Students’ ability to compare fractions related to proficiency in the four operations

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In this paper, we investigate the relationship between 99 fourth-grade students' ability to compare fractions ($\frac{1}{4} = \frac{2}{8}$ and $\frac{5}{11} < \frac{3}{5}$) and solve tasks in each of the four operations (68 + 753, 547 − 64, 12 × 72 78 ÷ 3) in a Danish primary school. The students faced more challenges answering the question where the fractions were equal compared to the non-equal fraction items (32% vs 49% correct answers) although the probability of solving the two types of questions were significantly positively associated (odds ratio = 11.0). Relative to the four operations items, solving the non-equal fraction item associated significantly positively with solving the multiplication (odds ratio = 4.5) and division item (odds ratio = 3.9) while the correct solving of the equal fraction associated significantly positively with the solving of the division item (odds ratio = 2.6). These results support the notion that the understanding of fractions is closely connected to multiplicative reasoning.

Keywords: Arithmetic, whole number division, fractions.

Introduction

Research has shown that students' knowledge of fractions in middle school greatly affects their overall progression in general mathematics (e.g., Bailey, Siegler, & Geary, 2014). There is also evidence that students need to have mathematical proficiency in the four operations of whole numbers to enable them to effectively comprehend rational numbers (Behr & Post, 1988). Additionally, multiplicative reasoning seems especially connected to the development of students' understanding of fractions (Thompson & Saldanha, 2003). Therefore, it is relevant to further investigate the variety of associations between the students' proficiency in the four operations and in fractions. This paper aims to uncover the fourth grade students' abilities to answer equations that require an understanding of fractions as well as the relationship between their abilities to answer tasks and their understanding of the four operations. The primary research question for this paper is: How do students' abilities to solve arithmetic tasks in the four operations (e.g., division) relate to their abilities to answer items that require them to compare fractions?

We expect that the students' proficiency in division and multiplication, but not their capability related to addition and subtraction, will show a strong association to their ability to answer fraction equations because we assume that an understanding of fractions is closely connected to multiplicative reasoning. This paper primarily focuses on students' difficulties with comparing fractions and how this is associated with the four operations. Therefore, the theoretical framework is based on theories of fractions and how these are connected to, for example, multiplicative reasoning.

Theoretical framework

Over the last three decades, several researchers, such as Bailey, Siegler and Geary (2014), Behr and Post (1988), and Ni (2001), have studied aspects of students' understanding of fractions and the complexities related to teaching and learning these concepts. For example, Kieren (1976) described how fractions can be seen as a multifaceted construct. He characterized the understanding and knowledge of rational numbers as sets of sub-constructs, which have since been further developed by Behr, Lesh, Post, and Silver (1983). They defined these sub-constructs as: Part-whole, decimal, ratio, quotient, operator, and measure. They recommended that the sub-construct part-whole should be considered to be distinct from the others as a fundamental scheme. Although we are aware of the limitation, in this paper we choose to focus on the sub-construct of quotient even though rational numbers are a more complex system of sub-constructs, and we are aware of for example the importance of ratio in the proportional understanding of fraction (Behr, Lesh, Post, & Silver, 1983; Behr & Post, 1988).

Quotient

For the sub-construct quotient, the denominator designates the amount of recipients, and the numerator represents the amount of quantities that have to be shared (Behr & Post, 1988; Toluk & Middleton, 2001). Thus, the sub-construct of quotient has a connection to partitive division, also known as sharing division, where an object or a group of objects are divided into a number of equal parts (Fischbein, Deri, Nello, & Marino, 1985). Moreover, division is the only one of the four operations where a rational number can be the outcome. Therefore, division can be regarded as an integral part of an understanding of rational numbers; hence students must develop an understanding of the connections between division and fractions (Behr & Post, 1988; Toluk & Middleton, 2001). Toluk and Middleton (2001) conducted a case study of students' development of fraction schemes and the importance of division in the progression from the part-whole sub-construct to the conceptualisation of the quotient sub-construct. They illustrated the developmental process from Fraction-as-Part-Whole and Whole Number Quotient Division-as-Operation to the final stage, which was Division-as-Number ($a \div b = a/b, \forall\ a/b$) (Toluk & Middleton, 2001). When comparing fractions, it is necessary to understand that the magnitude depends on the relation between the two quantities; the denominator and the nominator (Ni & Zhou, 2005), rather than simply viewing the fraction notation as two independent cardinal numbers above and below a bar (Stafylidou & Vosniadou, 2004).

