DELSARTE'S EQUATION FOR CAPUTO'S OPERATORS

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ABSTRACT. Delsarte's equation is investigated for Caputo's differential operators. Solvability of the resulting fractional hyperbolic Cauchy problem is achieved in the sense of distributions. A regularity result shows that the solution may be a function of time. Rigorous Delsarte's representations are established. The symmetry between the fractional operators acting on space and time, induced by Delsarte's equation, opens the door to new type of fractional PDE's.

1. INTRODUCTION

Let B be a linear operator acting on a space X of complex functions of one independent variable. In [Del38], J. Delsarte introduced an interesting representation of the solution to the Cauchy problem

$$B_t u = B_x u, \quad u_{|t=0} = f \in X. \tag{1.1}$$

Here, $u : [0, \infty[\times[0, \infty[\to \mathbb{C}$, $(t, x) \mapsto u(t, x)$, and B_x denotes the operator acting on a function $u(t, \cdot) : x \mapsto u(t, x)$. The equation in (1.1) is called *Delsarte's equation* for the operator B.

In effect, let us assume that the operator B has a continuous point spectrum, namely that there exists $\phi : [0, \infty[\times\mathbb{C} \to \mathbb{C}$ such that

$$B\phi(\cdot, \lambda) = \lambda\phi(\cdot, \lambda), \quad \forall\lambda \in \mathbb{C}. \tag{1.2}$$

Also, assume that $\phi(0, \lambda) = 1$ for each $\lambda \in \mathbb{C}$ and $\phi(t, \cdot)$ is analytic on $\mathbb{C}$, for $t \geq 0$. Writing the series expansion of $\phi(t, \cdot)$ under the form

$$\phi(t, \lambda) = \sum_{k\geq 0} \phi_k(t)\lambda^k, \tag{1.3}$$

J. Delsarte gave the following (formal) representation of the solution u to (1.1)

$$u(t, x) = \sum_{k\geq 0} \phi_k(t)B_x^k f(x). \tag{1.4}$$

We refer to Subsection 4.3 for more details.

In the particular case where $B_t = \frac{\mathrm{d}}{\mathrm{d}t}$, Delsarte's equation is the *transport equation.* Moreover

$$\phi(t, \lambda) = e^{\lambda t} = \sum_{k\geq 0} \frac{t^k}{k!}\lambda^k, \quad \phi_k(t) = \frac{t^k}{k!},$$

and *Delsarte's representation* (1.4) takes the form

$$u(t, x) = \sum_{k\geq 0} \frac{t^k}{k!} f^{(k)}(x) = f(t + x).$$

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That explain why the operator

$$T_t : f \mapsto \sum_{k\geq 0} \phi_k(t) B_x^k f$$

is called a *generalized translation operator*. We have to mention that various operators can play the role of B in the formalism of Delsarte. For instance in [Che74], B is a second order differential operator; and in [Lev51], B is a self-adjoint operator on a Hilbert space. In these studies, the generalized translation operator is said to be *continuous*. *Discrete* versions have been developed by Löfström and Peetre [LP69]. There, the operator B have a countable set of eigenmodes.

In this paper, we consider the continuous generalized translation operator where $B_t := {}^{\mathrm{c}}\mathrm{D}_t^\alpha$ is the *Caputo operator*. Then one has

$$\phi(t,\lambda) = E_\alpha(\lambda t^\alpha) := \sum_{k\geq 0} \frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\lambda^k, \qquad \phi_k(t) = \frac{t^{\alpha k}}{\Gamma(\alpha k+1)},$$

and Delsarte's equation reads

$$ {}^{\mathrm{c}}\mathrm{D}_t^\alpha u = {}^{\mathrm{c}}\mathrm{D}_x^\alpha u, \qquad u_{|t=0} = f \in X. \tag{1.5}$$

The goal of that paper is to give an existence and uniqueness result for (1.5) (Theorem 3.2), and also a precise meaning to the formal represention (1.4) when $B = {}^{\mathrm{c}}\mathrm{D}^\alpha$. The latter issue is addressed in Subsection 4.3.

Let us emphasize that we consider Caputo's derivative as an operator acting on functions of the space variable. It seems that this approach is new. The symmetry we introduce between spacial and time fractional derivatives opens the door to new type of fractional PDE's otained by replacing ∂_{x_j} by ${}^{\mathrm{c}}\mathrm{D}_{x_j}^\alpha$. For instance, the one dimensional heat operator $\partial_t - \partial_{xx}$ is replaced by

$$\partial_t^\alpha - \left(\partial_x^\alpha\right)^2,$$

where $\partial_x^\alpha := {}^{\mathrm{c}}\mathrm{D}_x^\alpha$. That will be the subject of subsequent works.

We may find studies where the *negative* of some fractional derivative is considered as a spatial operator. In [JK02], spacial Caputo's operator is considered but the minus sign destroys the symmetry between the domain of spacial and time operators. In [Baz19], the symmetry is weaker since the spacial operator is the negative of a Riemann-Liouville derivative and the time operator is a Caputo's operator.

We shall call (1.5) the *fractional Delsarte equation*. From a spectral point of view, it is an *hyperbolic* equation. Indeed, let us consider the abstract Cauchy problem

$$ {}^{\mathrm{c}}\mathrm{D}_t^\alpha u = Au, \qquad u_{|t=0} = f \in X, \tag{1.6}$$

where $A : D(A) \subseteq X \to X$ is an unbounded linear operator. In the standard case (i.e. $\alpha = 1$), the critical angle for the resolvent of A is $\frac{\pi}{2}$. In the fractional case ($0 < \alpha < 1$), the critical angle for the resolvent is $\frac{\pi}{2}\alpha$. Roughly speaking, if the angle of the resolvent of A is larger than $\frac{\pi}{2}\alpha$, Equation (1.6) is called *parabolic*. If it is equal to $\frac{\pi}{2}\alpha$ then (1.6) is said to be *hyperbolic*. This is precisely the angle we get in our framework where $A := {}^{\mathrm{c}}\mathrm{D}_x^\alpha$.

Besides, let us emphasize that, for $0 < \alpha < 1$, the angle of the resolvent of $A := -\mathrm{i}\Delta$ is not critical. Hence, the equation

$$ {}^{\mathrm{c}}\mathrm{D}_t^\alpha u = -\mathrm{i}\Delta u$$

is not hyperbolic, but rather parabolic. If $X = L^2(\Omega)$ then it is dissipative and possesses a (limited) smoothing effect: see [ER18]. Therefore, this equation should not be called a *time fractional equation Schrödinger*: see [ER19].

Therefore, the methods of solving time fractional parabolic problems (resolvent familes, complex contour integrals) do not apply here. *Subordination principle* is not usefull aswell since ${}^{\mathrm{c}}\mathrm{D}^{\alpha}$ seems not to be the α^{th}-power of some operator generating a semi-group.

Our main tool is the *Laplace transform of vector valued functions.* Besides, due to the hyperbolic type of (1.5) and the "bad" resolvent estimate, we are led to consider time distributional solutions in the sense of Definition 3.1.

The outline is as follows. Caputo's operator is introduced in the forcoming section. The underlying phase space is $X = L^p(0,T)$, where T may be infinite, and p is large enough w.r.t. α. Focusing on the case $T=\infty$, we compute the resolvent set in Corollary 2.9 and give an optimal resolvent estimate in Proposition 2.7. In Section 3, Theorem 3.2 is devoted to the solvability of the fractional Delsarte equation (1.5). In Corollary 3.3, we give a L^p-regularity result in time. Finally, Section 4 is concerned with representations of the solutions, in particular to Delsarte's representation.

2. Caputo's operator on L^p spaces

2.1. **Definition and basic properties.** Let $T \in (0,\infty]$, $p \in [1,\infty)$, $\alpha \in (0,1)$, $\beta \in (0,\infty)$, and $X_T := L^p(0,T)$. Denoting by g_β the function of $L^1_{\mathrm{loc}}([0,\infty))$ defined for a.e. $t>0$ by

$$g_\beta(t) = \frac{1}{\Gamma(\beta)} t^{\beta-1}, \tag{2.1}$$

the domain of the Caputo operator is a follows.

