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PROGRESSIVE DAMAGE IN QUASI-BRITTLE SOLIDS

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Abstract. *We present a model of material degradation relying upon a local damage law supplemented by convex constraints. This results in a damage model with bounded variation that is shown to share the same features of the so-called Thick Level Set approach. Unlike the original model, in the present formulation the level set-based representation is abandoned in favor of an implicit description of damaged regions, whereby one arrives at a non-local Generalized Standard Model with convex constraints. The solution of a one-dimensional problem demonstrate the capabilities of the proposed approach when simulating initiation and growth of damage in quasi-brittle materials.*

1 INTRODUCTION

Regularized damage formulations have become increasingly popular in the last decades for dealing with problems in Mechanics suffering from spurious mesh sensitivity induced by strain softening [1]. In short, the idea underlying almost all such models is that of using regularized constitutive equations in which some suitably defined length parameters bring to the macro level information about material structure at the fine scale.

Classical regularizations are formulated via gradient or averaging operators. They provide globally smoothed solutions by enforcing a greater regularity either on strains or internal variables that, as a consequence, are no longer defined at the local level.

The same concepts are implicitly present into the so-called *Thick Level Set* (TLS) approach to quasi-brittle fracture [2, 3], whereby progressive damage takes place in a region of finite thickness whose size is an explicit model parameter. Within this framework one possible way to follow the evolution of damage in the solid amounts to continuously tracking the position of layers in a state of progressive damage. In the original formulation [2] this was achieved based on distance functions and level sets, whose knowledge requires to solve an eikonal equation.

In the TLS model one prescribes the shape of the damage function d within the moving layer of thickness l_c where the transition between the sound material and the completely damaged one occurs. In particular, progressive damage that takes place in the transition zone is given as an explicit function of the distance ϕ to the boundary Γ_o of the undamaged portion of the domain under consideration. The latter turns out to be partitioned into three regions: the undamaged part Ω_o, the transition region Ω_c where the damage variable ranges between 0 and 1, and the completely damaged zone Ω_1, see e.g. Figure 1.

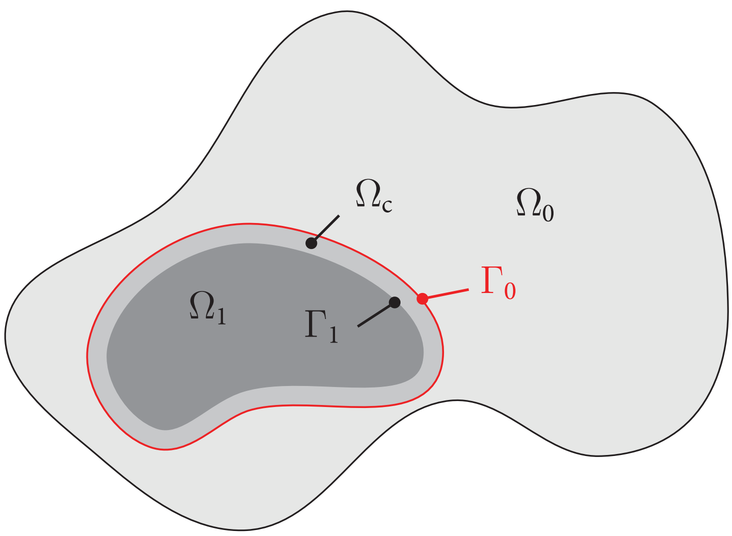

Figure 1: Domain partition in the TLS approach.

The boundary of the transition zone is denoted $\partial\Omega_c = \Gamma_o \cup \Gamma_1$, and points M belonging to it have the following properties:

$$M \in \Gamma_o, \quad d(M) = 0, \quad \phi(M,t) = 0; \tag{1}$$

$$M \in \Gamma_1, \quad d(M) = 1, \quad \phi(M,t) \geq l_c. \tag{2}$$

The zero-level set surface, i.e. the one implicitly defined by equation $\phi(M,t)=0$, describes the motion of the boundary Γ_o, whereas damage is given as an explicit function of the distance ϕ within the transition zone:

$$\begin{aligned} d(\phi) &= 0, & \phi(M,t) &\leq 0, & M &\in \Omega_o \\ \dot{d} &> 0, & 0 \leq \phi(M,t) &\leq l_c, & M &\in \Omega_c \\ d(\phi) &= 1, & \phi(M,t) &\geq l_c, & M &\in \Omega_1 \end{aligned} \tag{3}$$

