Improvement of Technical Efficiency of Firm Groups

Walter Briec, Stéphane Mussard

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Walter Briec[†]
Lamps
Université de Perpignan

Stéphane Mussard[‡]
Chrome
Université de Nîmes

Abstract

Cooperation between firms can never improve the technical efficiency of any firm coalition. The directional distance function, by virtue of its additive nature, is a useful tool that outlines this impossibility. In this paper, the additive aggregation scheme of input/output vectors is generalized according to an aggregator. Accordingly, cooperation between firms may increase the technical efficiency of the firm group. This improvement is shown to be compatible with nonjoint semilattice technologies that bring out either output or input (weak) complementarity. Firm games are investigated to show that firms may merge on the basis of their inputs due to constraints imposed on outputs. Conversely, they may merge with respect to the outputs they can produce because of limitations imposed on inputs.

Keywords: Productivity and competitiveness, Aggregation, Cooperative games, Distance functions, Technical efficiency.

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1 Introduction

The cooperation between firms can never improve the technical efficiency of any given firm coalition (industry). This impossibility has become a standard result in the productivity measurement literature, see Briec, Dervaux

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[†]Lamps and IAE, Université de Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan Cedex ; Email: briec@univ-perp.fr

[‡]Chrome Université de Nîmes, Rue du Dr Georges Salan, 30000 Nîmes, - e-mail: stephane.mussard@unimes.fr, Research fellow Grédi Université de Sherbrooke, Liser Luxembourg.

and Leleu (2003) or Färe, Grosskopf and Zelenyuk (2008). The *firm game* is the transferable utility game (TU-game) that exhibits this impossibility (see Briec and Mussard, 2014), in other words, the core interior of the firm game is empty (see also *DEA production games* introduced by Lozano (2012, 2013) in which different organizations have the possibility to merge). The result is derived from the directional distance function applied to the technology at the industry level, which relies on the standard sum of technology sets, see *e.g.* Färe, Grosskopf and Li (1992) and Li and Ng (1995). Li and Ng (1995) demonstrate that the standard sum of technology sets may yield different results, particularly a bad representation of the technical efficiency of the industry. In other words, the choice of the aggregation process, for aggregating input/outputs vectors, or equivalently aggregating technologies of firms, has a crucial impact on productivity measurement. In *firm games*, Briec and Mussard (2014) show that any given firm coalition may always increase its allocative efficiency if the input/output vectors are simply aggregated with the standard sum. However, the impossibility of improving technical efficiency holds true, *i.e.*, the inefficiency of the industry is always greater than the sum of the inefficiencies of each firm. In other words, the *technical bias* that represents the difference between the two aforementioned inefficiencies is always positive. Coalitions of firms are said to be sub-efficient because firm cooperation increases the technical inefficiency of the group. As a consequence, the core interior of the firm game is empty: no firm can improve its technical efficiency by joining any given coalition. As pointed out by a referee, there is another interpretation when the sum of the individual firms' inefficiencies is lower than the inefficiency of the aggregate operating point measured on the aggregate technology. This means that "the cooperation associated to the aggregate technology allows finding efficient operation points that can lead to more inefficiencies removal and hence to a larger efficiency improvement than what is possible for the individual firms." Following this interpretation, when the core interior is empty, then there exists no blocking coalition for which the aggregate technology exhibits efficient operations points.

