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Abstract

We prove a monotonicity property of the Hurwitz zeta function which, in turn, translates into a chain of inequalities for polygamma functions of different orders. We provide a probabilistic interpretation of our result by exploiting a connection between Hurwitz zeta function and the cumulants of the exponential-beta distribution.

1 Main result

Let $\zeta(x,s) = \sum_{k=0}^{+\infty}(k+s)^{-x}$ be the Hurwitz zeta function (Berndt, 1972) defined for $(x,s) \in (1,+\infty)\times(0,+\infty)$, and, for any $a>0$ and $b>0$, consider the function

$$x \mapsto f(x,a,b) = \left(\zeta(x,b) - \zeta(x,a+b)\right)^{\frac{1}{x}}, \tag{1}$$

defined on $[1,+\infty)$, where $f(1,a,b)$ is defined by continuity as

$$f(1,a,b) = \sum_{k=0}^{\infty}\left(\frac{1}{k+b} - \frac{1}{k+a+b}\right) = \sum_{k=0}^{\infty}\frac{a}{(k+b)(k+a+b)}. \tag{2}$$

The function $f(x,a,b)$ can be alternatively written, with a geometric flavour, as

$$f(x,a,b) = \left(\|\boldsymbol{v}_{a+b}\|_x^x - \|\boldsymbol{v}_b\|_x^x\right)^{\frac{1}{x}},$$

where, for any $s > 0$, $\boldsymbol{v}_s$ is an infinite-dimensional vector whose k^{th} component coincides with $(k-1+s)^{-1}$.

The main result of the paper establishes that the function $x \mapsto f(x,a,b)$ is monotone on $[1,+\infty)$ with variations only determined by the value of a. More specifically,

Theorem 1. *For any $b > 0$, the function $x \mapsto f(x,a,b)$ defined on $[1,+\infty)$ is increasing*[1] *if $0 < a < 1$, decreasing if $a > 1$, and constantly equal to $\frac{1}{b}$ if $a = 1$.*

Theorem 1 and the derived inequalities in terms of polygamma functions (see Equation (5)) add to the current body of literature about inequalities and monotonicity properties of the Hurwitz zeta function (Berndt, 1972; Srivastava et al., 2011; Leping and Mingzhe, 2013) and polygamma functions (Alzer, 1998, 2001; Batir, 2005; Qi et al., 2010; Guo et al., 2015), respectively. The last part of the statement of Theorem 1 is immediately verified as, when $a = 1$, $f(x,a,b)$ simplifies to a telescoping series which gives $f(x,1,b) = \frac{1}{b}$ for every $x \in [1,+\infty)$. The rest of the proof is presented in Section 2 while Section 3 is dedicated to an application of Theorem 1 to the study of the so-called exponential-beta distribution (Gupta and Kundu, 1999; Nadarajah and Kotz, 2006), obtained by applying a log-transformation to a beta distributed random variable. More specifically, the dichotomy observed in Theorem 1, determined by the position of a with respect to 1, is shown to hold for the exponential-beta distribution at the level of (i) its cumulants (whether function (4) is increasing or not), (ii) its dispersion (Corollary 2), (iii) the shape of its density (log-convex or log-concave, Proposition 1 and Figure 1) and (iv) its hazard function (increasing or decreasing, Proposition 2).

2 Proof of Theorem 1

The proof of Theorem 1 relies on Lemma 1, stated below. Lemma 1 considers two sequences and establishes the monotonicity of a third one, function of the first two, whose direction depends on how the two original sequences compare with each other. The same dichotomy, in Theorem 1, is driven by the position of the real number a with respect to 1.