The function $d(\phi)$ is assumed to be continuously increasing with distance ϕ from the sound material, whereby the inverse function $\phi(d)$ exists. Moreover, the damage derivative along ϕ is bounded by a positive function $f(d)$ and damage evolution is associated to the motion of the interface Γ_o. The above conditions can be summarized as follows:

$$\begin{aligned} d &= d(\phi) \\ ||\nabla\phi|| &= 1 \\ d'(\phi) &\leq f(d) \end{aligned} \tag{4}$$

The present contribution aims at studying an isotropic elastic-damageable model whose constitutive law includes a scalar damage variable subject to two internal constraints. The first one is local and expresses the classical bounds for the order parameter that describes the state of the material within the transition zone:

$$0 \leq d \leq 1. \tag{5}$$

The second condition is non-local and non-classical and amounts to bound the norm of the spatial gradient of damage:

$$||\nabla d|| \leq f(d) \tag{6}$$

that characterizes the present damage model with bounded variation.

It is worth emphasizing that the above inequality embodies the three relationships (4) provided that the function ϕ can be characterized as a signed distance function.

2 THE LOCAL STATE

In the present model the local state of the solid is described by the infinitesimal deformation measure $\boldsymbol{\varepsilon}$, the damage parameter d and its spatial gradient ∇d; the latter enters only the internal constraint equation (6).

The state equations stem from a convex free energy density $w(\boldsymbol{\varepsilon}, d)$; in particular, the thermodynamic forces that are work-conjugate to the state variables $\boldsymbol{\varepsilon}, d$ are the Cauchy stress tensor $\boldsymbol{\sigma}$ and the damage energy release rate Y:

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \qquad Y = -\frac{\partial w}{\partial d}. \tag{7}$$

For an isotropic damage model the free energy density function typically reads:

$$w(\boldsymbol{\varepsilon}, d) = \frac{1}{2} g(d) \mathbb{E}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, \tag{8}$$

where $\mathbb{E}$ is the elastic moduli tensor and $g(d)$ is the scalar degradation function that transforms the sound material into a damaged one.

2.1 The convex constraints

In the present model the damage variable d has to comply with two internal constraints. By its very definition it is subject to the conditions (5), which can in turn be expressed via a unique convex function g_1 embodying the two inequalities:

$$g_1(d) = d(d-1) \leq 0. \tag{9}$$

The second constraint is provided by the condition (6). We shall denote g_2 the function expressing the bound on the gradient of damage:

$$g_2(d) = ||\nabla d|| - f(d) \leq 0. \tag{10}$$

The inequality (10) defines a convex set provided that the function g_2 is convex, i.e. if $f(d)$ is concave, which is equivalent to $-f(d)$ convex. We shall assume in the remainder that $f(0) > 0$. For any concave function f, if it exists a point d_o such that $f(d_o) > 0$, then the set $\mathcal{C} = \{(d, \nabla d)/g_2(d) \leq 0\}$ is nonempty and convex. The proof is straightforward. Let $(d, \nabla d), (d^*, \nabla d^*)$ be two elements of $\mathcal{C}$; their convex combination is by definition an element of $\mathcal{C}$ and meets the condition:

$$||\theta \nabla d + (1-\theta)\nabla d^*|| \leq \theta ||\nabla d|| + (1-\theta)||\nabla d^*|| \leq \theta f(d) + (1-\theta) f(d^*), \qquad \theta \in [0,1] \tag{11}$$

whereby for a concave function f one has:

$$\theta f(d) + (1-\theta) f(d^*) \leq f(\theta d + (1-\theta) d^*) \tag{12}$$

Q.E.D.

