leq n$) introduced in the description of F in Proposition 3.2.11.

3.4.1. Lemma. For any $1 \leq \ell \leq n$, we have the following identities:

$$[k_\ell, y_m] = \begin{cases} y_\ell & \text{if } m = \ell, \\ 2^{m-\ell-1} y_m & \text{if } m > \ell, \end{cases} \quad \text{and} \quad [k_\ell, y_m] = 0 \text{ if } m < \ell. \quad (87)$$

$$\begin{cases} [k_\ell, q_{\ell,j}] = -q_{\ell,j} & [k_\ell, p_{\ell,j}] = p_{\ell,j} & \text{for any } j > \ell, \\ [k_\ell, q_{i,\ell}] = q_{i,\ell} & [k_\ell, p_{i,\ell}] = -p_{i,\ell} & \text{for any } i < \ell, \\ [k_\ell, q_{i,j}] = 0 & [k_\ell, p_{i,j}] = 0 & \text{for any } i \neq \ell \neq j. \end{cases} \quad (88)$$

Proof. We start with the action of k_ℓ on b_i^+ and $a_{i,j}^+$ for any $1 \leq i < j \leq n$ given by relations (32) and (33). Using the iterative definitions (54) and (55), we obtain for $1 \leq m < \ell$ the identities:

$$[k_\ell, x_{m,i}] = \begin{cases} x_{m,\ell} & \text{if } i = \ell, \\ 0 & \text{if } i \neq \ell, \end{cases} \quad \text{and} \quad [k_\ell, u_{m,i,j}] = \begin{cases} u_{m,\ell,j} & \text{if } i = \ell, \\ u_{m,i,\ell} & \text{if } j = \ell, \\ 0 & \text{if } i \neq \ell \neq j. \end{cases}$$

Similarly for $m = \ell$:

$$[k_\ell, x_{\ell,i}] = \begin{cases} x_{\ell,\ell} & \text{if } i = \ell, \\ 0 & \text{if } i \neq \ell, \end{cases} \quad \text{and} \quad [k_\ell, u_{\ell,i,j}] = \begin{cases} u_{\ell,\ell,j} & \text{if } i = \ell, \\ 0 & \text{if } i \neq \ell \neq j. \end{cases}$$

Finally for $m > \ell$: $[k_\ell, x_{m,i}] = 2^{m-\ell-1}x_{m,i}$ and $[k_\ell, u_{m,i,j}] = 2^{m-\ell}u_{m,i,j}$. Assertion (87) follows trivially from these relations since $y_m = x_{m,m}$. Moreover, for $1 \leq i, j \leq n$, we have by definition (77) and (82):

$$[k_\ell, q_{i,j}] = [k_\ell, y_i^{-2}u_{i,i,j}] = -2y_i^{-3}[k_\ell, y_i]u_{i,i,j} + y_i^{-2}[k_\ell, u_{i,i,j}], \quad (89)$$

$$[k_\ell, p_{i,j}] = [k_\ell, t_{i,j}] - \sum_{m=1}^{i-1} [k_\ell, q_{m,i}]p_{m,j} - \sum_{m=1}^{i-1} q_{m,i}[k_\ell, p_{m,j}]. \quad (90)$$

We deduce (88) applying case by case the previous identities. $\square$

3.4.2. *Notation.* An obvious observation arising from (88) is:

$$[k_\ell, p_{i,j}q_{i,j}] = 0 \quad \text{for all } 1 \leq \ell \leq n, \ 1 \leq i < j \leq n. \quad (91)$$

These relations lead us to introduce the following elements of H :

$$h_\ell := k_\ell + \sum_{m=\ell+1}^n q_{\ell,m}p_{\ell,m} - \sum_{m=1}^{\ell-1} q_{m,\ell}p_{m,\ell} \quad \text{for any } 1 \leq \ell \leq n, \quad (92)$$

which satisfy the relations described in the following lemma.

3.4.3. *Lemma.* For all $1 \leq \ell, m \leq n$ and $1 \leq i < j \leq n$, we have:

$$[h_\ell, q_{i,j}] = [k_\ell, p_{i,j}] = [h_\ell, h_m] = 0 \quad \text{and} \quad [h_\ell, y_m] = [k_\ell, y_m].$$

Proof. By (92) we have:

$$[h_\ell, q_{i,j}] = [k_\ell, q_{i,j}] + \sum_{m=\ell+1}^n q_{\ell,m}[p_{\ell,m}, q_{i,j}] - \sum_{m=1}^{\ell-1} q_{m,\ell}[p_{m,\ell}, q_{i,j}].$$

If $i \neq \ell$ and $j \neq \ell$, all brackets appearing in the right hand are zero by Corollary 3.2.10 and relation (88). If $i = \ell$, the equality becomes $[h_\ell, q_{\ell,j}] = [k_\ell, q_{\ell,j}] + q_{\ell,j}[p_{\ell,j}, q_{\ell,j}] = 0$. If $j = \ell$, we obtain $[h_\ell, q_{i,\ell}] = [k_\ell, q_{i,\ell}] - q_{i,\ell}[p_{i,\ell}, q_{i,\ell}] = 0$. The calculations are similar for $p_{i,j}$. It is trivial that $[(p_{i,j}q_{i,j}), (p_{i',j'}q_{i',j'})] = 0$ for any $1 \leq i < j \leq n$ and $1 \leq i' < j' \leq n$. Then $[h_\ell, h_m] = 0$ follows from the commutativity of the Cartan subalgebra and observation (91). By Proposition 3.2.11, the generators $y_1, \dots, y_n$ commute with all $p_{i,j}$'s and $q_{i,j}$'s; then the last identity of the lemma is clear. $\square$

3.4.4. *Proposition.* Let W be the subalgebra of F generated over $\mathbb{C}$ by the $n(n-1)$ elements p_{ij} and q_{ij} . Let P be the subalgebra of H generated over $\mathbb{C}$ by $y_1, \dots, y_n$ and $h_1, \dots, h_n$. Then $H \simeq \text{Frac}(W \otimes P)$.