In this paper, the aggregation of technology sets is generalized thanks to a Φ_α-aggregator inspired from Ben-Tal (1977) who studied its algebraic properties underlying the generalized mean introduced by Hardy, Littlewood and Pólya (1934) and characterized by Aczél (1966) and Eichorn (1979). The aggregation bias, *i.e.* the difference between the inefficiency of the firm coalition and the sum of each firm's inefficiency, takes different forms with respect to the nature of the aggregator Φ_α. (i) If the bias is positive, the cooperation between firms is impossible. (ii) If the bias is negative, the aggregate inefficiency of any given coalition of firms decreases with cooperation. To study the sign of the aggregation bias, two limit cases – in the neighborhood of infinity – allow Kholi's (1983) nonjoint technology to be characterized for a group of firms. This coalitional technology is shown to be consistent with the traditional assumptions of the literature. It is an upper (lower) semilattice satisfying a free disposal assumption with either complementarity in outputs or in inputs. It is shown that this aggregated technology enables the paradox of the positive technical bias to be solved. Indeed, the negative bias is obtained by specifying two firm games, the input fixed firm game and the output fixed firm game. The input fixed firm game postulates that the cooperation between firms is related to the use of inputs only, because some production constraints are imposed on the industrial sector to limit the number of outputs (production quota). The output fixed firm game allows the amount of outputs to be improved when the firms are limited by a given amount of inputs (resource limitations). These results are derived using the directional distance function, introduced by Chambers, Chung and Färe (1996, 1998), applied on aggregated data.[1] The input [output] fixed firm game defined on aggregated upper semilattice technologies yields a negative [positive] bias. On the contrary, the output [input] fixed firm game defined on aggregated lower semilattice technologies yields a negative [positive] bias. Finally, if the directional distance function defined on semilattice technologies is submodular, then the core of the firm game may be partitioned in order to find aggregate operating points measured on the aggregate semilattice technology for which the cooperation between firms improves technical efficiency (negative bias). In this sense, the aim of this paper is to find theoretical conditions, *i.e.* aggregators for technologies of firm groups giving rise to particular directional distance functions, that provide an improvement of technical efficiency (that is the non-vacuity of the core of the game).

The outline of the paper is as follows. Section 2 sets the notations and the motivations. Section 3 is devoted to the definitions of the aggregated technologies with examples of negative technical bias. Section 4 presents the characterization of nonjoint aggregated semilattice technologies with either input or output (weak) complementarity. Section 5 explores the negative bias supported by the directional distance function in firm games. Section 6 provides an example of firm games that outlines a negative technical bias. Section 7 closes the article.

[1] See also the benefit function of Luenberger (1992, 1995).

2 Setup and Motivations

The set of firms (players) is $\mathcal{K} := \{1, \dots, |\mathcal{K}|\}$, where $|\mathcal{K}| \equiv \#\{\mathcal{K}\}$. The subsets of the grand coalition $\mathcal{K}$ are denoted by $\mathcal{S}$ such that $\mathcal{S} \subseteq \mathcal{K}$. The interior of a set E is $\overset{\circ}{E}$, $\mathbb{R}_+$ is the non-negative part of the real line and $\mathbb{R}_{++}$ its positive part (with $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ its n-dimensional representation). 0_n $[0_m]$ is the n-dimensional [m-dimensional] vector of zeros, $\mathbb{1}_d$ is the d-dimensional vector of ones, $\mathbb{N}$ the set of weakly positive integers, $\geq$ [$\leq$] denotes inequalities over scalars and $\geqslant$ [$\leqslant$] over vectors, and finally $[d] := \{1, \dots, d\}$ with $d \in \mathbb{N} \setminus \{1\}$.

The firms use inputs and produce outputs. Let $x \in \mathbb{R}^n_+$ and $y \in \mathbb{R}^m_+$ be the input and output vectors, respectively. These vectors will be sometimes denoted by $z := (x, y) \in \mathbb{R}^d_+$ with $d = n + m$. The firm technology T satisfies the following basic assumptions:

(T1): $(0_n, 0_m) \in T$, $(0_n, y) \in T \Longrightarrow y = 0_m$ *i.e.*, no free lunch;

(T2): the set $A(x) = \{(u, y) \in T : u \leqslant x\}$ of dominating observations is bounded for all $x \in \mathbb{R}^n_+$, *i.e.*, infinite outputs cannot be obtained from a finite input vector;

(T3): T is closed;

(T4): $\forall z = (x, y) \in T$, $(x, -y) \leqslant (u, -v) \Longrightarrow (u, v) \in T$, *i.e.*, fewer outputs can always be produced with more inputs;

(T5): $\forall \beta \geq 0$, if $(x, y) \in T$ then $(\beta x, \beta y) \in T$, *i.e.* the technology satisfies constant returns to scale. Although this assumption is not central in the paper, we will prove that the technology proposed in Section 4 may respect this requirement.