Equivalent fraction

Overall, there is substantial support that students' whole number multiplicative understanding is an important resource for developing fraction knowledge (Hackenberg & Tillema, 2009). In addition, multiplicative reasoning appears to be essential for the effective understanding and accurate representation of equivalent fractions. One basic fraction perception is to understand fraction equivalence (Ni, 2001). Fraction equivalence can be explained as the constancy of a quotient and as the constancy of the multiplicative relationship between the numerator and the denominator (Behr, Harel, Post, & Lesh, 1992). Here, the quotient is involved in an understanding of equivalence; for example, $\frac{1}{2}$ can be viewed as a division, which represents the equivalence between $\frac{1}{2}$ and 0,5 (Behr, Lesh, Post, & Silver, 1983). Each fraction belongs to a unique equivalence class represented by a multiplicative equation. Each class denotes a distinct rational number; for example $\frac{1}{2} = \frac{2}{4}$. Difficulties related to the understanding of equivalent fractions have been connected to the students' lack of multiplicative reasoning (Ni, 2001). If the students do not develop this understanding of equivalence, they will view for example tasks that involve adding two fractions with different denominators as a purely technical algorithm (Arnon, Nesher, & Nirenburg, 2001). Therefore, both multiplication and division of whole numbers might be more closely connected to understanding fractions than additional reasoning. Hence, we expect to find an association between the fraction equations and the multiplication and division items, meaning that if the students can solve items involving the two operations correctly, they will more likely be able to solve the fraction equations. However, if the students have not developed a proficiency with multiplication and division, we expect that those students will experience difficulties answering the fraction equations correctly, especially the equivalent fraction equations. We will analyse how the two fraction items differ in proportion of correct answers as well as whether and how they differ in association with the four operations.

Methods and analysis

Data consisted of answers provided by fourth-grade students (ages 10–11) on a computerised test. The test consisted of 110 items, including problems on the four operations, fractions, and algebra as well as word problems. The test was time restricted (45 minutes), so not all students finished all of the items. The questions on fractions included in the analysis were items 50 and 52, and only the students who answered these items within the timeframe were included in the analysis. However, we are aware of the constraints of interpreting the students' answers on a computerised test in which the student just enters the answer to the item. This only allows us to review a student's correct or incorrect answers and not the working process itself. In addition, we are aware that neither a whole number multiplication nor a division item can be seen as a strong indicator of student's multiplicative reasoning. A quick review of these answers showed that the students who answered this section were generally quick at calculating and, on average, had the highest scores in the first section of the test (whole number arithmetic operation tasks). In other words, the students who reached the fraction items (items 50 and 52) scored above average on the test (N = 99). The items we chose to analyse were items where the students were asked to compare $\frac{1}{4}$ to $\frac{1}{5}$, $\frac{5}{11}$ to $\frac{3}{5}$, and $\frac{1}{4}$ to $\frac{2}{8}$. The students could choose between the following three symbols: <, >, and = (Figure 1).

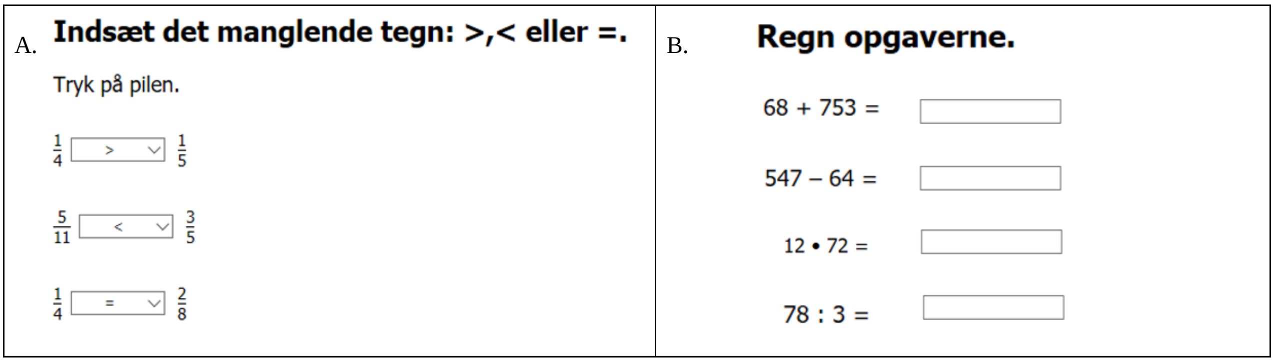

Figure 1: The items used in the analysis. Translated to: A. Choose the right symbol: >, < or =. (No symbol was preselected). B. Calculate the task