2.1 A technical lemma

Lemma 1. *Let $(s_n)_{n\geq 1}$ and $(r_n)_{n\geq 1}$ be two sequences in $(0,1)$ and define, for $N \geq 1$,*

$$u_N \overset{\text{def}}{=} \left(1+\sum_{n=1}^{N}(s_n - r_n)\right)\ln\left(1+\sum_{n=1}^{N}(s_n - r_n)\right) - \sum_{n=1}^{N}(s_n \ln s_n - r_n \ln r_n).$$

[1]Throughout the paper, we say that a function f is increasing (resp. decreasing) if $x < y$ implies $f(x) < f(y)$ (resp. $f(x) > f(y)$), and that a quantity A is positive (resp. negative) if $A > 0$ (resp. $A < 0$).

We define by convention $u_0 = 0$. Then two cases are considered:

1. *if, for any $n \geq 1$, $r_n \leq s_n$ then, for all $N \geq 0$, we have $u_{N+1} \geq u_N$, with the equality holding if and only if $s_{N+1} = r_{N+1}$;*

2. *if, for any $n \geq 1$, $s_{n+1} \leq r_{n+1} \leq s_n \leq r_n$ then, for all $N \geq 0$, we have $u_{N+1} \leq u_N$, with the equality holding if and only if $s_{N+1} = r_{N+1}$.*

Moreover, if $\sum_{n=1}^{\infty} |s_n - r_n| < \infty$ (implying absolute convergence of the series $\sum_{n=1}^{\infty}(s_n \ln s_n - r_n \ln r_n)$) then

$$u_\infty \stackrel{\text{def}}{=} \lim_{N\to+\infty} u_N = \left(1 + \sum_{n=1}^{\infty}(s_n - r_n)\right) \ln \left(1 + \sum_{n=1}^{\infty}(s_n - r_n)\right) - \sum_{n=1}^{\infty}(s_n \ln s_n - r_n \ln r_n)$$

*exists and satisfies $u_\infty \geq 0$ in case **1**, while $u_\infty \leq 0$ in case **2**. In both cases, $u_\infty = 0$ if and only if the two sequences $(r_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ equal each other.*

Remark 1. *Note that, in case **2**, we have*

$$1 + \sum_{n=1}^{N}(s_n - r_n) = (1 - r_1) + s_N + \sum_{n=1}^{N-1}(s_n - r_{n+1}) \geq (1 - r_1) + s_N > 0,$$

so that all quantities defined in the lemma make sense. The absolute convergence of $\sum_{n=1}^{\infty}(s_n \ln s_n - r_n \ln r_n)$, stated in Lemma 1, follows directly from the trivial inequalities

$$0 \leq s \ln s - r \ln r \leq s - r \,,\ \forall\, 0 < r \leq s < 1.$$

Proof of Lemma 1

Proof for $N = 0$. We first study the case $N = 0$ and define

$$h_{r_1}(s_1) = (1 + s_1 - r_1)\ln(1 + s_1 - r_1) - s_1 \ln s_1 + r_1 \ln r_1.$$

For $s_1 = r_1$ we trivially have $h_{r_1}(r_1) = 0$. A straightforward computation shows that

$$h'_{r_1}(s_1) = \ln(1 + s_1 - r_1) - \ln s_1 = \ln((1 - r_1) + s_1) - \ln s_1 > 0,$$

since $r_1 < 1$. Hence h_{r_1} is an increasing function on $(0,1)$. Since $h_{r_1}(r_1) = 0$, we immediately get that h_{r_1} is positive on $(r_1, 1)$ and negative on $(0, r_1)$, thus proving both cases for $N = 0$.

Proof for $N \geq 1$. We now consider the case $N \geq 1$ and define

$$\begin{aligned} h_{r_1,\dots,r_{N+1},s_1,\dots,s_N}(s_{N+1}) &= u_{N+1} - u_N \\ &= \left(1 + \sum_{n=1}^{N+1}(s_n - r_n)\right)\ln\left(1 + \sum_{n=1}^{N+1}(s_n - r_n)\right) \\ &\quad - \left(1 + \sum_{n=1}^{N}(s_n - r_n)\right)\ln\left(1 + \sum_{n=1}^{N}(s_n - r_n)\right) \\ &\quad - s_{N+1}\ln s_{N+1} + r_{N+1}\ln r_{N+1}. \end{aligned}$$