Given a production set one can define an input correspondence $L : \mathbb{R}^m_+ \longrightarrow 2^{\mathbb{R}^n_+}$ and an output correspondence $P : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^m_+}$ such that:

$$T = \big\{(x, y) \in \mathbb{R}^{n+m}_+ : x \in L(y)\big\} = \big\{(x, y) \in \mathbb{R}^{n+m}_+ : y \in P(x)\big\}. \tag{2.1}$$

The literature on the aggregation of technologies (see *e.g.* Li, 1995; Li and Ng, 1995) defines the technology of the grand coalition as a standard sum of input and output vectors. Let $(x^k, y^k) \in \mathbb{R}^{n+m}_+$ be the input-output vectors of firm k whose technology is T^k. Following this specification, the technology of any given coalition $\mathcal{S}$ is the standard sum of the technologies T^k of each firm $k \in \mathcal{S}$:

$$T^{\mathcal{S}} := \sum_{k \in \mathcal{S}} T^k = \Big\{ \big(\sum_{k \in \mathcal{S}} x^k, \sum_{k \in \mathcal{S}} y^k \big) : (x^k, y^k) \in T^k, k \in \mathcal{S} \Big\}. \tag{2.2}$$

Any given technology enables the technical efficiency of the firms to be measured thanks to distance functions (the less the distance of the couple (x, y) to the technology frontier, the more the efficiency). The directional distance function introduced by Chambers, Chung and Färe (1996, 1998) is expressed as, for a single firm k,

$$D_T(x^k, y^k; g) = \sup_{\delta} \left\{ \delta \in \mathbb{R} : (x^k - \delta g_i, y^k + \delta g_o) \in T^k \right\}, \tag{2.3}$$

which gauges input and output variation in the direction of a pre-assigned vector $g = (g_i, g_o) \in \mathbb{R}_+^{n+m}$. In the sequel, the directional distance function is such that $D_T(x^k, y^k; g) \geq 0$, *i.e.*, the cases of infeasibilities for which $(x^k, y^k) \notin T^k$ are not reported. For a group of $|\mathcal{S}|$ firms with technology $T^{\mathcal{S}}$, the technical aggregation bias is defined as follows (see Briec and Mussard, 2014):

$$AB(\mathcal{S}; g) := D_{T^{\mathcal{S}}} \left(\sum_{k \in \mathcal{S}} (x^k, y^k); g \right) - \sum_{k \in \mathcal{S}} D_{T^k}(x^k, y^k; g). \tag{2.4}$$

It provides the loss of technical efficiency due to the cooperation between the firms of group $\mathcal{S}$, for any given $\mathcal{S} \subseteq \mathcal{K}$. The aggregation bias may be null. In this case, the exact aggregation condition is,

$$D_{T^{\mathcal{S}}} \left(\sum_{k \in \mathcal{S}} (x^k, y^k); g \right) = \sum_{k \in \mathcal{S}} D_{T^k}(x^k, y^k; g). \tag{2.5}$$

Under the assumptions (T1)-(T4), the exact aggregation is possible whenever $T^{\mathcal{S}} = \sum_{k \in \mathcal{S}} T^k$:

(i) if the technologies T^k are identical and the input set is one-dimensional;

(ii) or if the firms use the same technique and (T5) holds.[2]

In this paper, we are particularly interested in the impossibility related to directional distance functions: the collaboration between firms cannot improve the technical efficiency of the firm group. This impossibility was proven independently by Briec *et al.* (2003) and Färe, Grosskopf and Zelenyuk (2008). Their result, which may be formalized as $AB(\mathcal{K}) \geq 0$, provides the loss of technical efficiency due to cooperation. In other words, the distance of the group is, for any given type of technology, always greater than the sum of the individual distances. The same conclusion holds true for all possible coalitions $\mathcal{S} \subseteq \mathcal{K}$, $AB(\mathcal{S}) \geq 0$. This result reports a sub-efficiency inherent to the cooperation between firms in a cost-sharing problem (in which cooperation never reduces the cost of technical inefficiency).[3] The problem