Definition 2.1. We denote by $D(A_{\alpha,T})$ the space of functions u in $L^p(0,T)$ such that

(i) there exists $u_0 \in \mathbb{C}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big\{g_{1-\alpha} * (u-u_0)\big\} \in L^p(0,T);$$

(ii) the function $g_{1-\alpha} * (u-u_0)$ evaluated at $x=0$ vanishes, i.e.

$$\big(g_{1-\alpha} * (u-u_0)\big)(0) = 0.$$

Let us notice that, if T is finite then Young inequality implies that

$$D(A_{\alpha,T}) = \big\{u \in X_T \mid \exists\, u_0 \in \mathbb{C},\ g_{1-\alpha} * (u-u_0) \in {}_0W^{1,p}(0,T)\big\}, \tag{2.2}$$

where ${}_0W^{1,p}(0,T)$ denotes the subspace of $W^{1,p}(0,T)$ whose elements vanish at $x=0$. That representation is not true when $T=\infty$, since $g_{1-\alpha} * (u-u_0)$ does not always belong to $L^p(0,\infty)$ when u lies in $D(A_{\alpha,\infty})$. However, its restriction for any finite T belongs to ${}_0W^{p}(0,T)$.

Proposition 2.1. *Let $T \in (0,\infty]$, $p \in [\frac{1}{\alpha},\infty)$ and $u \in D(A_{\alpha,T})$. Then there exists a unique $u_0 \in \mathbb{C}$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\{g_{1-\alpha} * (u-u_0)\} \in L^p(0,T).$$

Proof. Assume that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big\{g_{1-\alpha} * (u-u_1)\big\} \in L^p(0,T),$$

for another $u_1 \in \mathbb{C}$. Let $\tau \in (0,T)$. Then

$$g_{1-\alpha} * (u_0-u_1) = g_{1-\alpha} * (u-u_1) - g_{1-\alpha} * (u-u_0)$$

belongs to $W^{1,p}(0,\tau)$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}\big\{g_{1-\alpha}*(u_0-u_1)\big\}=(u_0-u_1)g_{1-\alpha}\in L^p(0,\tau).$$

However, $g_{1-\alpha}$ does not belong to $L^p(0,\tau)$ since $p\geq 1/\alpha$. Hence, $u_0=u_1$. □

We are now in position to define *Caputo's operators* in $L^p(0,T)$.

Definition 2.2. Let $T\in(0,\infty]$ and $p\in[\frac{1}{\alpha},\infty)$. Then we definie the operator

$$A_{\alpha,T}:D(A_{\alpha,T})\subset L^p(0,T)\to L^p(0,T),\qquad u\mapsto\frac{\mathrm{d}}{\mathrm{d}x}\big\{g_{1-\alpha}*(u-u_0)\big\},\tag{2.3}$$

where u_0 is the unique complex number given by Proposition 2.1. For each $u\in D(A_{\alpha,T})$, $A_{\alpha,T}u$ is called the *fractional Caputo's derivative* of u of order α. Also we denote ${}^{\mathrm{c}}\mathrm{D}^\alpha:=A_{\alpha,T}$, and, for simplicity, we will sometimes write A_α instead of $A_{\alpha,T}$.

Lemma 2.2. *Let $\alpha\in(0,1)$ and $p\in[\frac{1}{\alpha},\infty)$. Let also T be finite and $u\in D(A_{\alpha,T})$. Then*

$$u=u_0+g_\alpha*{}^{\mathrm{c}}\mathrm{D}^\alpha u\qquad\text{in}\ \ L^p(0,T).\tag{2.4}$$

Moreover, if in addition, $p>\alpha^{-1}$ then u is continuous on $[0,T]$ and $u_0=u(0)$.

Proof. Starting from the definition

$$\frac{\mathrm{d}}{\mathrm{d}x}\big\{g_{1-\alpha}*(u-u_0)\big\}={}^{\mathrm{c}}\mathrm{D}^\alpha u\qquad\text{in}\ \ L^p(0,T),$$

convoluting by g_1, and using $g_{1-\alpha}*(u-u_0)(0)=0$, we get

$$g_{1-\alpha}*(u-u_0)=g_1*{}^{\mathrm{c}}\mathrm{D}^\alpha u\qquad\text{in}\ \ W^{1,p}(0,T).\tag{2.5}$$

Hence (2.4) follows. By [HL28, Theorem 12], $g_\alpha*{}^{\mathrm{c}}\mathrm{D}^\alpha u$ is continuous on $[0,T]$ if $p>\alpha^{-1}$; thus $g_\alpha*{}^{\mathrm{c}}\mathrm{D}^\alpha u(0)=0$. Hence (2.4) yields the continuity of u and $u_0=u(0)$. □

In view of Lemma 2.2, we will set $u(0):=u_0$ for all $u\in D(A_{\alpha,T})$ and $T\in(0,\infty]$. The following result is a consequence of Lemma 2.2.

Corollary 2.3. *Let $T\in(0,\infty]$ and $p\in[\frac{1}{\alpha},\infty)$. Then A_α is a closed operator on $L^p(0,T)$.*

Proof. When T is finite, the assertion follows easily from (2.4). On the other hand, for each $T\in(0,\infty)$, the restriction to $[0,T]$, of any element of $D(A_{\alpha,\infty})$ lies in $D(A_{\alpha,T})$. Hence, if

$$D(A_{\alpha,\infty})\ni u_n\to u,\quad A_{\alpha,\infty}u_n\to f\qquad\text{in}\ \ L^p(0,\infty),$$

then $g_{1-\alpha}*(u-u_0)(0)=0$ and

$$A_{\alpha,T}u=f\qquad\text{in}\ \ L^p(0,T).$$

Since f belongs to $L^p(0,\infty)$, we infer that $u\in D(A_{\alpha,\infty})$ and $A_{\alpha,\infty}u=f$. □

Proposition 2.4. *Let $T\in(0,\infty]$ and $p\in(\frac{1}{\alpha},\infty)$. Then $D(A_\alpha)$ is dense in $L^p(0,T)$.*

Proof. It is enough to show that $\mathscr{D}(0,T)\subset D(A_\alpha)$. Since that embbeding is clear for finite T, we will assume that $T=\infty$. Let $\varphi\in\mathscr{D}(0,\infty)$ and $M\in\mathbb{R}$ be such that $\varphi=0$ on $[M,\infty)$. Since φ is smooth, one has

$$g_{1-\alpha}*\big(\varphi-\varphi(0)\big)(0)=0\qquad\text{and}\qquad\frac{\mathrm{d}}{\mathrm{d}x}\big\{g_{1-\alpha}*\big(\varphi-\varphi(0)\big)\big\}=g_{1-\alpha}*\varphi'.\tag{2.6}$$

We claim that $g_{1-\alpha} * \varphi' \in L^p(0,\infty)$. Indeed, this function lies in $L^p(0,M+1)$. Hence, it remains to prove that $g_{1-\alpha} * \varphi' \in L^p(M+1,\infty)$. For each $x \geq M+1$,

$$g_{1-\alpha} * \varphi'(x) = \int_{x-M}^{x} g_{1-\alpha}(y)\varphi'(x-y)\,\mathrm{d}y.$$

Let

$$\widetilde{g} := \begin{cases} g_{1-\alpha} & \text{on } [1,\infty) \\ 0 & \text{on } (-\infty,1) \end{cases}.$$

Since $x \geq M+1$, $\widetilde{g} = g_{1-\alpha}$ on $[x-M,x]$; hence

$$|g_{1-\alpha} * \varphi'(x)| \leq \int_{\mathbb{R}} \widetilde{g}(y)|\varphi'(x-y)|\,\mathrm{d}y = \big(\widetilde{g} *_{\mathbb{R}} |\varphi'|\big)(x).$$

Since $\alpha p > 1$, $\widetilde{g}$ lies in $L^p(\mathbb{R})$. Hence, by Young inequality,

$$\|\widetilde{g} *_{\mathbb{R}} |\varphi'|\|_{L^p(\mathbb{R})} \leq \|\widetilde{g}\|_{L^p(\mathbb{R})}\|\varphi'\|_{L^1(\mathbb{R})}.$$

Therefore $g_{1-\alpha} * \varphi' \in L^p(M+1,\infty)$. Hence $\varphi \in D(A_{\alpha,\infty})$, which completes the proof. □

Proposition 2.5. *Let T be finite, $p \in [\frac{1}{\alpha},\infty)$ and $f \in L^p(0,T)$. Then $g_\alpha * f \in D(A_{\alpha,T})$, $(g_\alpha * f)(0) = 0$ and*

$$^{\mathrm{c}}\mathrm{D}^{\alpha}(g_\alpha * f) = f \qquad \text{in } \; L^p(0,T).$$

Proof. One has $g_{1-\alpha} * g_\alpha * f = g_1 * f \in {}_0W^{1,p}(0,T)$. Hence, in view of (2.2), $g_\alpha * f$ lies in $D(A_{\alpha,T})$ with $u_0 = 0$. Also Definition 2.2 yields $^{\mathrm{c}}\mathrm{D}^{\alpha}(g_\alpha * f) = f$. □

2.2. **The resolvent of $A_{\alpha,\infty}$.** Corollary 2.9 states that the resolvante set of $A_{\alpha,\infty}$ is the sector

$$S_\alpha := \big\{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| < \frac{\pi\alpha}{2}\big\}, \tag{2.7}$$

where the *argument function* is defined on $\mathbb{C}\setminus\{0\}$ with values in $[-\pi,\pi)$. A resolvent estimate is given in Proposition 2.7.