The constraints (9) and (10) are introduced in the formulation via two fields of Lagrange multipliers γ_1, γ_2 and the relevant Karush-Kuhn-Tucker conditions:

$$\gamma_i \geq 0; \qquad g_i(d) \leq 0; \qquad \gamma_i g_i(d) = 0. \tag{13}$$

The above relationships put forward the non-dissipative character of the constraints, whereby a potential of the constraints themselves can be defined as:

$$w_\gamma(d, \gamma_i) = \gamma_1 g_1(d) + \gamma_2 g_2(d) \tag{14}$$

which turns out to be convex owing to the convexity of the functions g_1 and g_2.

3 EQUILIBRIUM

We study the equilibrium problem of a solid whose boundary is partitioned into two parts, that is $\partial\Omega_u$, where displacements $\mathbf{u} = \mathbf{u}^d$ are prescribed, and $\partial\Omega_t$ where the surface tractions are given. The problem variables are the displacement field $\mathbf{u}$, the damage field d and the fields of Lagrange multipliers γ_i.

The total potential energy of the system reads:

$$\mathcal{E}(\mathbf{u}, d, \gamma_i) = \int_\Omega w \, \mathrm{d}\Omega - \int_{\partial\Omega_t} \mathbf{t}.\mathbf{u} \, \mathrm{d}S + \int_\Omega w_\gamma \, \mathrm{d}\Omega \tag{15}$$

The functional (15) is defined over the set of kinematically admissible displacements $\mathcal{K}$:

$$\mathcal{K} = \left\{ \mathbf{u}^* | \mathbf{u}^*(M) = \mathbf{u}^d(M), M \in \partial\Omega_u \right\}. \tag{16}$$

3.1 Variation wrt $\mathbf{u}$

For a given damage field, an equilibrium state $\mathbf{u}^{sol}$ is a minimizer for the potential energy over $\mathcal{K}$:

$$\frac{\partial \mathcal{E}}{\partial \mathbf{u}} \cdot \delta \mathbf{u} = 0, \qquad \forall \delta \mathbf{u} \in \{\mathbf{v} | \mathbf{v}(M) = 0, M \in \partial \Omega_u\}. \tag{17}$$

The above condition is equivalent to differential equilibrium and the relevant boundary conditions:

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \quad \operatorname{div} \boldsymbol{\sigma} = 0, \text{ in } \Omega_o; \qquad \boldsymbol{\sigma} \mathbf{n} = \mathbf{t}, \text{ on } \partial \Omega_t. \tag{18}$$

3.2 Variations wrt γ_i

Variations of the potential energy with respect to the Lagrange multipliers define a partition of the domain Ω that reflects the state of the internal constraints:

$$\frac{\partial \mathcal{E}}{\partial \gamma_1} \delta \gamma_1 = \int_\Omega d(d-1) \delta \gamma_1 \, \mathrm{d}\Omega = 0, \tag{19}$$

$$\frac{\partial \mathcal{E}}{\partial \gamma_2} \delta \gamma_2 = \int_\Omega (||\nabla d|| - f(d)) \delta \gamma_2 \, \mathrm{d}\Omega = 0. \tag{20}$$

For a given damage state d, the domain is decomposed into three parts $\Omega = \Omega_o \cup \Omega_c \cup \Omega_1$, that is:

- on Ω_o and Ω_1, $g_1 = 0$ and $\gamma_1 > 0$,
- on Ω_o and Ω_1, $\gamma_2 = 0$ since $g_2 \leq 0$,
- on Ω_c, $\gamma_1 = 0$ since $g_1 < 0$

The domain Ω_c is in turn partitioned into two sub-domains:

- Ω_c^- where $g_2 < 0$ and $\gamma_2 = 0$,
- Ω_c^o where $g_2 = 0$ and $\gamma_2 \geq 0$.

3.3 Variation wrt d

Variation of the potential energy with respect to d and use of the divergence theorem yields:

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial d} \delta d &= \int_\Omega \frac{\partial w}{\partial d} \delta d \, \mathrm{d}\Omega + \int_\Omega \gamma_1 (2d-1) \delta d \, \mathrm{d}\Omega + \int_\Omega \gamma_2 (\frac{\nabla d}{||\nabla d||} \cdot \nabla \delta d - f'(d) \delta d) \, \mathrm{d}\Omega \\ &= -\int_\Omega G \, \delta d \, \mathrm{d}\Omega + \int_{\partial \Omega} \gamma_2 \frac{\nabla d}{||\nabla d||} \cdot \mathbf{n} \, \delta d \, \mathrm{d}S + \int_S [\![\gamma_2 \frac{\nabla d}{||\nabla d||}]\!]_S . \mathbf{n} \, \delta d \, \mathrm{d}S \end{aligned} \tag{21}$$

In the above equation one can recognize three contributions, i.e. a volume integral, a surface integral over the external boundary and an integral over internal discontinuity surfaces.