[2] If the firms of coalition $\mathcal{S}$ use the same technique, then $T^1 = \cdots = T^{|\mathcal{S}|}$ such that, for all $k \in \mathcal{S}$, $x_i^k = \alpha_{i,j} x_j^k$, $y_m^k = \beta_{m,n} y_n^k$ and $y_m^k = \gamma_{m,i} x_i^k$, with $\alpha_{i,j}, \beta_{l,q}, \gamma_{l,i}$ constants for $i, j \in \{1, \ldots, m\}$ and $l, q \in \{1, \ldots, n\}$.

[3] It is noteworthy that technology sets are represented by distance functions (primal representation) and also by cost functions (dual representations).

of sub-efficiency is inherent to the standard additive form of the aggregation of production vectors. To be precise, the measurement of technical efficiency and its bias may depend on particular aggregators of input/output vectors that confer the technology some particular algebraic structures such as semilattice technologies studied in Section 4. These production technologies exhibit either output or input (weak) complementarity. Accordingly, it is shown that the impossibility of improving technical efficiency does not hold any more (Section 5).

3 Aggregated Technologies: Definitions and Examples

Andriamasy *et al.* (2017) investigate the characterization of the limits of generalized convex technologies dealing with Constant-Elasticity-Substitution and Constant-Elasticity-Transformation (CES-CET) models.[4] In what follows, generalized convex technologies are presented thanks to power functions. Some examples show that the directional distance function applied to aggregated data is relevant to negative technical bias for these technologies.

3.1 Power Functions

For all $\alpha \in (0, +\infty)$, let $\phi_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be the map defined by:

$$\phi_\alpha(\lambda) = \begin{cases} \lambda^\alpha & \text{if } \lambda \geq 0 \\ -|\lambda|^\alpha & \text{if } \lambda \leq 0. \end{cases} \tag{3.1}$$

For all $\alpha \neq 0$, the reciprocal map is $\phi_\alpha^{-1} := \phi_{\frac{1}{\alpha}}$. It is first quite straightforward to state that: (i) ϕ_α is defined over $\mathbb{R}_+$; (ii) ϕ_α is continuous over $\mathbb{R}_+$; (iii) ϕ_α is bijective over $\mathbb{R}_+$. Throughout the section, for any vector $z = (z_1, \ldots, z_d) \in \mathbb{R}_+^d$ we use the following notations:

$$\Phi_\alpha(z) = \big(\phi_\alpha(z_1), \ldots, \phi_\alpha(z_d)\big). \tag{3.2}$$

It is then natural to introduce the following algebraic operation over $\mathbb{R}_+^d$:

$$z \overset{\alpha}{+} u = \Phi_\alpha^{-1}\big(\Phi_\alpha(z) + \Phi_\alpha(u)\big) \quad \text{and} \quad \lambda \overset{\alpha}{\cdot} z = \Phi_\alpha^{-1}\left(\phi_\alpha(\lambda)\Phi_\alpha(z)\right). \tag{3.3}$$

In this case $(\phi_\alpha(\mathbb{R}), +, \cdot)$ is a scalar field since $\phi_\alpha(\mathbb{R}) = \mathbb{R}$.

Let us focus on the case $\alpha \in (-\infty, 0)$. The map $\lambda \mapsto \lambda^\alpha$ is not defined at point $\lambda = 0$. Thus, it is not possible to construct a bijective endomorphism

[4]On this ground, Ravelojaona (2019) introduces a generalized directional distance functions for the measurement of technical efficiency based on transformed data.

on $\mathbb{R}$. However, it is possible to construct an operation preserving at least associativity. For all $\alpha \in (-\infty, 0)$ we consider the function ϕ_α defined by:

$$\phi_\alpha(\lambda) = \begin{cases} \lambda^\alpha & \text{if } \lambda > 0 \\ -(|\lambda|)^\alpha & \text{if } \lambda < 0 \\ +\infty & \text{if } \lambda = 0. \end{cases} \tag{3.4}$$