In order to prove this result, we need to consider finite intervalls $[0,T]$ as well. Let $\alpha \in (0,1)$, $p \in [\frac{1}{\alpha},\infty)$ and $T \in (0,\infty]$. For each $\lambda \in \mathbb{C}$ and $f \in X_T$, we consider the *resolvent equation*:

$$\text{Find } u \in D(A_{\alpha,T}) \text{ such that } \lambda u - {}^{\mathrm{c}}\mathrm{D}^{\alpha}u = f \qquad \text{in } \; L^p(0,T). \tag{2.8}$$

Using the well-known notations, for $\beta \in [0,\infty)$, let

$$E_{\alpha,\beta} : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto \sum_{k\geq 0} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

be the *generalized Mittag-Leffler function* and $E_\alpha := E_{\alpha,1}$. For each $\lambda \in \mathbb{C}$, we define

$$e_\alpha : [0,\infty) \to \mathbb{C}, \qquad x \mapsto E_\alpha(\lambda x^\alpha). \tag{2.9}$$

For $\lambda \in \mathbb{C}\setminus\{0\}$, let

$$n : [0,\infty) \to \mathbb{C}, \qquad x \mapsto \frac{e_\alpha(x)}{\lambda} = \sum_{k\geq 1} \lambda^{k-1} g_{\alpha k+1}(x), \tag{2.10}$$

and denote by n' the first order derivative of n.

The following result describes the *solution set* of the resolvent equation (2.8) for finite T.

Proposition 2.6. *Let T be finite, $p \in [\frac{1}{\alpha},\infty)$, $\lambda \in \mathbb{C}$, and $f \in L^p(0,T)$.*

(i) *For each* $u_0 \in \mathbb{C}$*, the function* $u := u_0 e_\alpha - f * n'$ *lies in* $D(A_{\alpha,T})$*,* $u(0) = u_0$*, and* u *solves* (2.8)*.*
(ii) *Conversely, if* $u \in D(A_{\alpha,T})$ *solves* (2.8) *then* $u = u(0)e_\alpha - f * n'$ *in* $L^p(0,T)$*.*

The above results are well-known; however we will give simple proofs that do not rely on Laplace transforms nor on Wright functions but rather on basic properties of series. Compare with [Die10, Theorem 7.2] or [Zho14, Lemma 4.21].

Proof of Proposition 2.6. Let us start to show that, for $p \geq \alpha^{-1}$, the function $u := f * n'$ belongs to $D(A_{\alpha,T})$, satisfies $u(0) = 0$, and $A_{\alpha,T} u = \lambda u + f$. In effect, we have

$$n' = \sum_{k \geq 1} \lambda^{k-1} g_{\alpha k} \qquad \text{in} \quad L^1(0,T).$$

Thus

$$f * \sum_{k=1}^{N} \lambda^{k-1} g_{\alpha k} \xrightarrow[N\to\infty]{} f * n' = u \qquad \text{in} \quad L^1(0,T). \tag{2.11}$$

Then, on one hand,

$$g_\alpha * f * \sum_{k=1}^{N} \lambda^{k} g_{\alpha k} \xrightarrow[N\to\infty]{} \lambda g_\alpha * u.$$

On the other hand,

$$g_\alpha * f * \sum_{k=1}^{N} \lambda^{k} g_{\alpha k} = f * \sum_{k=1}^{N} \lambda^{k} g_{\alpha(k+1)} = f * \sum_{k=1}^{N+1} \lambda^{k-1} g_{\alpha k} - f * g_\alpha \xrightarrow[N\to\infty]{} u - f * g_\alpha,$$

with (2.11). Hence $\lambda g_\alpha * u = u - f * g_\alpha$; that is to say $u = g_\alpha * (\lambda u + f)$. Using Proposition 2.5 with $\lambda u + f$ instead of f, we derive that the function $f * n'$, denoted by u, lies in $D(A_\alpha)$, satisfies $u(0) = 0$ and

$${}^{\mathrm{c}}\mathrm{D}^\alpha u = \lambda u + f. \tag{2.12}$$

Then we prove (i) by using (2.12) and ${}^{\mathrm{c}}\mathrm{D}^\alpha e_\alpha = \lambda e_\alpha$. Let us prove (ii). Lemma 2.2 yields that any $u \in D(A_{\alpha,T})$ satisfying ${}^{\mathrm{c}}\mathrm{D}^\alpha u = \lambda u$, is equal to $u(0)e_\alpha$: see for instance [ER17, Prop. 5.1]. Then we conclude in a standard way with (2.12). ☐

Let us notice that if T is finite then e_α is in the kernel of $\lambda - A_\alpha$ for every $\lambda \in \mathbb{C}$. Hence the resolvante of A_α is empty. This is consistent with the case $\alpha = 1$ since

$$D(A_{1,T}) = W^{1,p}(0,T), \qquad A_{1,T} = \frac{\mathrm{d}}{\mathrm{d}x}.$$

In fact the domain of $A_{\alpha,T}$ is specially designed for the case $T = \infty$, that we are in position to consider now.

Proposition 2.7. *Let* $\alpha \in (0,1)$*,* $p \in (\frac{1}{\alpha}, \infty)$*,* $\lambda \in S_\alpha$*,* $f \in L^p(0,\infty)$ *and*

$$u_0 := \lambda^{\frac{1}{\alpha}-1} \int_0^\infty f(y) \exp\big(-\lambda^{\frac{1}{\alpha}} y\big)\,\mathrm{d}y. \tag{2.13}$$

Then the function $u := u_0 e_\alpha - f * n'$ *belongs to* $L^p(0,\infty)$ *and*

$$\|u\|_p \leq C\|f\|_p |\lambda|^{-1} \Big(|\lambda|^{\frac{1}{\alpha}} \big[\mathrm{Re}(\lambda^{\frac{1}{\alpha}})\big]^{-1} + |\lambda|^{\frac{1}{\alpha p'}} \big[\mathrm{Re}(\lambda^{\frac{1}{\alpha}})\big]^{-\frac{1}{p'}} + 1 \Big), \tag{2.14}$$

where the constant C *is independent of* u*,* λ *and* f*. Also* p' *denotes the conjugate exponent of* p*.*

Remark 2.1. Under the assumptions and notation of Proposition 2.7, let us assume in addition that $\lambda > 0$. Then (2.14) reads

$$\|u\|_p \leq \frac{3C}{\lambda}\|f\|_p. \tag{2.15}$$

Since our computations lead to $3C > 1$, such a *resolvent estimate* is not enough to obtain an existence result for the following Cauchy problem, for all time T.

$$\begin{cases} \text{Find } u \in C^1\big((0,T); L^p(0,\infty)\big) \cap C\big([0,T]; D(A_{\alpha,\infty})\big) \ \text{ such that} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}u = A_{\alpha,\infty}u \qquad \text{on } \ (0,T) \\ u(0) = u_0 \in D(A_{\alpha,\infty}). \end{cases}$$

See [Are87, Example 3.2] for a counter example. However, according to [Paz83, Theorem 1.2, Chap 4], it is sufficent for uniqueness.

Moreover, in (2.15), the constant $3C$ cannot be replaced by 1: see Section 4 or Remark 2.2 for details.