We trivially get that $h_{r_1,\dots,r_{N+1},s_1,\dots,s_N}(r_{N+1}) = 0$. Moreover, we have

$$\begin{aligned} h'_{r_1,\dots,r_{N+1},s_1,\dots,s_N}(s_{N+1}) &= \ln\left(1 + \sum_{n=1}^{N+1}(s_n - r_n)\right) - \ln s_{N+1}, \\ &\overset{(1)}{=} \ln\left(s_{N+1} + (1 - r_{N+1}) + \sum_{n=1}^{N}(s_n - r_n)\right) - \ln s_{N+1}, \\ &\overset{(2)}{=} \ln\left(s_{N+1} + (1 - r_1) + \sum_{n=1}^{N}(s_n - r_{n+1})\right) - \ln s_{N+1}. \end{aligned}$$

Equality (1) shows that $h'_{r_1,\dots,r_{N+1},s_1,\dots,s_N}$ is positive on $(r_{N+1},1)$ for conditions of case **1**, while equality (2) shows that $h'_{r_1,\dots,r_{N+1},s_1,\dots,s_N}$ is positive on $(0,r_{N+1})$ for conditions of case **2**. Since $h_{r_1,\dots,r_{N+1},s_1,\dots,s_N}(r_{N+1}) = 0$ we get that $h_{r_1,\dots,r_{N+1},s_1,\dots,s_N}$ is positive on $(r_{N+1},1)$ for conditions of case **1**, while $h_{r_1,\dots,r_{N+1},s_1,\dots,s_N}$ is negative on $(0,r_{N+1})$ for conditions of case **2**. This ends the proof of monotonicity of $(u_N)_{N\geq 0}$ and conditions for strict monotonicity in both cases. Extending results from finite N to $N \to \infty$ follows directly from these results and Remark 1.

2.2 Proof of Theorem 1

We want to study the variations of

$$x \mapsto f(x,a,b) = (\zeta(x,b) - \zeta(x,a+b))^{\frac{1}{x}}$$

on $[1,\infty)$, for which it is enough, by continuity, to focus on $(1,\infty)$. Since f is positive, its variations are equivalent to those of

$$\begin{aligned} F(x,a,b) &\overset{\text{def}}{=} \ln f(x,a,b) \\ &= \frac{1}{x}\ln(\zeta(x,b) - \zeta(x,a+b)) = \frac{1}{x}\ln\left(\sum_{k=0}^{\infty}\frac{1}{(k+b)^x} - \frac{1}{(k+a+b)^x}\right) \end{aligned}$$

$$= -\ln b + \frac{1}{x}\ln\left(\sum_{k=0}^{\infty}\left(\frac{b}{k+b}\right)^x - \left(\frac{b}{k+a+b}\right)^x\right).$$

A straightforward computation shows that

$$\partial_x F(x,a,b) = \frac{H(x,a,b)}{x^2\left(\sum_{k=0}^{\infty}\left(\frac{b}{k+b}\right)^x - \left(\frac{b}{k+a+b}\right)^x\right)},$$

hence the sign of the derivative $\partial_x F(x,a,b)$ is the same as that of $H(x,a,b)$ defined by

$$\begin{aligned}
H(x,a,b) &\stackrel{\text{def}}{=} \sum_{k=0}^{\infty}\left(\frac{b}{k+b}\right)^x \ln\left(\left(\frac{b}{k+b}\right)^x\right) - \left(\frac{b}{k+b+a}\right)^x \ln\left(\left(\frac{b}{k+b+a}\right)^x\right) \\
&- \left(\sum_{k=0}^{\infty}\left(\frac{b}{k+b}\right)^x - \left(\frac{b}{k+a+b}\right)^x\right)\ln\left(\sum_{k=0}^{\infty}\left(\frac{b}{k+b}\right)^x - \left(\frac{b}{k+a+b}\right)^x\right) \\
&= \sum_{k=1}^{\infty}\left(\frac{b}{k+b}\right)^x \ln\left(\left(\frac{b}{k+b}\right)^x\right) - \left(\frac{b}{k+b+a-1}\right)^x \ln\left(\left(\frac{b}{k+b+a-1}\right)^x\right) \\
&- \left(1+\sum_{k=1}^{\infty}\left(\frac{b}{k+b}\right)^x - \left(\frac{b}{k+a-1+b}\right)^x\right)\ln\left(1+\sum_{k=1}^{\infty}\left(\frac{b}{k+b}\right)^x - \left(\frac{b}{k+a-1+b}\right)^x\right),
\end{aligned}$$