In such a case $M := \phi_\alpha(\mathbb{R}) = \mathbb{R}\backslash\{0\} \cup \{+\infty\}$. Moreover, let us construct the application $\Phi_\alpha : \mathbb{R}^d \longrightarrow M^d$, defined by $\Phi_\alpha(z_1, \ldots, z_d) = (\phi_\alpha(z_1), \ldots, \phi_\alpha(z_d))$. For all $\alpha < 0$, the algebraic operators $\overset{\alpha}{+}$ and $\overset{\alpha}{\cdot}$ are also defined by:

$$z \overset{\alpha}{+} u = {\Phi_\alpha}^{-1}\left(\Phi_\alpha(z) + \Phi_\alpha(u)\right) \text{ and } \lambda \overset{\alpha}{\cdot} z = {\Phi_\alpha}^{-1}\left(\phi_\alpha(\lambda) \cdot \Phi_\alpha(z)\right). \tag{3.5}$$

In such a case $(\mathbb{R}, \overset{\alpha}{+}, \overset{\alpha}{\cdot})$ is not a scalar field because there is not a neutral element. Notice that $(\mathbb{R}, \overset{\alpha}{+}, \overset{\alpha}{\cdot})$ admits 0 as an absorbing element. It is easy to check that for all $\lambda \in \mathbb{R}$, $0 \overset{\alpha}{+} \lambda = 0$. This comes from the fact that for all $\mu \in M$, $\mu + \infty = \infty \in M$. Thus $\left(\mathbb{R}^d, \overset{\alpha}{+}, \overset{\alpha}{\cdot}\right)$ is not a Φ_α-vector space. However, the addition $\overset{\alpha}{+}$ is well defined over $\mathbb{R}^d$ and it is trivial to check that associativity holds.

3.2 Definitions

According to the properties of the power function, let us investigate the Φ_α-aggregator introduced by Ben-Tal (1977). For all $z = (x, y) \in \mathbb{R}^d_+$, an one-dimensional aggregator is given by:

$$\sum_{j\in[d]}^{\phi_\alpha} z_j := \begin{cases} \phi_\alpha^{-1}\left(\sum_{j\in[d]} \phi_\alpha(z_j)\right) & \forall \alpha \neq 0 \\ \prod_{j\in[d]} z_j & \alpha = 0. \end{cases} \tag{3.6}$$

In particular, if $\alpha < 0$,

$$\sum_{j\in[d]}^{\phi_\alpha} z_j = \begin{cases} \phi_\alpha^{-1}\left(\sum_{j\in[d]} \phi_\alpha(z_j)\right) & \text{if } \min_j z_j > 0 \\ 0 & \text{if } \min_j z_j = 0. \end{cases} \tag{3.7}$$

In order to aggregate technologies, *i.e.* input/output vectors, the aggregation is made dimension by dimension, taking recourse to the Φ_α-aggregator.

Definition 3.1 – **Φ_α-Aggregator** – *Let $\Phi_\alpha : \mathbb{R}^d_+ \longrightarrow M^d_+$ defined for all $\alpha \in \mathbb{R}$ such that $\Phi_\alpha(z_1, \ldots, z_d) = (\phi_\alpha(z_1), \ldots, \phi_\alpha(z_d))$. For all collections $Z := \{z^k : k \in \mathcal{S}\} \in \mathbb{R}^d_+$, the Φ_α-aggregator is:*

$$\sum_{k\in\mathcal{S}}^{\Phi_\alpha} z^k := \left(\sum_{k\in\mathcal{S}}^{\phi_\alpha} z_1^k, \ldots, \sum_{k\in\mathcal{S}}^{\phi_\alpha} z_d^k\right).$$

Aggregated technologies based on the Φ_α-aggregator are defined as $T \overset{\alpha}{+} T = \{z \overset{\alpha}{+} u : z, u \in T\}$. Equivalently $T \overset{\alpha}{+} T = \Phi_\alpha^{-1}(\Phi_\alpha(T) + \Phi_\alpha(T))$. For several firms $k \in \mathcal{S}$, with different technologies, the aggregated technology of coalition $\mathcal{S}$ is defined as follows.