Proof of Proposition 2.7. Since $\lambda \in S_\alpha$, one has $\mathrm{Re}(\lambda^{\frac{1}{\alpha}}) > 0$, thus u_0 is a well defined complex number, and by Hölder inequality

$$|u_0| \leq C\|f\|_p|\lambda|^{\frac{1}{\alpha}-1}\big[\mathrm{Re}(\lambda^{\frac{1}{\alpha}})\big]^{-\frac{1}{p'}}. \tag{2.16}$$

According to [Pod99, Theorem 1.3, p32], we have the following asymptotic expansions for λ, $z \in S_\alpha$ and $|z|$, $x \to \infty$.

$$e_\alpha(x) = \frac{1}{\alpha}\exp\big(\lambda^{\frac{1}{\alpha}}x\big) + R_0(\lambda x^\alpha), \qquad R_0(z) = O\Big(\frac{1}{z}\Big) \tag{2.17}$$

$$n'(x) = \frac{1}{\lambda x}E_{\alpha,0}(\lambda x^\alpha) = \frac{1}{\alpha}\lambda^{\frac{1}{\alpha}-1}\exp\big(\lambda^{\frac{1}{\alpha}}x\big) + \frac{R_1(\lambda x^\alpha)}{\lambda x}, \qquad R_1(z) = O\Big(\frac{1}{z}\Big). \tag{2.18}$$

Plugging these expansions into u, we get, for all $x \in (0,\infty)$,

$$u(x) = \frac{\lambda^{\frac{1}{\alpha}-1}}{\alpha}\int_x^\infty f(y)\exp\big(\lambda^{\frac{1}{\alpha}}(x-y)\big)\,\mathrm{d}y + u_0R_0(\lambda x^\alpha) - f * \frac{R_1(\lambda x^\alpha)}{\lambda x}. \tag{2.19}$$

Let us estimate the three terms in the right hand side of (2.19), starting with the first one. Extending f by zero on $[-\infty, 0]$ and exp by zero on $[0,\infty]$, and denoting these extensions by $\widetilde{f}$ and $\widetilde{\exp}$, respectively, one gets

$$\int_x^\infty f(y)\exp\big(\lambda^{\frac{1}{\alpha}}(x-y)\big)\,\mathrm{d}y = \int_{\mathbb{R}} \widetilde{f}(y)\widetilde{\exp}\big(\lambda^{\frac{1}{\alpha}}(x-y)\big)\,\mathrm{d}y = \widetilde{f} *_{\mathbb{R}} \widetilde{\exp}(\lambda^{\frac{1}{\alpha}}x).$$

Then, by Young inequality,

$$\|\widetilde{f} *_{\mathbb{R}} \widetilde{\exp}(\lambda^{\frac{1}{\alpha}}\cdot)\|_p \leq \|f\|_p\big[\mathrm{Re}\big(\lambda^{\frac{1}{\alpha}}\big)\big]^{-1}.$$

so what the first term in (2.19) is bounded in $L^p(0,\infty)$, by

$$\frac{1}{\alpha}\|f\|_p|\lambda|^{\frac{1}{\alpha}-1}\big[\mathrm{Re}\big(\lambda^{\frac{1}{\alpha}}\big)\big]^{-1}.$$

Regarding the second term, for $x \in (0, |\lambda|^{-\frac{1}{\alpha}})$, one has, for some constant C independent of x and λ,

$$|R_0(\lambda x^\alpha)| = \Big|E_\alpha(\lambda x^\alpha) - \frac{1}{\alpha}\exp\big(\lambda^{\frac{1}{\alpha}}x\big)\Big| \leq C.$$

On the other hand, if $x \geq |\lambda|^{-\frac{1}{\alpha}}$ then (2.17) yields

$$|R_0(\lambda x^\alpha)| \leq \frac{C}{|\lambda| x^\alpha}.$$

Hence, since $\alpha p > 1$,

$$\Big(\int_0^\infty \big|R_0(\lambda x^\alpha)\big|^p \, \mathrm{d}x\Big)^{\frac{1}{p}} \leq C|\lambda|^{-\frac{1}{\alpha p}}. \tag{2.20}$$

Thus, using also (2.16), the second term of (2.19) is bounded by

$$C\|f\|_p |\lambda|^{\frac{1}{\alpha p'}-1}\big[\mathrm{Re}\big(\lambda^{\frac{1}{\alpha}}\big)\big]^{-\frac{1}{p'}}.$$

We procced in the same way for estimating the third term. Starting from (2.18), we get

$$\int_0^{|\lambda|^{-\frac{1}{\alpha}}} \big|\frac{R_1(\lambda x^\alpha)}{\lambda x}\big| \leq \int_0^{|\lambda|^{-\frac{1}{\alpha}}} |n'(x)| + \frac{|\lambda|^{\frac{1}{\alpha}-1}}{\alpha} \exp\big(\mathrm{Re}\,(\lambda^{\frac{1}{\alpha}} x)\big) \, \mathrm{d}x.$$

Moreover, by considering the analytic expansion of $E_{\alpha,0}$, we show that $|E_{\alpha,0}(z)| \leq C|z|$ for all $|z| \leq 1$. Thus, for each $x \in (0, |\lambda|^{-\frac{1}{\alpha}})$,

$$|n'(x)| = \big|\frac{E_{\alpha,0}(\lambda x^\alpha)}{\lambda x}\big| \leq C x^{\alpha-1}.$$

Thus

$$\int_0^{|\lambda|^{-\frac{1}{\alpha}}} \big|\frac{R_1(\lambda x^\alpha)}{\lambda x}\big| \leq \frac{C}{|\lambda|}. \tag{2.21}$$

Moreover, in view of (2.18),

$$\int_{|\lambda|^{-\frac{1}{\alpha}}}^\infty \big|\frac{R_1(\lambda x^\alpha)}{\lambda x}\big| \leq \frac{C}{|\lambda|}.$$

By Young inequality, we derive that the third term of (2.19) is bounded by $C\|f\|_p/|\lambda|$. We then conclude that u belongs to $L^p(0,\infty)$ and satisfies the *resolvent estimate* (2.14). □

Now we can solve the resolvent equation (2.8) for $\lambda \in S_\alpha$ and $T = \infty$.

Theorem 2.8. *Let $\alpha \in (0,1)$, $p \in (\frac{1}{\alpha}, \infty)$, $\lambda \in S_\alpha$ and $f \in L^p(0,\infty)$. Then these propositions are equivalent.*
(i) *$u \in D(A_{\alpha,\infty})$ and $\lambda u - {}^{\mathrm{c}}\mathrm{D}^\alpha u = f$ in $L^p(0,\infty)$.*
(ii) *$u = u_0 e_\alpha - f * n'$ where u_0 is given by (2.13).*

Proof. Let us show that (ii) implies (i). For, we consider $\varphi \in \mathscr{D}(0,\infty)$ and $M \in \mathbb{R}$ such that the support of φ is included in $[0, M]$. Let

$$u_M := \begin{cases} u & \text{on } [0, M] \\ 0 & \text{on } (M, \infty) \end{cases}. \tag{2.22}$$

Then

$$\begin{aligned} \big\langle \frac{\mathrm{d}}{\mathrm{d}x}\big\{g_{1-\alpha} * (u - u_0)\big\}, \varphi\big\rangle_{\mathscr{D}'(0,\infty),\mathscr{D}(0,\infty)} &= -\big\langle g_{1-\alpha} * (u_M - u_0), \varphi'\big\rangle \\ &= \int_0^M A_{\alpha,M} u_M(x) \varphi(x) \, \mathrm{d}x \\ &= \big\langle \lambda u_M - f, \varphi \big\rangle \qquad \text{(by Prop. 2.6 (i))} \\ &= \big\langle \lambda u - f, \varphi \big\rangle. \end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\{g_{1-\alpha} * (u-u_0)\} = \lambda u - f \qquad \text{in } \mathscr{D}'(0,\infty).$$

Moreover, by Proposition 2.7, u lies in $L^p(0,\infty)$, and by Proposition 2.6 (i), $g_{1-\alpha} * (u - u_0)(0) = 0$. Hence $u \in D(A_{\alpha,\infty})$ and ${}^{\mathrm{c}}\mathrm{D}^\alpha u = \lambda u - f$ in $L^p(0,\infty)$.

Conversely, let u satisfy the assumption (i). For each positive integer m, let us denote by u_m the function defined thru (2.22) with $M = m$. Proposition 2.6 (ii) gives

$$u_m = u_m(0)e_\alpha - f * n' \qquad \text{in } L^p(0,m), \tag{2.23}$$

for each $m \geq 1$. According to Proposition 2.6 (i), we have $(f * n')(0) = 0$. Thus $u_m(0) = u(0)$, so that the right hand side of (2.23) is independent of m. Since $u_m \to u$ in $L^p(0,\infty)$, we deduce that $u = u(0)e_\alpha - f * n'$.