which can be rewritten as

$$\sum_{n=1}^{\infty}(s_n \ln s_n - r_n \ln r_n) - \left(1+\sum_{n=1}^{\infty}(s_n - r_n)\right)\ln\left(1+\sum_{n=1}^{\infty}(s_n - r_n)\right),$$

where, for all $n \geq 1$, we have defined

$$s_n = \left(\frac{b}{n+b}\right)^x \quad \text{and} \quad r_n = \left(\frac{b}{n+a-1+b}\right)^x.$$

Note that, for any values of $a > 0$ and $b > 0$, we have $\sum_{n=1}^{\infty}|s_n - r_n| < \infty$ as $x > 1$ implies that $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$. Moreover, when $a > 1$, we have $0 < r_n < s_n < 1$, while when $0 < a < 1$, we have $0 < r_{n+1} < s_n < r_n < 1$ for all $n \geq 1$ and $x > 1$. We can then apply Lemma 1 to obtain that $H(x,a,b)$, and thus $\partial_x F(x,a,b)$, is negative if $a > 1$ (case **1** of Lemma 1) and is positive if $0 < a < 1$ (case **2** of Lemma 1), which concludes the proof.

3 Probabilistic interpretation: application to the exponential-beta distribution

The aim of this section is to identify a connection between Theorem 1 and the exponential-beta distribution. The latter arises by taking a log-transformation of a beta random variable. More specifically, let V be a beta random variable with parameters $a > 0$ and $b > 0$, then we say that X is an exponential-beta random variable with parameters a and b if $X = -\ln(1-V)$, and use the notation $X \sim \mathrm{EB}(a,b)$ (see Gupta and Kundu, 1999; Nadarajah and Kotz, 2006). The corresponding density function is given by

$$g(x;a,b) = \frac{1}{B(a,b)}(1-\mathrm{e}^{-x})^{a-1}\mathrm{e}^{-bx}\mathbf{1}_{(0,+\infty)}(x), \tag{3}$$

where $B(a,b)$ denotes the beta function[2]. The cumulant-generating function of X can be written as

$$K(t) \stackrel{\text{def}}{=} \ln \mathbb{E}(\exp(tX)) = \ln\Gamma(a+b) + \ln\Gamma(b-t) - \ln\Gamma(b) - \ln\Gamma(a+b-t),$$

provided that $t < b$ (see Section 3 of Nadarajah and Kotz, 2006). This implies that, for any $n \geq 1$, the n^{th} cumulant of X, denoted $\kappa_n(a,b)$, is given by

$$\kappa_n(a,b) = (-1)^n \left(\psi^{(n-1)}(b) - \psi^{(n-1)}(b+a)\right),$$

where $\psi^{(m)}$ for $m \in \mathbb{N}$ denotes the polygamma function of order m, which is defined as the derivative of order $m+1$ of the logarithm of the gamma function. An interesting relation across cumulants of different orders is then obtained as a straightforward application of Theorem 1. Before stating the result, and for the sake of compactness, we define on $\mathbb{N}\setminus\{0\}$, for any $a > 0$ and $b > 0$, the function

$$n \mapsto f_{\mathrm{EB}}(n,a,b) = \left(\frac{\kappa_n(a,b)}{(n-1)!}\right)^{\frac{1}{n}}. \tag{4}$$