Definition 3.2 – **Φ_α-Aggregated technologies** – *For all aggregator $\Phi_\alpha : \mathbb{R}_+^{n+m} \longrightarrow M_+^{n+m}$, an aggregated technology $T_{\Phi_\alpha}^{\mathcal{S}}$ is, for all $\mathcal{S} \subseteq \mathcal{K}$ and $|\mathcal{S}| \geq 2$,*

$$T_{\Phi_\alpha}^{\mathcal{S}} := \sum_{k \in \mathcal{S}}^{\Phi_\alpha} T^k. \tag{3.8}$$

When $\alpha = 0$, a multi-output Cobb-Douglas technology is designed *i.e.* the product of input vectors provides a multidimensional output, see Andriamasy *et al.* (2017). When $\alpha = 1$, the well-known aggregation over sets is obtained, see Li (1995) and Li and Ng (1995).

3.3 Negative technical bias: examples

The Φ_α-aggregator enables the technical bias inherent to the directional distance function to be negative, *i.e.*, the improvement of technical efficiency of the firm group is due to the cooperation between firms. The technical aggregation bias is defined as, for all $\mathcal{S} \subseteq \mathcal{K}$ and $|\mathcal{S}| \geq 2$,

$$AB_\alpha(\mathcal{S}; g) := D_{T^{\mathcal{S}}}\left(\sum_{k \in \mathcal{S}}^{\Phi_\alpha}(x^k, y^k); g\right) - \sum_{k \in \mathcal{S}}^{\phi_\alpha} D_{T^k}(x^k, y^k; g). \tag{3.9}$$

Example 3.1 *Suppose that $\mathcal{K} = \{1, 2\}$ and that $T^k = \{(x, y) \in \mathbb{R}_+^3 : y^\alpha - (x_1)^\alpha - (x_2)^\alpha \leq 0\}$ for $k = 1, 2$. Assume moreover that for $k = 1$, we have $(x^1, y^1) = (2, 1, 1)$ and for $k = 2$, $(x^2, y^2) = (1, 2, 1)$. Setting $T^1 = T^2$, it is possible to find some α such that $AB_\alpha(\mathcal{S}; g) \lesseqqgtr 0$.*

Set $T^0 := T^1 = T^2$. By construction T^0 is quasi-linear and satisfies the constant returns to scale assumption (T5), then:

$$\sum_{k=1,2}^{\Phi_\alpha} T_\alpha^k = T^0.$$

Setting $g = (1, 1, 0)$, we have $D_{T^0}(2, 1, 1; 1, 1, 0) = 1$ and $D_{T^0}(1, 2, 1; 1, 1, 0) = 1$. It follows that:

$$\sum_{k=1,2}^{\alpha} D_{T_\alpha^k}(x^k, y^k; 1, 1, 0) = \big((D_{T^0}(2, 1, 1; 1, 1, 0))^\alpha + (D_{T^0}(1, 2, 1; 1, 1, 0))^\alpha\big)^{\frac{1}{\alpha}} = 2^{\frac{1}{\alpha}}.$$

Moreover,

$$\sum_{k=1,2}^{\Phi_\alpha}(x^k,y^k)=\Big((2^\alpha+1^\alpha)^{\frac{1}{\alpha}},(1^\alpha+2^\alpha)^{\frac{1}{\alpha}},(1^\alpha+1^\alpha)^{\frac{1}{\alpha}}\Big)=\Big((1+2^\alpha)^{\frac{1}{\alpha}},(1+2^\alpha)^{\frac{1}{\alpha}},2^{\frac{1}{\alpha}}\Big).$$