The volume integral defines the energy release rate G:

$$G = Y - \gamma_1 (2d-1) + \gamma_2 f'(d) + \operatorname{div} \left(\gamma_2 \frac{\nabla d}{||\nabla d||} \right) \tag{22}$$

Whenever $\gamma_2 = 0$ the thermodynamic force G is a local quantity. On the contrary, when $\gamma_2 \neq 0$ the G becomes non-local because of the divergence term originating from the constraint (10).

Concerning the possible discontinuity surfaces, if they are non-dissipative one has:

$$[\![\gamma_2 \frac{\nabla d}{||\nabla d||}]\!]_S \cdot \mathbf{n} = 0. \tag{23}$$

In particular, equation (23) has to be fulfilled over the surface Γ_o where $d = 0^+$ and $||\nabla d|| = f(0^+) > 0$, which in turn implies $\gamma_2 = 0$. Along a discontinuity surface for the gradient of d, damage is continuous and $g_2 = 0$. In this particular case one has:

$$d^+ = d^-, \qquad ||\nabla d^+|| - f(d^+) = ||\nabla d^-|| - f(d^-) = 0, \tag{24}$$

which in turn imply:

$$0 = \mathbf{n}.\nabla d^+ + \mathbf{n}.\nabla d^-, \qquad 0 = (\gamma_2^+ \nabla d^+ - \gamma_2^- \nabla d^-).\mathbf{n}, \qquad 0 < \gamma_2^+, \qquad 0 < \gamma_2^-, \tag{25}$$

whereby $\gamma_2^+ = \gamma_2^- = 0$.

Boundary conditions A major difference of the present formulation compared to classical gradient-enhanced models is that on the boundary $\partial\Omega \cap \partial\Omega_c^-$ the relationship $g_2 < 0$ holds, which in turn implies $\gamma_2 = 0$. The net result is that the boundary condition on the normal derivative of damage is generally no longer homogeneous for the present model of damage with bounded variation.

4 DISSIPATION

Since the constraints (9) et (10) are non-dissipative, the only contribution to the total dissipation of the system stems from the local damage energy release rate Y, that is:

$$-\frac{\partial \mathcal{E}}{\partial d}\, \dot{d} = \int_\Omega G\dot{d}\, \mathrm{d}\Omega = \int_\Omega Y\dot{d}\, \mathrm{d}\Omega \geq 0 \tag{26}$$

Following [4] we assume that damage evolution emanates from a pseudo-potential of dissipation, which is a convex and degree-one positively homogenous function of $\dot{d}$. Whenever $\mathcal{D}$ is a smooth function one arrives at a kinetic equation for d:

$$G = \frac{\partial \mathcal{D}}{\partial \dot{d}} \tag{27}$$

which is equivalent to the classical Biot equation [5]:

$$\frac{\partial \mathcal{E}}{\partial d} + \frac{\partial \mathcal{D}}{\partial \dot{d}} = 0 \tag{28}$$

For the non-smooth case the dissipation pseudo-potential reads:

$$\mathcal{D}(d^*) = \begin{cases} Y_c\, d^*, & \text{if } d^* \geq 0, \\ +\infty, & \text{otherwise} \end{cases} \tag{29}$$

and damage evolution is governed by a normality rule expressed as:

$$G - Y_c \leq 0, \qquad \dot{d} \geq 0, \qquad (G - Y_c)\dot{d} = 0. \tag{30}$$

Analogous to Linear Elastic Fracture Mechanics [6] a dissipated energy $\mathcal{G}(d)$ can be defined whenever the threshold Y_c depends from a unique parameter

$$\mathcal{G}(d) = \int_0^d Y_c(\alpha)\,\mathrm{d}\alpha \tag{31}$$

In this case the total energy of the system reads:

$$\mathcal{W}(\mathbf{u}, d, \gamma_i) = \mathcal{E}(\mathbf{u}, d, \gamma_i) + \mathcal{G}(d) \tag{32}$$

and normal damage evolution can be recast in the form of a variational inequality:

$$\dot{d} \geq 0, \qquad \frac{\partial \mathcal{W}}{\partial d}(\delta d - \dot{d}) \geq 0, \qquad \forall \delta d \geq 0. \tag{33}$$