Corollary 1. *For any $b > 0$, the function $n \mapsto f_{EB}(n,a,b)$ defined on $\mathbb{N}\setminus\{0\}$, is increasing if $0 < a < 1$, decreasing if $a > 1$, and constantly equal to $\frac{1}{b}$ if $a = 1$.*

Proof. The proof follows by observing that, when $n \in \mathbb{N}\setminus\{0\}$, $f_{\mathrm{EB}}(n,a,b) = f(n,a,b)$, with the latter defined in (1) and (2). This can be seen, when $n > 1$, by applying twice the identity $\psi^{(n-1)}(s) = (-1)^n(n-1)!\zeta(n,s)$, and, when $n = 1$, by applying twice the identity $\psi^{(0)}(s) = -\gamma + \sum_{k=0}^{\infty} \frac{s-1}{(k+1)(k+s)}$, where γ is the Euler-Mascheroni constant, which holds for any $s > -1$ (see Identity 6.3.16 in Abramowitz and Stegun, 1965). □

[2] The beta function is defined in this article as

$$B(a,b) = \int_0^{+\infty} (1-\mathrm{e}^{-x})^{a-1}\mathrm{e}^{-bx}\mathrm{d}x.$$

Corollary 1 can alternatively be expressed as a chain of inequalities in terms of polygamma functions of different orders, which might be of independent interest. Namely, for any $b > 0$ and any $0 < a_1 < 1 < a_2$, the following holds:

$$\begin{cases} \psi^{(0)}(b+a_1) - \psi^{(0)}(b) < \ldots < \left(\frac{\psi^{(n)}(b+a_1)-\psi^{(n)}(b)}{n!}\right)^{\frac{1}{n+1}} < \ldots < \frac{1}{b}, \\ \psi^{(0)}(b+a_2) - \psi^{(0)}(b) > \ldots > \left(\frac{\psi^{(n)}(b+a_2)-\psi^{(n)}(b)}{n!}\right)^{\frac{1}{n+1}} > \ldots > \frac{1}{b}. \end{cases} \tag{5}$$

Corollary 1, as well as (5), highlights the critical role played by the exponential distribution with mean $\frac{1}{b}$, special case of the exponential-beta distribution recovered from (3) by setting $a = 1$. In such special instance, the cumulants simplify to $\kappa_n(1,b) = b^{-n}(n-1)!$, which makes $f_{EB}(n,1,b) = \frac{1}{b}$ for every $n \in \mathbb{N} \setminus \{0\}$. Within the exponential-beta distribution, the case $a = 1$ then creates a dichotomy by identifying two subclasses of densities, namely $\{g(x;a,b) : 0 < a < 1\}$, whose cumulants $\kappa_n(a,b)$ make $f_{\mathrm{EB}}(n,a,b)$ an increasing function of n, and $\{g(x;a,b) : a > 1\}$ for which $f_{\mathrm{EB}}(n,a,b)$ a decreasing function of n. The left panel of Figure 1 displays the function $b \mapsto f_{\mathrm{EB}}(n,a,b)$ for some values of n and a: it can be appreciated that, for any b in the considered range, the order of the values taken by $f_{\mathrm{EB}}(n,a,b)$ is in agreement with Corollary 1.

The first two cumulants of a random variable X have a simple interpretation in terms of its first two moments, namely $\kappa_1 = \mathbb{E}[X]$ and $\kappa_2 = \mathrm{Var}[X]$. A special case of Corollary 1, focusing on the case $n \in \{1,2\}$, then provides an interesting result relating the dispersion of the exponential-beta distribution with its mean. Specifically,

Corollary 2. *For any $b > 0$, the exponential-beta random variable $X \sim EB(a,b)$ is characterized by over-dispersion $\left(\sqrt{\mathrm{Var}[X]} > \mathbb{E}[X]\right)$ if $0 < a < 1$, under-dispersion $\left(\sqrt{\mathrm{Var}[X]} < \mathbb{E}[X]\right)$ if $a > 1$, and equi-dispersion $\left(\sqrt{\mathrm{Var}[X]} = \mathbb{E}[X]\right)$ if $a = 1$.*

The behaviour of the cumulants is not the only distinctive feature characterizing the two subclasses of density functions corresponding to $0 < a < 1$ and $a > 1$. For any b, the value of a determines the shape of the density as displayed in the right panel of Figure 1 and summarized by the next proposition, whose proof is trivial and thus omitted.