Besides, according to Proposition 2.7, the function $u_0 e_\alpha - f * n'$ lies in $L^p(0,\infty)$. Thus by difference, $(u(0) - u_0)e_\alpha \in L^p(0,\infty)$. Since $\lambda \in S_\alpha$, we must have, in view of (2.17), $u(0) = u_0$; so that $u = u_0 e_\alpha - f * n'$. The proof of the theorem is now completed. ☐

Corollary 2.9. *Let $\alpha \in (0,1)$ and $p \in (\frac{1}{\alpha},\infty)$. Then the resolvent set of $A_{\alpha,\infty}$, denoted by $\rho(A_{\alpha,\infty})$, is equal to the sector S_α, that is*

$$\rho(A_{\alpha,\infty}) = \big\{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| < \frac{\pi\alpha}{2}\big\},$$

Proof. Proposition 2.7 and Theorem 2.8 yield that $S_\alpha \subseteq \rho(A_{\alpha,\infty})$. Moreover, we claim that each $\lambda \in \mathbb{C}\setminus\overline{S_\alpha}$ belongs to the *point spectrum* of $A_{\alpha,\infty}$. Indeed, according to [Pod99, Theorem 1.4], e_α given by (2.9), lies in $L^p(0,\infty)$, since $\alpha p > 1$. Moreover, ${}^{\mathrm{c}}\mathrm{D}^\alpha e_\alpha = \lambda e_\alpha$. Hence the claim follows. Since the spectrum is closed, we then obtain $\rho(A_{\alpha,\infty}) = S_\alpha$. ☐

Remark 2.2. Corollary 2.9 yields that $A_{\alpha,\infty}$ does not generate a C_0-semigroup on $L^p(0,\infty)$. Indeed, by [Paz83, Th. 5.3 and Remark 5.4 p 20], the resolvent set of a C_0-semigroup with exponential growth contains a complex half plane of the form $[\operatorname{Re} z > \omega]$. Moreover, [Paz83, Th. 2.2 p4] states that a C_0-semigroup has always an exponential growth.

Hence the *subordination principle* (which gives an integral representation of the solution operator of fractional problems in terms of the semigroup, see [Baz19]) can not be implemented in our setting.

2.3. **Resolvent estimate for more regular functions.** For $|\lambda|$ large enough, we may improve (2.14) provided f is more regular.

Proposition 2.10. *Let $\alpha \in (0,1)$, $p \in (\frac{1}{\alpha},\infty)$, $T = \infty$, and $\lambda \in S_\alpha$ with $\operatorname{Re}(\lambda^{\frac{1}{\alpha}}) \geq 1$. Let us assume in addition that f belongs to $W^{1,p}(0,\infty)$. Then*

$$\|(\lambda - {}^{\mathrm{c}}\mathrm{D}^\alpha)^{-1} f\|_{L^p(0,\infty)} \leq C|\lambda|^{-1}\|f\|_{W^{1,p}(0,\infty)},$$

where the constant C is independent of λ and f.

Proof. Since $\lambda \in S_\alpha$ and $f \in L^p(0,\infty)$, Corollary 2.9 tells us that the resolvent equation (2.8) has a unique solution u. Moreover, by Theorem 2.8, $u = u_0 e_\alpha - f * n'$ where u_0 is given by (2.13). For ease of reference, we recall that (2.19) reads

$$u(x) = \frac{\lambda^{\frac{1}{\alpha}-1}}{\alpha}\int_x^\infty f(y)\exp\big(\lambda^{\frac{1}{\alpha}}(x-y)\big)\,\mathrm{d}y + u_0 R_0(\lambda x^\alpha) - f * \frac{R_1(\lambda x^\alpha)}{\lambda x}, \tag{2.24}$$

with R_0 and R_1 defined by (2.17) and (2.18).

Let us estimate the fisrt term in the right hand side of (2.24). For, integration by part gives

$$\int_x^\infty f(y)\exp\big(\lambda^{\frac{1}{\alpha}}(x-y)\big)\,\mathrm{d}y = -\int_x^\infty f'(y)\lambda^{-\frac{1}{\alpha}}\exp\big(\lambda^{\frac{1}{\alpha}}(x-y)\big)\,\mathrm{d}y - \lambda^{-\frac{1}{\alpha}}f(x).$$

Using $\mathrm{Re}(\lambda^{\frac{1}{\alpha}}) \geq 1$, we derive that the first term of (2.24) is bounded in $L^p(0,\infty)$, by

$$C|\lambda|^{-1}\big(\|f\|_p + \|f'\|_p\big).$$

Regarding the second term, integration by part in (2.13) gives

$$u_0 = \lambda^{-1}\int_0^\infty f'(y)\exp\big(-\lambda^{\frac{1}{\alpha}}y\big)\,\mathrm{d}y + \lambda^{-1}f(0).$$

Hence, since $W^{1,p}(0,\infty)$ is continuously embedded into $L^\infty(0,\infty)$, we get

$$|u_0| \leq C|\lambda|^{-1}\big(\|f\|_p + \|f'\|_p\big).$$

Using also (2.20), the second term of (2.24) is bounded by

$$C|\lambda|^{-1}\big(\|f\|_p + \|f'\|_p\big),$$

since $\mathrm{Re}\,\lambda^{\frac{1}{\alpha}} \geq 1$ yields $|\lambda| \geq 1$. The third term is estimated as in the proof of Proposition 2.7, so that

$$\|u\|_p = \|(\lambda - {}^{\mathrm{c}}\mathrm{D}^\alpha)^{-1}f\|_p \leq C|\lambda|^{-1}\big(\|f\|_p + \|f'\|_p\big).$$

□

3. Distributional Delsarte's equations

Our main tools is the *Laplace transform* of vector valued distributions, for which we refer to [DL92, Chap. XVI] and [Ama03]. A distribution $u \in \mathscr{D}'(\mathbb{R}, X)$, with values into a complex Banach space X is said to be *Laplace transformable* if there exists some $\xi_0 \in \mathbb{R}$ such that $e^{-\xi\cdot}u$ lies in $\mathcal{S}'(\mathbb{R}, X)$ for all $\xi > \xi_0$, where $e^{-\xi\cdot}$ denotes the real function $t \mapsto e^{-\xi t}$. For such u, the Fourier transform $\mathcal{F}(e^{-\xi\cdot}u)$ of $e^{-\xi\cdot}u$ appears to be a function if $\xi > \xi_0$. The *Laplace transform* of u is then

$$\mathcal{L}u : [\mathrm{Re}\,s > \xi_0] \to X, \qquad s \mapsto \mathcal{F}\big(e^{-\mathrm{Re}\,s\cdot}u\big)(\mathrm{Im}\,s),$$

where $[\mathrm{Re}\,s > \xi_0] := \{s \in \mathbb{C} \mid \mathrm{Re}\,s > \xi_0\}$. It turns out that $\mathcal{L}u$ is an holomorphic function on $[\mathrm{Re}\,s > \xi_0]$ with values in X. See the above references for details.