The input set is by definition $L^0(2^{\frac{1}{\alpha}})=\{(x_1,x_2)\in\mathbb{R}^2_+:(x_1)^\alpha+(x_2)^\alpha\geq 2^{\frac{1}{\alpha}}\}$.
It follows that:

$$D_{T^{\mathcal{S}}_\alpha}\Big(\sum_{k=1,2}^{\Phi_\alpha}(x^k,y^k);1,1,0\Big)=D_{T^{\mathcal{S}}_\alpha}\Big((1+2^\alpha)^{\frac{1}{\alpha}},(1+2^\alpha)^{\frac{1}{\alpha}},2^{\frac{1}{\alpha}};1,1,0\Big)=(1+2^\alpha)^{\frac{1}{\alpha}}-1.$$

Thus,

$$AB_\alpha=(1+2^\alpha)^{\frac{1}{\alpha}}-1-2^{\frac{1}{\alpha}}.$$

- *If* $\alpha=1$*, we have*
$$AB_\alpha=3-1-2=0.$$
- *If* $\alpha=1/2$*, we have*
$$AB_\alpha=(1+\sqrt{2})^2-1-2^2=(1+\sqrt{2})^2-5>0.$$
- *If* $\alpha=2$*, we have*
$$AB_\alpha=\sqrt{5}-1-\sqrt{2}<0.$$

It is also possible to show, when $\alpha=0$, that the technical bias is either positive or negative.

Example 3.2 *Let* $z^2:=(x^2,y^2)=\mathbb{1}_d$ *and* $\mathcal{S}$ *a coalition of two firms,* $k=1,2$*, such that* $T^{\mathcal{S}}_\alpha=T^1_\alpha$*. Setting* $\alpha=0$*, it can be shown that* $AB_0(\mathcal{S};g)\lesseqqgtr 0$*. For* $\alpha=0$*,*

$$AB_0(\mathcal{S};g)=D_{T^{\mathcal{S}}_0}\left(\prod_{k=1,2}z^k;g\right)-D_{T^1_0}\left(z^1;g\right)\cdot D_{T^2_0}\left(z^2;g\right).$$

Since $T^{\mathcal{S}}_0=T^1_0$*, it comes that*

$$D_{T^{\mathcal{S}}_0}\left(\prod_{k=1,2}z^k;g\right)=D_{T^1_0}\left(z^1;g\right).$$

By definition, $D_{T^{\mathcal{S}}_0}(\cdot)\geq 0$*. If* $D_{T^2_0}\left(z^2;g\right)\gtreqqless 1$ *then* $AB_0(\mathcal{S};g)\lesseqqgtr 0$*.*

shown in Briec and Mussard (2014), if the allocative efficiency corresponds to a positive aggregation bias, then a coalition $\mathcal{S}$ may improve its allocative efficiency. However, in the transformed space $\Phi_\alpha^{-1}(\mathbb{R}^d)$, the bias is not always positive as in the Euclidean vector space of dimension $d=n+m$. Then, the improvement of allocative efficiency holds true with the Φ_α-aggregator if $AB_\alpha(\mathcal{S};g)\geq 0$.

4 Nonjoint Aggregated Technologies with Complementarity

In the following, the limit of the Φ_α-aggregator is investigated in order to deal with aggregated semilattice technologies being nonjoint technologies exhibiting complementarity either in inputs or in outputs.

In production economics, a complementary output is an output with a negative cross elasticity of supply, in contrast to a substitute output. This means that an output supply is increased when the price of another output is decreased (not weakly). Along this line an output is weakly complementary if the output supply is not decreased when the price of another output is decreased. This is typically the case of Kohli technologies, analyzed by Kohli (1983), where the output set has a cubic structure.

Paralleling this definition, a complementary input is an input with a negative cross elasticity of the demand factor. Hence an input demand is increased when the price of another input is decreased. Then, an input is *weakly complementary* if the input demand is not decreased when the price of another input is decreased. For example Leontief production functions imply input weak complementarity of production factors.