Earlier in the test, the students were asked to solve a whole number equation using the same symbols. There were not any significant differences found in the items containing the equal sign. The first item was created to see the students' answers on a unit fraction equation. Often, students have whole number biases in their learning processes of fractions such that they look at the biggest digit and conclude that this is the largest number; for example 5 is larger than 4 (Ni & Zhou, 2005). The second item, $\frac{5}{11}$ compared to $\frac{3}{5}$, was likewise designed to examine the students' answers on an item where there is a whole number bias since both 5 and 11 are bigger than 3 and 5. In the solving process of this item, the student might use a different strategy, such as the benchmark strategy where $\frac{5}{11}$ is smaller than a half and $\frac{3}{5}$ is bigger than a half (Clarke & Roche, 2009). We considered this item to be the most difficult item for the students to solve because of the different numerator and denominator as well as the fact that the item did not contain any unit fraction. The last item was the equal fraction item where $\frac{1}{4}$ is equal to $\frac{2}{8}$. This item included the unit fraction $\frac{1}{4}$, which we considered to be a commonly used fraction in the fourth-grade curriculum. Therefore, we assumed that the students might have a god mental representation of this fraction. As for the analysis of the relationship between the four operations, we have selected one whole-number problem from each of the operations ($68 + 753$, $547 - 64$, 12×72, $78 \div 3$) based on the following criteria: The item where the students had the fewest correct answers, the most wrong answers, and the most times it was not answered, in that order. Each of the arithmetic items was designed to evoke solution strategies based on number sense approaches. For example, in the case of item 12×72, it involves breaking up one of the factors 12 into a 10 and a 2 and then multiply it in parts $(10 \times 72) + (2 \times 72)$.

Analysis

The analysis focused on how the students' answers to one fraction equation associated to the other fraction equation and to answers on the four operations items. The association between the probability of returning a correct answer as opposed to an incorrect one or omitting an answer for one item in relation to whether the student answered another item correctly was analysed as a binary logistic regression (Logistic procedure in SAS 9.5) function. Logistic regression models yield similar *p*-values for statistical significance of associations as conventional 2 x 2 contingency $\chi 2$-tests as well as additionally producing predictions of the conditional probabilities of returning a correct answer for one item in relation to whether the student had answered the other item correctly. The difference between these two predicted probabilities on a logit scale (coefficient of the difference between the two levels in the regression equation) furthermore serves as a coefficient of association in the response patterns between the two items.

The anti-log of this coefficient (the odds ratio 'OR') equals how many more times a student is likely to answer item Y (e.g., $\frac{1}{4} = \frac{2}{8}$) correctly if he/she has answered item X (e.g., $78 \div 3 =$) correctly as compared to if he/she has not answered X correctly. Hence, an association coefficient of 1.0 indicates that the students answering item X correctly are exp. (1.0) = 2.71 times more likely to answer item Y correctly as compared to the students who did not answer item X correctly. Note that the ORs derived from the logistic regression equation exceeds the ORs calculated from the arithmetic values of the probabilities because of the infinite state space on the logit scale, whereas values are on the arithmetic scale $\in [0;1]$. The conditional probabilities of getting correct answers as a function of correct vs. incorrect or missing answers in other item types were modelled and illustrated graphically for the item $\frac{1}{4} = \frac{2}{8}$ as a function of the item $\frac{5}{11} < \frac{3}{5}$ and $\frac{1}{4} > \frac{1}{5}$. The same was modelled and illustrated for the item $\frac{5}{11} < \frac{3}{5}$ as a function of item $\frac{1}{4} > \frac{1}{5}$. A matrix table of association coefficients between items that represent the different concepts of fractions, addition, subtraction, multiplication, and division was calculated for each of the items.

Results

When analysing data from the test, it was found that the students showed difficulties solving items involving equivalent fractions. Only 32% of the students answered $\frac{1}{4} = \frac{2}{8}$ correctly while 49% answered $\frac{5}{11} < \frac{3}{5}$ correctly (Figure 2). Overall, the two non-equal fraction equations were similar in their answer patterns since both had about 50% correct, about 30% incorrect, and 20% with no answer.