In order to introduce Caputo's derivative of vector valued distributions, let

$$\mathscr{D}'_{+,0}(\mathbb{R}, X) = \mathscr{D}'_{+,0}(X) := \{u \in \mathscr{D}'(\mathbb{R}, X) \mid \mathrm{supp}\,u \subseteq [0,\infty)\}, \tag{3.1}$$

where $\mathrm{supp}\,u$ stands for the support of u. Let $H \in L^\infty(\mathbb{R})$ be the *Heaviside function*, and for each $u_0 \in X$, denote by $H \otimes u_0$ the distribution of $\mathscr{D}'_{+,0}(\mathbb{R}, X)$ defined by

$$\big\langle H \otimes u_0, \varphi\big\rangle_{\mathscr{D}'(\mathbb{R},X),\mathscr{D}(\mathbb{R})} = \big\langle H, \varphi\big\rangle_{\mathscr{D}'(\mathbb{R}),\mathscr{D}(\mathbb{R})}u_0 = \int_0^\infty \varphi(y)\,\mathrm{d}y\,u_0. \tag{3.2}$$

The definition of Caputo's derivative of distributions relies on the following issue: find a convinient and simple transformation of a function $u \in H^1(0,\infty; X)$ into an element of $\mathscr{D}'_{+,0}(\mathbb{R}, X)$. The basic idea is to set

$$\widetilde{u} := \begin{cases} u & \text{on } [0,\infty) \\ 0 & \text{on } (-\infty, 0) \end{cases}.$$

However, when computing the derivative of $\widetilde{u}$, we find

$$\langle \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{u}, \varphi \rangle_{\mathscr{D}'(\mathbb{R},X),\mathscr{D}(\mathbb{R})} = \int_0^\infty u'(y)\varphi(y)\,\mathrm{d}y + \varphi(0)u(0).$$

In order to get rid of the interfacial singular term $\varphi(0)u(0)$, we notice that, in view of (3.2),

$$\varphi(0)u(0) = \langle \frac{\mathrm{d}}{\mathrm{d}t}\{H \otimes u(0)\}, \varphi \rangle_{\mathscr{D}'(\mathbb{R},X),\mathscr{D}(\mathbb{R})}.$$

Thus the substitution of u by $\widetilde{u} - H \otimes u(0)$ gives

$$\langle \frac{\mathrm{d}}{\mathrm{d}t}\{\widetilde{u} - H \otimes u(0)\}, \varphi \rangle_{\mathscr{D}'(\mathbb{R},X),\mathscr{D}(\mathbb{R})} = \int_0^\infty u'(y)\varphi(y)\,\mathrm{d}y.$$

We choose that substitution in the definition of Caputo's derivatives.

Let us notice that the extension

$$v := \begin{cases} u & \text{on } [0,\infty) \\ u(0) & \text{on } (-\infty,0) \end{cases}$$

is simple as well. However, the support of v is not limited to the left. Since that support condition is essential to perform the convolution of distributions, the latter extension is not convinient.

For any $f \in L^1_{\mathrm{loc}}(0,\infty;X)$, let us set

$$\widetilde{f} := \begin{cases} f & \text{on } [0,\infty) \\ 0 & \text{on } (-\infty,0) \end{cases}. \tag{3.3}$$

Now, following [LL18, Def. 2.2], we are in position to define *Caputo's derivatives of distributions.*

Definition 3.1. Let $\alpha \in (0,1)$, $u_0 \in X$ and $u \in \mathscr{D}'_{+,0}(\mathbb{R},X)$. Then the *Caputo's derivative of u associated with u_0* is the distribution of $\mathscr{D}'_{+,0}(\mathbb{R},X)$, denoted by ${}^{\mathrm{c}}\mathrm{D}^{\alpha}_{u_0}u$, and defined thru

$${}^{\mathrm{c}}\mathrm{D}^{\alpha}_{u_0}u := \frac{\mathrm{d}}{\mathrm{d}t}\big\{\widetilde{g_{1-\alpha}} * (u - H \otimes u_0)\big\}.$$

That definition extends the one of Caputo's derivatives of functions given in Definition 2.2. More precisely, this result holds.

Proposition 3.1. *Let $X := \mathbb{C}$, $p \in [\frac{1}{\alpha},\infty)$, and $u \in D(A_{\alpha,\infty})$. Then*

$${}^{\mathrm{c}}\mathrm{D}^{\alpha}_{u(0)}\widetilde{u} = \widetilde{A_{\alpha,\infty}u} \qquad \text{in } \mathscr{D}'_{+,0}(\mathbb{R},\mathbb{C}). \tag{3.4}$$

Proof. According to Definition 2.2, for all $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\big\langle \widetilde{A_{\alpha,\infty}u}, \varphi \big\rangle_{\mathscr{D}'(\mathbb{R},\mathbb{C}),\mathscr{D}(\mathbb{R})} = -\int_0^\infty g_{1-\alpha} * (u - u_0)\varphi'(y)\,\mathrm{d}y - \big(g_{1-\alpha} * (u-u_0)\big)(0)\varphi(0).$$

By Definition 2.1 of $D(A_{\alpha,\infty})$, we have on the one hand

$$\big(g_{1-\alpha} * (u-u_0)\big)(0) = 0.$$

On the other hand,

$$\int_0^\infty (g_{1-\alpha} * u_0)(y)\varphi'(y)\,\mathrm{d}y = \big\langle \widetilde{g_{1-\alpha}} * (H \otimes u_0), \varphi' \big\rangle_{\mathscr{D}'(\mathbb{R},\mathbb{C}),\mathscr{D}(\mathbb{R})},$$

and

$$\begin{aligned}\int_0^\infty g_{1-\alpha} * u(y)\varphi'(y)\,\mathrm{d}y &= \int_{\mathbb{R}} \widetilde{g_{1-\alpha}} * \widetilde{u}(y)\varphi'(y)\,\mathrm{d}y \\ &= \big\langle \widetilde{g_{1-\alpha}} * \widetilde{u}, \varphi' \big\rangle_{\mathscr{D}'(\mathbb{R},\mathbb{C}),\mathscr{D}(\mathbb{R})}.\end{aligned}$$

Thus

$$\big\langle \widetilde{A_{\alpha,\infty} u}, \varphi \big\rangle_{\mathscr{D}'(\mathbb{R},\mathbb{C}),\mathscr{D}(\mathbb{R})} = -\big\langle \widetilde{g_{1-\alpha}} * (\widetilde{u} - H \otimes u_0), \varphi' \big\rangle_{\mathscr{D}'(\mathbb{R},\mathbb{C}),\mathscr{D}(\mathbb{R})}.$$

Hence, in view of Definition 3.1, (3.4) follows. □

Let $\alpha \in (0,1)$, $p \in [\alpha^{-1}, \infty)$, and $u_0 \in X$. Then the *Delsarte equation* for Caputo's operator on $L^p(0,\infty)$ reads

$$\begin{aligned}&\text{Find } u \in \mathscr{D}'_{+,0}\big(\mathbb{R}, D(A_{\alpha,\infty})\big) \text{ such that} \\ &{}^{\mathrm{c}}\mathrm{D}^\alpha_{u_0} u = A_{\alpha,\infty} u \qquad \text{in } \mathscr{D}'\big(\mathbb{R}, L^p(0,\infty)\big).\end{aligned} \tag{3.5}$$

According to [DL92, Def. 4, p 220], $A_{\alpha,\infty} u$ is defined thru

$$\big\langle A_{\alpha,\infty} u, \varphi \big\rangle_{\mathscr{D}'(\mathbb{R},X_\infty),\mathscr{D}(\mathbb{R})} := A_{\alpha,\infty}\left(\langle u, \varphi \rangle_{\mathscr{D}'(\mathbb{R},D(A_{\alpha,\infty})),\mathscr{D}(\mathbb{R})}\right).$$

Problem (3.5) may be rewritten formally in a more Delsarte's fashion as

$$ {}^{\mathrm{c}}\mathrm{D}^\alpha_t u = {}^{\mathrm{c}}\mathrm{D}^\alpha_x u, \qquad u_{|t=0} = u_0. \tag{3.6}$$

Remark 3.1. It is easy to check that any Laplace transformable *strong solution* in the sense of Definition 4.1 below, is a solution to (3.5).

Theorem 3.2. *Let $\alpha \in (0,1)$, $p \in (\frac{1}{\alpha}, \infty)$, and $u_0 \in L^p(0,\infty)$. Then* (3.5) *admits a unique solution which is Laplace transformable.*

Proof. It relies on the one of [DL92, Theorem 1, p 226], for which we refer for the details. Let us start to show the uniqueness of the solution. For, let $u \in \mathscr{D}'_{+,0}\big(\mathbb{R}, D(A_\alpha)\big)$ be a Laplace transformable solution to (3.5). Then using [Ama03, Theorem 1.10.11] and $\mathcal{L}g_{1-\alpha}(s) = s^{\alpha-1}$ for each $s \in \mathbb{C}$ such that $\operatorname{Re} s > 0$, one has

$$\mathcal{L}u(s) = s^{\alpha-1}(s^\alpha - A_\alpha)^{-1} u_0 \qquad \text{in } L^p(0,\infty). \tag{3.7}$$