The normality law (30) amounts to a partial differential equation that provides γ_2 on the region Ω_c^o, where $\gamma_1 = 0$ since $g_1 < 0$. In particular, for $\dot{d} > 0$ equation (30) yields:

$$G = Y + \gamma_2 f'(d) + \mathrm{div}\left(\gamma_2 \frac{\nabla d}{||\nabla d||}\right) = Y_c. \tag{34}$$

Over the domain Ω_c^o one has $g_2 = 0$ whereby the above equation becomes

$$(Y - Y_c) f(d) + \mathrm{div}(\gamma_2 \nabla d) = 0. \tag{35}$$

Now consider an iso-damage surface and define the function $\phi(d)$ such that

$$\nabla d(\phi) = f(d)\,\nabla\phi, \qquad ||\nabla\phi|| = 1 \tag{36}$$

Evidently, the function ϕ is a signed distance from the surface $d = 0^+$, and $d'(\phi) = f(d)$. Let $M_o(\alpha, \beta)$ be a point of Γ_o; any point $M \in \Omega_c^o$ has coordinates (α, β, z) such that

$$M(\alpha, \beta, z) = M_o(\alpha, \beta) + z\,\nabla\phi \qquad M_o \in \Gamma_o \tag{37}$$

Now consider the integral of (35) over a truncated cone of axis ∇d and delimited by surfaces $\mathrm{d}S(z = 0)$ and $\mathrm{d}S(z = l) = j(z)\ \mathrm{d}S(z = 0)$. The term $j(z)$ accounts for area change due to geometric curvature when the surfaces are described in a local basis attached to point $M_o \in \Gamma_o$. The integral of the divergence term $\mathrm{div}(\gamma_2 \nabla d)$ reduces to the only contributions over surfaces ($\mathrm{d}S(0),\ \mathrm{d}S(l)$) where $\gamma_2 = 0$. In this way one obtains the energy release rate $\hat{G}$ that is associated to the motion of the transition layer of finite thickness $\phi(d) = l_c$ originally introduced in the TLS model [2]:

$$\hat{G} = \int_{z=0}^{z=l} Y f(d) j(\phi)\,\mathrm{d}\phi \tag{38}$$

whereby damage evolution takes place under the condition:

$$\hat{G} = \int_{z=0}^{z=l} Y_c f(d) j(\phi)\,\mathrm{d}\phi = \hat{G}_c. \tag{39}$$

The above arguments show that the coupling of a local damage model with the constraint equation (6) allows one to recover the features of the Thick Level Set approach in its original form. However, the present model of damage with bounded variation can be implemented without using level sets since equation (35) that provides damage evolution is completely independent from the notion of distance function.

5 THE TRACTION BAR

We consider a one-dimensional bar of length L subject to an increasing elongation. In order to study the influence of the constraint equations we choose the following form for the free energy density:

$$w = \frac{1}{2}(1-d)E\boldsymbol{\varepsilon}^2 + \frac{k}{2}||\nabla d||^2, \tag{40}$$

where $k > 0$ is a regularization parameter with dimensions of a force.

The constraint function g_2 is taken as:

$$g_2(d) = ||\nabla d|| - \frac{1}{l_c} \leq 0. \tag{41}$$

which characterizes a linear damage distribution.

The study of the traction bar whose free energy density contains a quadratic term in ∇d allows to determine the conditions that the parameter k has to comply with in order to obtain a solution which is coherent with the one of the non-regularized problem.