Proof. It is clear by (86), (92) and Proposition 3.2.11 that the subfield of H generated over F by $h_1, \dots, h_n$ is equal to H . Then the result follows from Lemma 3.4.3 and Proposition 3.2.11. $\square$

3.4.5. *Proposition.* Let us define $y'_1, \dots, y'_n$ in Q by:

$$y'_1 = y_1 \quad \text{and} \quad y'_m = y_1^{-1}y_2^{-1} \cdots y_{m-1}^{-1}y_m \quad \text{for any } 2 \leq m \leq n. \quad (93)$$

Let P' be the subalgebra of H generated over $\mathbb{C}$ by $y'_1, \dots, y'_n$ and $h_1, \dots, h_n$. Then $P' \simeq \tilde{U}_n(\mathbb{C})$ and $H \simeq \text{Frac}(W \otimes P')$.

Proof. It is clear by (93) that $\text{Frac } P' \simeq \text{Frac } P$ and $H \simeq \text{Frac } (W \otimes P')$. By tedious but simple calculations that we leave to the reader, we have:

$$y'_\ell y'_m = -y'_m y'_\ell, \quad [h_\ell, y'_m] = 0 \quad \text{and} \quad [h_\ell, y'_\ell] = y'_\ell \quad (94)$$

for all $1 \leq \ell \neq m \leq n$. We conclude with point (ii) of 1.2.1. $\square$

3.4.6. Corollary. We have $H \simeq \text{Frac } (A_{n(n-1)/2}(\mathbb{C}) \otimes \tilde{A}_n(\mathbb{C}))$, with center $\mathbb{C}$.

Proof. We have already observed in Corollary 3.2.12 that $W \simeq A_{n(n-1)/2}(\mathbb{C})$ and in Remark 1.2.4 that $\text{Frac } \tilde{A}_n(\mathbb{C}) \simeq \text{Frac } \tilde{U}_n(\mathbb{C})$. Then the isomorphism $H \simeq \text{Frac } (A_{n(n-1)/2}(\mathbb{C}) \otimes \tilde{A}_n(\mathbb{C}))$ follows from previous proposition. Using the usual notation $D_m(E)$ for the skewfield of fractions of a Weyl algebra $A_m(E)$ over a skewfield E , we have $H \simeq D_{n(n-1)}(E)$ for $E = \text{Frac } \tilde{A}_n(\mathbb{C})$. Its center satisfies $\mathcal{Z}(H) \simeq \mathcal{Z}(D_{n(n-1)}(E)) = \mathcal{Z}(E)$, and $\mathcal{Z}(E) = \mathbb{C}$ by Proposition 3.3.1 of [18]. $\square$

3.4.7. Conclusion. The second assertion of the main theorem is proved.

3.5. Enveloping skewfield of the Lie subalgebra $\mathfrak{b}_0^+$.

3.5.1. Notations. In order to determine the skewfield of fractions of the enveloping algebra of the even part $\mathfrak{b}_0^+$ of $\mathfrak{b}^+$ described in (50), we denote:

$$H_0 := \text{Frac } \mathcal{U}(\mathfrak{b}_0^+) = F_0(k_1; \Delta_1) \cdots (k_n; \Delta_n), \quad (95)$$

where F_0 is described by Proposition 3.3.3.

3.5.2. Proposition. We have $H_0 \simeq \text{Frac } (W \otimes P'_0)$, where P'_0 is the subalgebra of P' generated by $(y'_1)^2, \dots, (y'_n)^2, h_1, \dots, h_n$, which is isomorphic to the enveloping algebra $U_n(\mathbb{C})$ defined in point (i) of 1.2.1.

Proof. By Proposition 3.3.3 and (92), the subfield of H_0 generated over F_0 by $h_1, \dots, h_n$ is equal to H_0 . Since $\text{Frac } P'_0 = Q_0$ by (93), the result follows from Proposition 3.3.3. $\square$

3.5.3. Conclusion. Because of the isomorphism $\text{Frac } U_n(\mathbb{C}) \simeq \text{Frac } A_n(\mathbb{C})$ observed in Remark 1.2.4 and the canonical isomorphism $A_{n(n+1)/2}(\mathbb{C}) \simeq A_{n(n-1)/2}(\mathbb{C}) \otimes A_n(\mathbb{C})$, the previous proposition proves that:

$$\text{Frac } \mathcal{U}(\mathfrak{b}_0^+) \simeq \text{Frac } A_{n(n+1)/2}(\mathbb{C}).$$

Then we retrieve in the context of classical Gelfand Kirillov property the usual description of the enveloping skewfield of the solvable Borel subalgebra of the Lie algebra $\mathfrak{sp}(2n)$ as the Weyl skewfield $D_{n(n+1)/2}(\mathbb{C})$.

3.6. Final observation. A thorough reading of the previous proofs shows that the isomorphisms we obtain at the level of the skewfields of fractions are already realized at the level of the localization of the algebras under consideration by the multiplicative set generated by the n elements $y_1, \dots, y_n$. Since these elements are homogeneous, this result is consistent with the notion of skewfield of fractions in the category of superalgebras.

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