By injectivity of the Laplace transform, we infer that (3.5) has a unique Laplace transformable solution. For the existence, it is enough to show that, for each $\operatorname{Re} s > 1$, the right hand side of (3.7) is the Laplace transform of a distribution with values in $D(A_\alpha)$. For, such a s reads

$$s = \lambda^{\frac{1}{\alpha}} \qquad \text{for some } \lambda \in S_\alpha,$$

where S_α is defined by (2.7). Hence λ belongs to the resolvant set of A_α (by Corollary 2.9) and

$$\| s^{\alpha-1}(s^\alpha - A_\alpha)^{-1} u_0 \|_{D(A_\alpha)} = |s|^{\alpha-1} \| (\lambda - A_\alpha)^{-1} u_0 \|_{D(A_\alpha)}.$$

Since $A_\alpha(\lambda - A_\alpha)^{-1} = \lambda(\lambda - A_\alpha)^{-1} - 1$ and $\operatorname{Re}(\lambda^{\frac{1}{\alpha}}) > 1$, we derive from (2.14), that

$$\| (\lambda - A_\alpha)^{-1} u_0 \|_{D(A_\alpha)} \le C \|u_0\|_p |\lambda|^{\frac{1}{\alpha}}.$$

Thus

$$\| s^{\alpha-1}(s^\alpha - A_\alpha)^{-1} u_0 \|_{D(A_\alpha)} \le C \|u_0\|_p |s|^\alpha.$$

Whence [DL92, Theorem 1, p 224] yields the existence of some $u \in \mathscr{D}'_{+,0}\big(\mathbb{R}, D(A_\alpha)\big)$ satisfying (3.7). The proof of the theorem is now completed. □

Remark 3.2. The proof of Theorem 3.2 still provides an existence and uniqueness result for

$$\begin{aligned} &\text{Find } u \in \mathscr{D}'_{+,0}\big(\mathbb{R}, D(A_{\beta,\infty})\big) \text{ such that} \\ &{}^{\mathrm{c}}\mathrm{D}^{\alpha}_{u_0} u = A_{\beta,\infty} u \qquad \text{in } \mathscr{D}'_{+,0}\big(\mathbb{R}, L^p(0,\infty)\big), \end{aligned} \tag{3.8}$$

provided $\beta \in [\alpha, 1)$. However when $\beta > \alpha$, (3.8) is a parabolic problem tractable by holomorphic integration methods (see for intance [Baz98, Theorem 4.1]), which provide classical solutions.

If the initial condition of (3.5) is more regular, then its solution turns out to be a function of time. More precisely we have this result.

Corollary 3.3. *Let $\alpha \in (0,1)$, $u_0 \in W^{1,p}(0,\infty)$, and*

$$\frac{1}{\alpha} < p < \infty, \qquad p \geq 2.$$

Then the distributional solution u to (3.5) *is a causal function and*

$$u \in L^p_{\mathrm{loc}}\big(\mathbb{R}; L^p(0,\infty)\big).$$

Proof. Let $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Then the solution u to (3.5) given by Theorem 3.2 satisfies

$$\begin{aligned} \|\mathcal{L}u(s)\|_p &= |s|^{\alpha-1}\|(s^\alpha - A_\alpha)^{-1}u_0\|_p && \text{(by (3.7))} \\ &\leq C|s|^{-1}\|u_0\|_{W^{1,p}(0,\infty)} && \text{(by Prop. 2.10).} \end{aligned}$$

Moreover, $p' > 1$ since p is finite. Thus the function

$$\mathbb{R} \to L^p(0,\infty), \qquad \eta \mapsto \mathcal{F}\big(e^{-\operatorname{Re} s \cdot} u\big)(\eta)$$

lies in $L^{p'}\big(\mathbb{R}; L^p(0,\infty)\big)$. Hence, the *Hausdorff-Young Theorem* for vector valued functions (see [Pee69, Exemple 2.4]) yields that

$$e^{-\operatorname{Re} s \cdot} u \in L^p\big(\mathbb{R}; L^p(0,\infty)\big),$$

since $1 < p' \leq 2$ (due to $p \geq 2$). Thus u lies in $L^p_{\mathrm{loc}}(\mathbb{R}; X_\infty)$. □

4. Representation of particular solutions

Let us start by defining a solution to (3.6) in a more usual sense, namely in the sense of *semi-group theory.* This definition relies on [ER17, Definition 3.3 & 3.4].

Definition 4.1. Let $\alpha \in (0,1)$, $p \in [\frac{1}{\alpha}, \infty)$, and $u_0 \in D(A_{\alpha,\infty})$. A function u in $C([0,\infty); D(A_{\alpha,\infty}))$ is called a *strong solution* to (3.6) if

(i) u admits a *Caputo's derivatives of order α in* $C([0,\infty); X_\infty)$, that is to say,

$$g_{1-\alpha} * \big(u - u(0)\big) \in C^1\big([0,\infty); X_\infty\big);$$

(ii) $u(0) = u_0$ in $D(A_{\alpha,\infty})$;

(iii) ${}^{\mathrm{c}}\mathrm{D}^{\alpha}_t u = A_{\alpha,\infty} u$ in $C([0,\infty); X_\infty)$, where

$${}^{\mathrm{c}}\mathrm{D}^{\alpha}_t u := \frac{\mathrm{d}}{\mathrm{d}t}\big\{g_{1-\alpha} * (u - u_0)\big\}.$$

As already noticed in Remark 3.1, any Laplace transformable *strong solution* to (3.6) is a solution to (3.5).

The simple structure of the differential operator ${}^{\mathrm{c}}\mathrm{D}^{\alpha}_t - {}^{\mathrm{c}}\mathrm{D}^{\alpha}_x$, allows us to get explicit and integral representations of the solution to (3.6), for a certain class of initial conditions. Delsarte's representation is given in Subsection 4.3.

4.1. **explicit solutions.** In order to introduce explicit solutions to (3.6) in the sense of Definition 4.1, let $\alpha \in (0,1)$, $\beta \in (-2,2]$ be such that $\alpha < |\beta|$, and $\lambda > 0$. Also, let $E_\alpha := E_{\alpha,\beta}$ denote the *Mittag-Leffler function*. Then the function $u(\cdot,\cdot)$ defined for all t, $x \geq 0$, by

$$u(t,x) := E_\alpha(\mathrm{i}^\beta \lambda t^\alpha) E_\alpha(\mathrm{i}^\beta \lambda x^\alpha),$$

is solution to (3.6) provided $p > 1/\alpha$ and $u_0 := u(0,\cdot)$. Indeed, the only non trivial point to check is that $u(t,\cdot)$ belongs to $L^p(0,\infty)$. That follows from [Pod99, Theorem 1.4, p33].

Moreover, observe that, for the same reasons, the function $v(\cdot,\cdot)$ defined for all t, $x \geq 0$, by

$$v(t,x) := \exp(\mathrm{i}^\beta \lambda t) E_\alpha(\mathrm{i}^\beta \lambda x^\alpha),$$

is the strong solution to the standard Cauchy problem (the uniqueness comes from Remark 2.1)

$$\frac{\mathrm{d}}{\mathrm{d}t} v = A_{\alpha,\infty} v, \qquad v(0,\cdot) = u_0.$$

In addition, if $\alpha < \beta < 1$ then

$$\|v(t,\cdot)\|_{L^p(0,\infty)} = \exp\big(\cos(\tfrac{\pi}{2}\beta)\lambda t\big)\|u_0\|_{L^p(0,\infty)} \xrightarrow[t\to\infty]{} \infty.$$

That elementary observation shows that $A_{\alpha,\infty}$ does not generate a *contraction* semi-group on $L^p(0,\infty)$. Thus, according to the Hille-Yosida Theorem (see for instance [Jac01, Theorem 4.1.33]), $A_{\alpha,\infty}$ is not dissipative, and, in (2.15), the constant $3C$ cannot be replaced by 1.

4.2. **Integral representation.** The following result gives an integral representation of some solutions to Delsarte's equation.