We obtain two families of solutions. The first one is a homogeneous solution, whereas the second one gives the initiation and growth of a defect. We assume that such a defect nucleates at point $x = 0$. For the regularized model the solution depends on the value of the parameter k. Actually, the presence of the regularization term changes the definition of the energy release rate (22) as follows

$$G^k = G + k\Delta d \tag{42}$$

and also the boundary conditions turn out to be modified and now require

$$\nabla d \cdot \mathbf{n} = 0 \tag{43}$$

over the boundary of Ω_c. During damage growth two phases of evolution can be distinguished. The first one during which $||\nabla d|| < 1/l_c$ et $\gamma_2 = 0$, and a second one during which the constraint is satisfied with the equality. One can show that the second phase can take place only if k is sufficiently small.

Actually, condition $G^k = Y_c$ with $\gamma_2 = 0$ allows to obtain the damage distribution along the bar $d(x)$ by integrating

$$\frac{\Sigma^2}{2E(1-d)} - Y_c d + \frac{k}{2}(\nabla d)^2 = C. \tag{44}$$

The integration constant C is provided by the boundary conditions

$$\nabla d(x=0)\cdot \mathbf{e}_x = \nabla d(x=L)\cdot \mathbf{e}_x = 0; \qquad d(x=L) = 0. \tag{45}$$

Now assume $d(0) = d_m$. We infer that $C = \Sigma^2/2E$ over $[0, L]$ and obtain the load Σ as:

$$\Sigma^2 = 2EY_c(1-d_m) = \Sigma_c^2(1-d_m). \tag{46}$$

Such a solution holds provided that $||\nabla d|| < 1/l_c$, which in turn occurs if

$$K = \frac{k}{2\,l_c^2\,Y_c} < 1. \tag{47}$$

Under this last condition a region $[x_a, x_b]$ appears for which $d = d_a + \dfrac{x_b - x}{l_c}$. Over this segment $\gamma_2 \geq 0$, $\gamma_2(x_a) = \gamma_2(x_b) = 0$, and the gradient $\nabla d \cdot \mathbf{e}_x$ is continuous at such points.

We then obtain $K = d_a d_b$ with

$$d_a = \frac{1}{2}(d_m + K)(1 + \sqrt{1 - \frac{4K}{(d_m + K)^2}}) \tag{48}$$

$$d_b = \frac{1}{2}(d_m + K)(1 - \sqrt{1 - \frac{4K}{(d_m + K)^2}}) \tag{49}$$

The response curves of the traction bar are shown in Figure 2.

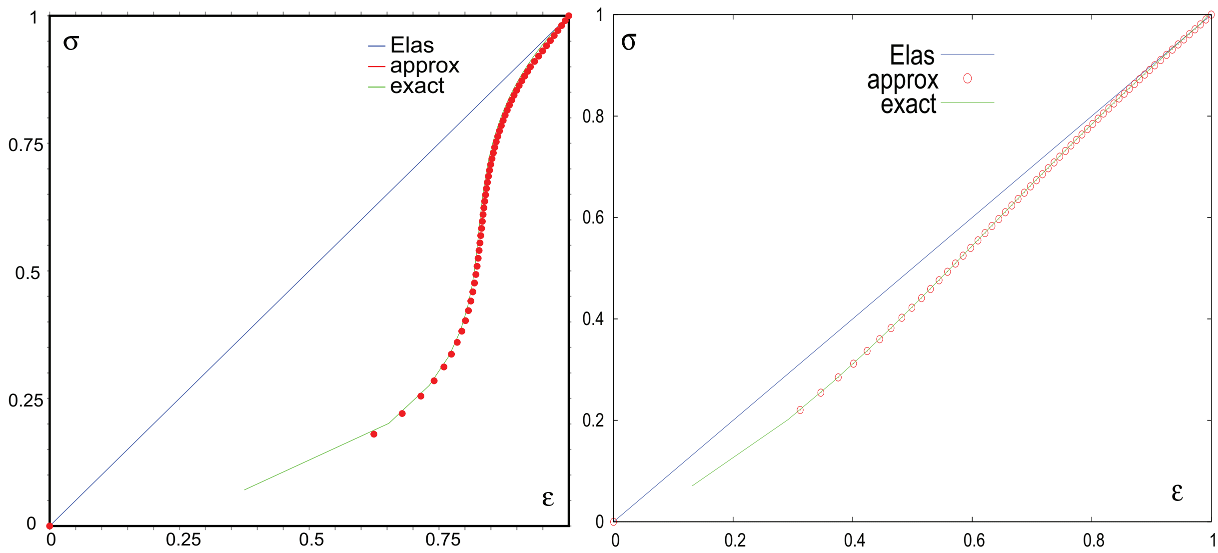

Figure 2: Response curves $\Sigma - \varepsilon$ for $K = 0.01$. $L = 1$ (left), and $L = 5$ (right).