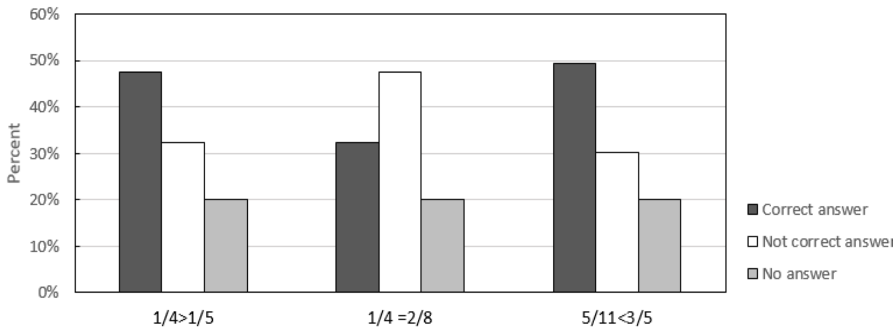

Figure 2: Percent of correct, incorrect or no answer to the three fraction items (*N = 99)*

When comparing, the probability of answering the three items representing equation problems correctly were all strongly and highly statistically significantly associated (Figure 3, association coefficient values: 2.4–4.0, i.e. ORs: 11–54).

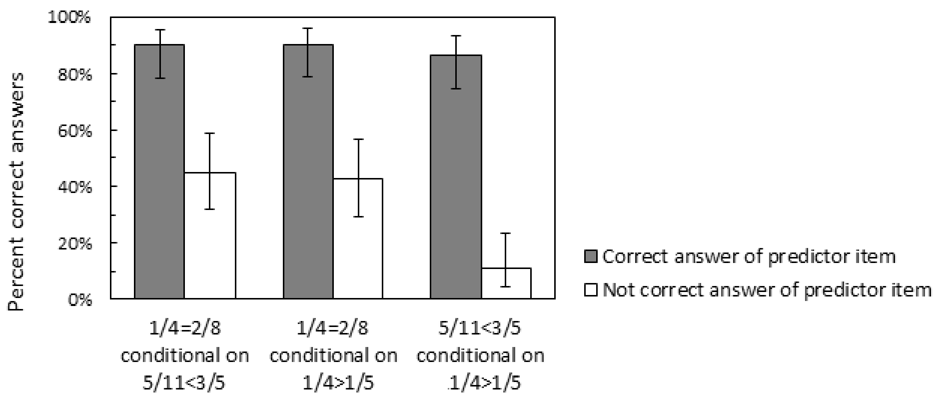

Figure 3: Percent of correct answers on one item type conditional to whether the student answered another item type. Error bars indicate 95% confidence intervals. All differences were highly significant (p < 0.0001)

The two non-equal fraction items showed the same pattern when compared to the equal fraction. Hence, in all comparisons, the probability of answering correctly on the item $\frac{1}{4} = \frac{2}{8}$ was about 90% if the student had answered the predictor item $\frac{5}{11} < \frac{3}{5}$ correctly compared to about 50% if they had answered it incorrectly (Figure3). When comparing the two non-equal fraction items, we found that if the students had answered the predictor item $\frac{1}{4} > \frac{1}{5}$ correctly, only about 10% could then not solve $\frac{5}{11} < \frac{3}{5}$. Therefore, we chose to focus on the equal fraction item $\frac{1}{4} = \frac{2}{8}$ and the non-equal fraction item $\frac{5}{11} < \frac{3}{5}$. When comparing all problem types (Table 1), correct solutions were most strongly associated with the two equation problems (OR = 11.0, $p < 0.0001$), followed by the association between the addition and subtraction (OR = 6.8, $p < 0.0001$), multiplication and division (OR = 5.4, $p < 0.0001$), and subtraction and multiplication (OR = 5.0, $p = 0.001$) items, respectively. The results of the equation item $\frac{5}{11} < \frac{3}{5}$ were highly significantly positively associated with the results of the multiplication (OR = 4.5, $p = 0.0009$) and division items (OR = 3.9, $p = 0.003$). The results of the equation item $\frac{1}{4} = \frac{2}{8}$ only associated positively with the division item (OR = 2.6 $p = 0.02$) (Table 1). The lowest associations (all non-significant) were found between the results of two equation items and the results of addition and subtraction items (all four ORs 1.7–2.4), between equation item $\frac{1}{4} = \frac{2}{8}$, and the multiplication item (OR = 2.1) (Table 1).