Proposition 1. *For any $b > 0$, the exponential-beta density $g(x;a,b)$ is log-convex if $0 < a < 1$ and log-concave if $a > 1$.*

The same dichotomy within the exponential-beta distribution is further highlighted by the behaviour of the corresponding hazard function, defined for an absolutely continuous random variable X as the function $x \mapsto \frac{f_X(x)}{1-F_X(x)}$, where f_X and F_X are, respectively, the probability density function and the cumulative distribution function of X.

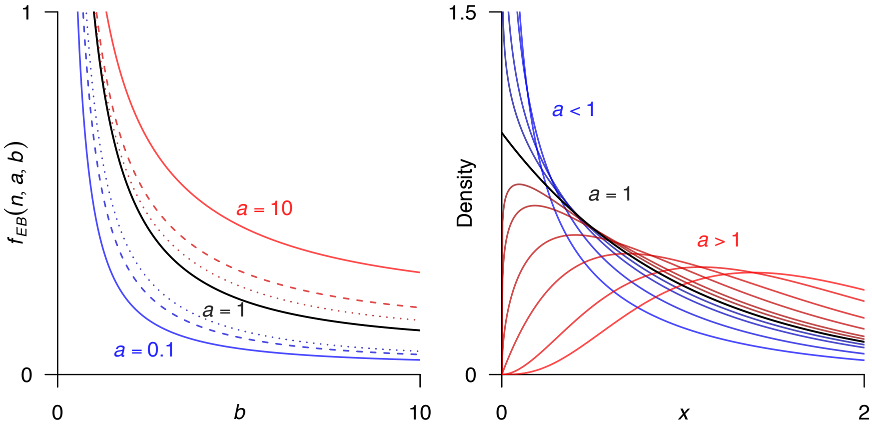

Figure 1: Left: illustration of Corollary 1 displaying curves $b \mapsto f_{\mathrm{EB}}(n,a,b)$, with a taking values 0.1 (blue curves), 1 (black curve) and 10 (red curves), and n taking values 1 (continuous curves), 2 (dashed curves) and 3 (dotted curves). Right: exponential-beta density function $g(x;a,1)$ for values of $a \in [0.4,4]$: densities are log-convex for $0 < a < 1$ (in blue), log-concave for $a > 1$ (in red), while $a = 1$ corresponds to the exponential distribution with mean 1 (in black).

Proposition 2. *For any $b > 0$, the hazard function of the exponential-beta distribution with parameters a and b is decreasing if $a < 1$, increasing if $a > 1$, and constantly equal to b if $a = 1$.*

Proof. The result follows from the log-convexity and log-concavity properties of $g(x;a,b)$ (see Barlow and Proschan, 1975).

□

Finally, it is worth remarking that an analogous dichotomy holds within the class of gamma density functions with $a > 0$ and $b > 0$ shape and rate parameters, and that once again the exponential distribution with mean $\frac{1}{b}$, special case recovered by setting $a = 1$, lays at the border between the two subclasses. The n^{th} cumulant of the gamma distribution is $\kappa_n = ab^{-n}(n-1)!$, which makes the function $n \mapsto \left(\frac{\kappa_n(a,b)}{(n-1)!}\right)^{\frac{1}{n}}$, defined on $\mathbb{N} \setminus \{0\}$, increasing if $0 < a < 1$, decreasing if $a > 1$ and constantly equal to $\frac{1}{b}$ if $a = 1$. Similarly, the gamma density is log-convex if $0 < a < 1$ and log-concave if $a > 1$ and, thus, the corresponding hazard function is decreasing if $a < 1$, increasing if $a > 1$ and constantly equal to b if $a = 1$.

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