A value $K = 0.1$ is not well-suited since the resulting damage distribution broadens too much. For a given K, the value d_o for which the constraint g_2 is fulfilled at a point is given by:

$$d_o = 2\sqrt{K} - K, \tag{50}$$

which implies that K must be lower than 1. The smaller K, the more localized is damage around the position $x = 0$ during the first phase.

It is worth emphasizing that the solution we presented here is an approximation of the exact solution. Actually, during the phase that precedes the fulfillment of the condition (41) with the equality, the damage evolution is characterized by a loading region $[0, x_c]$ and an unloading region $[x_c, x_M]$ over the segment of lengt x_M. The function $x_c(d_m)$ is first decreasing and then increasing beyond the value x_M. This is apparent on the damage response depicted in Figure 3 when the curves $d(x)$ do intersect each other.

The condition $\dot{d} > 0$ is fulfilled only when the region where the condition $g_2 = 0$ is firmly established. The smaller K, the more rapidly the condition $g_2 = 0$ is satisfied during the loading history. The value of K must be small enough in order to obtain an abscissa x_M much smaller than l_c so that the linear part of the damage distribution is the dominating one.

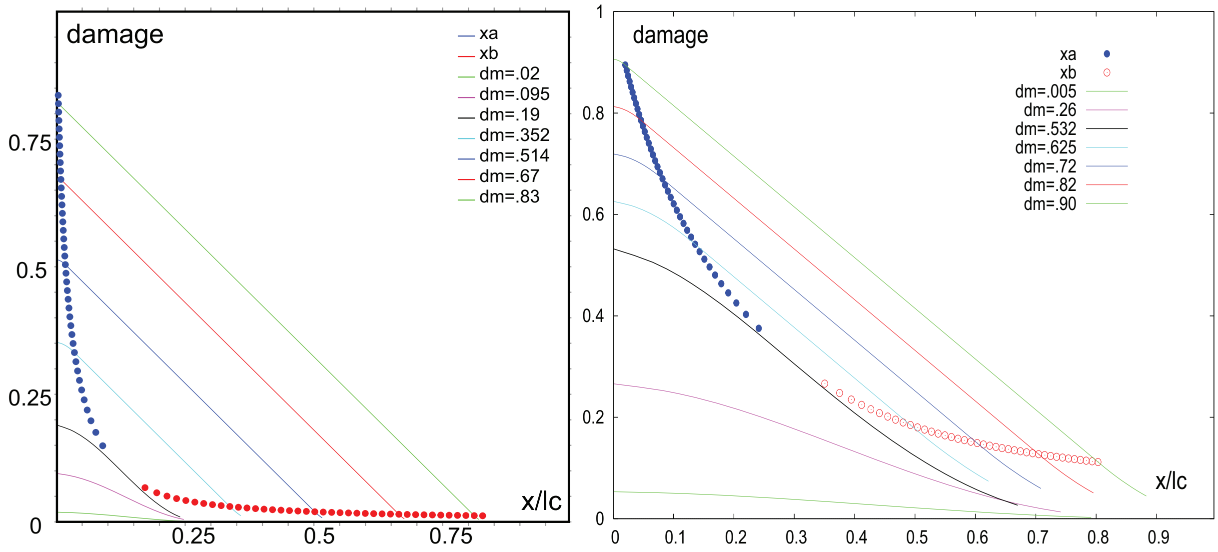

Figure 3: Damage profiles $d(x)$ for $K = 0.01$ (left) and $K = 0.1$ (right).

6 CLOSURE

We presented a new model of damage with bounded variation that fits in the *Thick Level Set* approach. However, in the present model the introduction of level sets with all consequent difficulties can be abandoned, since the information necessary to track the evolution of interphases where progressive damage occurs is implicitly contained in the Lagrange multipliers fields associated to two constraint equations described via convex functions.

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