	1/4 = 2/8		5/11<3/5		68+753		547–64		12×72		78÷3	
5/11<3/5	2.40	****										
68+753	0.85		0.53									
547–64	0.75		0.88		1.91	****						
12×72	0.72		1.51	***	0.83		1.60	***				
78÷3	0.96	*	1.36	**	1.02	*	0.68		1.68	****		
N items:	*99*		*99*		*142*		*142*		*142*		*142*	
N correct items:	*50*	*51%*	*67*	*68%*	*32*	*23%*	*49*	*35%*	*98*	*69%*	*87*	*61%*

Table 1: Coefficients of association (log odds ratios) of correct vs. incorrect/missing answers of different problem items: The higher the coefficient value, the more likely a correct answer of one item type associated with a correct answer of the other item type. Significance levels: * p < 0.05, ** p < 0.01, *** p < 0.001, **** p < 0.0001. (A log odds ratio of 1 equals a difference of exp (1) = 2.7)

Discussion

This statistical investigation of the two fraction items' differences and their associations to the four operations revealed the following three results: Firstly, when comparing the three items with each other, the students' answers showed that the equal fraction item (30% correct answers) was more difficult than the two non-equal fraction items (50% correct). This was unexpected since one of the fractions in the equal item was a commonly referenced unit fraction of $\frac{1}{4}$. It is a fraction which is common in the introduction of fractions in the fourth grade, whereas $\frac{5}{11}$ is a rare fraction notation in school curriculums. As referred to earlier, we therefore expected the item to be easier than the other one. The students' difficulties with the equal item could indicate that the students have difficulties understanding fractions as an *equivalence class* made by a multiplicative equation (Ni, 2001). This is emphasized by the results in Figure 3, which show that the probability of answering correctly on the item $\frac{1}{4} = \frac{2}{8}$ was still only 50% if the student had answered $\frac{5}{11} < \frac{3}{5}$ incorrectly. However, a probability of only 10% was found for answering $\frac{5}{11} < \frac{3}{5}$ correctly if they had answered $\frac{1}{4} > \frac{1}{5}$ incorrectly, which indicates that the comparison of equal fraction items differed. This could be due to the fact that when solving the equivalent fraction items, the students have a whole number bias (Ni & Zhou, 2005), the equivalence can be difficult for the students to comprehend because it is the first time they have experienced numbers that can be labelled differently and still refer to the same numerical quantity. Secondly, we found that the result of the non-equal $\frac{5}{11} < \frac{3}{5}$ fraction was highly associated with the result for both the multiplication (OR = 4.3) and the division (OR = 3.9) items, whereas there was no significant association with the results of the addition and subtraction items. This result coheres well with the theory that fractions have a stronger connection to multiplicative reasoning than to additive reasoning (Thompson & Saldanha, 2003). According to this theory, one should have expected a closer association with division than with multiplication (Toluk & Middleton, 2001). A possible explanation might be that the result patterns of the two operations were also closely related (OR = 5.4), suggesting that the basis of the understanding necessary for solving division and multiplication problems is so closely related that they could not be discriminated in the analysis.

Thirdly, we found that the results of the non-equal item correlated more strongly and significantly with the results of the division and multiplication items than was the case for the result of the equal fraction item (that only associated modestly with the result of the division item). Thus, there is a component of the equal fraction item making this more difficult for the students to answer. An explanation could be that students have not yet developed the conceptualisation of the *quotient* as *Division-as-Number* (Toluk & Middleton, 2001). Hence, when students look at equal fractions, they can see a *quotient* as a division, which represents the equivalence between both $\frac{1}{4}$, $\frac{2}{8}$ and 0.25 (Behr, Lesh, Post, & Silver, 1983). Overall, these results support the notion that the understanding of fractions is closely connected to multiplicative reasoning; however, it is essential to pursue further investigation since there seems to be a different pattern in the students' proficiency in the equivalence concept. Otherwise, they will experience difficulties in understanding other concepts, like common denominators (Arnon, Nesher, & Nirenburg, 2001). Further research should collect qualitative data to examine the students' working process to overcome the strong limitations of a computer test, which can only offer limited insights. We need to investigate further ways in which the students experience difficulties in the equal fraction task. It may be that this is strongly connected to their understanding of the equal sign or that there is a connection to a whole number bias. On the one hand, the students' whole number multiplicative understanding might support their understanding in some fraction contexts, and on the other hand, the students' understanding of whole numbers can distract them in other fraction contexts.

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