Proposition 4.1. *Let us assume that*

(i) $\alpha \in (0,1)$ *and* $p \in (\frac{1}{\alpha}, \infty)$;

(ii) $f : [0,\infty) \to \mathbb{C}$ *is a measurable function whose restriction to* $[0,1]$ *lies in* $L^\infty(0,1)$, *and which satisfies*

$$\int_0^\infty \lambda |f(\lambda)| \,\mathrm{d}\lambda < \infty. \tag{4.1}$$

Then the function $u : [0,\infty) \times [0,\infty) \to \mathbb{C}$ *defined by*

$$u(t,x) := \int_0^\infty E_\alpha(-\lambda t^\alpha) E_\alpha(-\lambda x^\alpha) f(\lambda) \,\mathrm{d}\lambda$$

provides a strong solution to the Delsarte equation (3.6), *for the initial condition*

$$u_0(x) := \int_0^\infty E_\alpha(-\lambda x^\alpha) f(\lambda) \,\mathrm{d}\lambda.$$

Proof. Let us start to show that u belongs to $C([0,\infty); D(A_\alpha))$. By the *generalized Minkowski inequality*, one has, for every $t \geq 0$,

$$\Big(\int_0^\infty |u(t,x)|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq \int_0^\infty E_\alpha(-\lambda t^\alpha)|f(\lambda)| \Big(\int_0^\infty E_\alpha(-\lambda x^\alpha)^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \mathrm{d}\lambda. \tag{4.2}$$

Using [Pod99, Theorem 1.4] for the decay rate of the Mittag-Leffler function and computing as in (2.20), we get

$$\|E_\alpha(-\lambda(\cdot)^\alpha)\|_{L^p(0,\infty)} \leq C\lambda^{-\frac{1}{\alpha p}}, \quad \forall \lambda > 0. \tag{4.3}$$

With $0 < E_\alpha(-\lambda t^\alpha) \leq 1$ and (4.3), we get

$$\|u(t)\|_p \leq C \int_0^\infty \lambda^{-\frac{1}{\alpha p}} |f(\lambda)| \, \mathrm{d}\lambda < \infty,$$

by (4.1), the boundedness of f near zero, and $\alpha p > 1$. Hence, $u(t)$ lies in $L^p(0, \infty)$ for each $t \geq 0$. The continuity is then obtained easily by Lebesgue Theorem and (4.3).

Let us show that $A_\alpha u \in C([0, \infty); X_\infty)$. For, let us set for simplicity

$$e_\lambda(x) := E_\alpha(-\lambda x^\alpha), \quad \forall x \geq 0. \tag{4.4}$$

Then Fubbini's Theorem leads to

$$g_{1-\alpha} * \big(u(t, \cdot) - u(t, 0)\big)(x) = \int_0^\infty E_\alpha(-\lambda t^\alpha) f(\lambda) g_{1-\alpha} * (e_\lambda - 1)(x) \, \mathrm{d}\lambda. \tag{4.5}$$

Moreover, the derivative of the latter integrand w.r.t. x is bounded by $\lambda |f(\lambda)|$. Thus with (4.1) and the Lebesgue differentiation Theorem, we get for each $x \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x} \big\{ g_{1-\alpha} * \big(u(t, \cdot) - u(t, 0)\big)(x) \big\} = -\int_0^\infty \lambda E_\alpha(-\lambda t^\alpha) E_\alpha(-\lambda x^\alpha) f(\lambda) \, \mathrm{d}\lambda.$$

For all purposes, let us make precise that the convolution acts on the space variable x. By (4.3), there results that $u(t, \cdot)$ lies in $D(A_\alpha)$. Then the continuity of u is easy to prove, and we have

$$A_\alpha\big(u(t)\big)(x) = -\int_0^\infty \lambda E_\alpha(-\lambda t^\alpha) E_\alpha(-\lambda x^\alpha) f(\lambda) \, \mathrm{d}\lambda. \tag{4.6}$$

Arguing as above, we show that u admits a Caputo's derivative of order α in $C([0, \infty); X_\infty)$, and that

$${}^{\mathrm{c}}\mathrm{D}^\alpha u(t, x) = -\int_0^\infty \lambda E_\alpha(-\lambda t^\alpha) E_\alpha(-\lambda x^\alpha) f(\lambda) \, \mathrm{d}\lambda.$$

Therefore, in view of (4.6), u solves Delsarte's equation. □

4.3. **Delsarte's representation.** Regarding Caputo's operator ${}^{\mathrm{c}}\mathrm{D}^\alpha$, Delsarte's representation (see the introduction) of the solution u to the Cauchy problem

$${}^{\mathrm{c}}\mathrm{D}_t^\alpha u = {}^{\mathrm{c}}\mathrm{D}_x^\alpha u, \quad u(0, x) = f(x),$$

is

$$u(t, x) = \sum_{k \geq 0} g_{\alpha k + 1}(t) \big({}^{\mathrm{c}}\mathrm{D}_x^\alpha\big)^k f(x).$$

For ease of reading, we will write B instead of ${}^{\mathrm{c}}\mathrm{D}^\alpha$, so that B_x denotes ${}^{\mathrm{c}}\mathrm{D}_x^\alpha$. Hence the above identity becomes

$$u(t, x) = \sum_{k \geq 0} g_{\alpha k + 1}(t) B_x^k f(x). \tag{4.7}$$

Here, $\alpha \in (0, 1]$ and $f : [0, \infty) \to \mathbb{C}$. Since, for $\beta > 1$,

$$B g_\beta = {}^{\mathrm{c}}\mathrm{D}^\alpha g_\beta = g_{\beta - \alpha}, \quad B g_1 = 0, \tag{4.8}$$

(4.7) can be formally justified as follows.

$$\begin{aligned}B_t u(t,x) =& \sum_{k\geq 1} g_{\alpha(k-1)+1}(t) B_x^k f(x)\\ =& \sum_{k\geq 0} g_{\alpha k+1}(t) B_x^{k+1} f(x)\\ =& B_x u(t,x).\end{aligned}$$

Hence the right hand side of (4.7) solves Delsarte's equation.

Now we would like to justify the above computations and give a precise meaning to (4.7). In the particular case where $\alpha = 1$, (4.7) reads

$$u(t,x) = \sum_{k\geq 0} \frac{D^k f(x)}{k!} t^k = f(x+t), \tag{4.9}$$

and Delsarte's equation reduces to the usual *transport equation* $D_t u = D_x u$, $u(0,x) = f(x)$. Moreover, (4.9) holds for all $x \in (0,\infty)$ and $t \geq 0$, close to zero, if and only if f is analytic on $(0,\infty)$.

In the case $\alpha \in (0,1)$, we will begin to show that (4.7) provides a continuous function on $[0,\infty)^2$, for every function $f : [0,\infty) \to \mathbb{C}$ defined by

$$f(x) := \sum_{k\geq 0} b_k g_{\alpha k+1}(x), \quad \forall x \geq 0, \tag{4.10}$$

where $(b_k)_{k\geq 0} \subset \mathbb{C}$ satisfies, for some positive constants B_0 and λ (depending on f),

$$|b_k| \leq B_0 \lambda^{\alpha k}, \quad \forall k \in \mathbb{N}. \tag{4.11}$$

Indeed, firstly, let us notive that f is a well defined continuous function on $[0,\infty)$ since the series (4.10) converges normally on each compact intervall $[0,R]$. Indeed,

$$|b_k g_{\alpha k+1}(x)| \leq B_0 g_{\alpha k+1}(\lambda R), \quad \forall x \in [0,R],$$

and

$$\sum_{k\geq 0} g_{\alpha k+1}(\lambda R) = E_\alpha\big((\lambda R)^\alpha\big).$$

Secondly, arguing in the same way and using (4.8), we show easily that, for each $n \in \mathbb{N}$, $B^n f$ is well defined and

$$B^n f = \sum_{k\geq 0} b_{k+n} g_{\alpha k+1} \qquad \text{in } C\big([0,\infty)\big).$$

Thus, with (4.11) again,

$$\|B^n f\|_{C([0,R])} \leq B_0 \lambda^{\alpha n} E_\alpha\big((\lambda R)^\alpha\big).$$

Hence the function u defined by (4.7) lies in $C([0,\infty)^2)$.

Finally, that estimate allows us to prove that u is a solution to the Delsarte equation in the following sense.

(1) u is continuous function on $[0,\infty)^2$;
(2) for each $x \geq 0$, the function $u(\cdot,x)$ has a Caputo derivative in $C([0,\infty))$ and $(t,x) \mapsto {}^{\mathrm{c}}\mathrm{D}_t^\alpha u(t,x)$ is continuous on $[0,\infty)^2$;
(3) for each $x \geq 0$, $u(t,\cdot)$ has a Caputo derivative in $C([0,\infty))$ and $(t,x) \mapsto {}^{\mathrm{c}}\mathrm{D}_x^\alpha u(t,x)$ is continuous on $[0,\infty)^2$;
(4) u satisfies ${}^{\mathrm{c}}\mathrm{D}_t^\alpha u = {}^{\mathrm{c}}\mathrm{D}_x^\alpha u$ in $C([0,\infty)^2)$.

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