Notice that in the case where the technology is derived from a production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ weak complementarity is often associated to supermodularity. This is defined with respect to the standard partial order over $\mathbb{R}^n$. A function f is weakly supermodular if $f(x \vee y) + f(x \wedge y) > f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$. If f is twice continuously differentiable, then weak supermodularity is equivalent to the condition $\frac{\partial^2 f}{\partial z_i \partial z_j} > 0$ for all $i \neq j$.

4.1 Semilattice Aggregators

The Φ_α-aggregator is defined, for all $z \in \mathbb{R}^d_+$, such that:

$$\sum_{j\in[d]}^{\phi_\alpha} z_j := \begin{cases} \min\limits_{j\in[d]} z_j & \text{if } \alpha = -\infty \\ \max\limits_{j\in[d]} z_j & \text{if } \alpha = \infty. \end{cases} \tag{4.1}$$

By construction $\sum_j^{\phi_\alpha} z_j = \phi_\alpha^{-1}\Big(\sum_j \phi_\alpha(z_j)\Big)$ when $\alpha \notin \{-\infty, \infty\}$. In such a case, Blackorby *et al.* (1981) axiomatically characterize this aggregator.[5]

[5] It is a generalized quasi-linear function that respects continuity, monotonicity, separability and symmetry.

This notation is justified by the fact that:

$$\begin{aligned} \lim_{\alpha \longrightarrow -\infty} \sum_{j\in[d]}^{\phi_\alpha} z_j &= \min_{j\in[d]} z_j \qquad \text{if } \alpha = -\infty \\ \lim_{\alpha \longrightarrow +\infty} \sum_{j\in[d]}^{\phi_\alpha} z_j &= \max_{j\in[d]} z_j \qquad \text{if } \alpha = \infty. \end{aligned} \tag{4.2}$$

In the case where $\alpha = -\infty$, if there is some j such that $z_j = 0$, then $\phi_\alpha(z_j) = +\infty$ and it follows that $\sum_{j\in[d]}^{\phi_\alpha} z_j = \min_{j\in[d]} z_j = 0$.

Definition 4.1 – **Semilattice Aggregators** – *For all collections* $Z = \{z^k : k \in \mathcal{S}\} \in \mathbb{R}^d_+$ *an upper-semilattice aggregator is given by:*

$$\bigvee_{k\in\mathcal{S}} z^k := \sum_{k\in\mathcal{S}}^{\Phi_\infty} z^k = \big(\max\{z_1^1, \ldots, z_1^{|\mathcal{S}|}\}, \ldots, \max\{z_d^1, \ldots, z_d^{|\mathcal{S}|}\}\big).$$

A lower-semilattice aggregator is given by:

$$\bigwedge_{k\in\mathcal{S}} z^k := \sum_{k\in\mathcal{S}}^{\Phi_{-\infty}} z^k = \big(\min\{z_1^1, \ldots, z_1^{|\mathcal{S}|}\}, \ldots, \min\{z_d^1, \ldots, z_d^{|\mathcal{S}|}\}\big).$$

On this ground, aggregated semilattice technologies may be characterized.

4.2 Aggregated semilattice technologies: characterization

Sets being semilattices are defined as follows.

Definition 4.2 – **Semilattices** – *A subset* T *of* $\mathbb{R}^{n+m}_+$ *is an upper semilattice if for all* $z, z' \in T$ *we have* $z \vee z' \in T$. *A subset* T *of* $\mathbb{R}^{n+m}_+$ *is a lower semilattice if for all* $z, z' \in T$ *we have* $z \wedge z' \in T$.

Semilattice technologies are related to $\mathbb{B}$-convex technologies.[6]

Example 4.1 *Let us recall the notion of* $\mathbb{B}$*-convex (*$\mathbb{B}^{-1}$*-convex) sets. A* $\mathbb{B}$*-convex hull of a set* $A = \{z^1, \ldots, z^{|\mathcal{S}|}\} \subset \mathbb{R}^{n+m}_+$ *is*

$$\mathbb{B}(A) = \Big\{ \bigvee_{k\in\mathcal{S}} t_k z^k,\ \max_{k=1,\ldots,|\mathcal{S}|} t_k = 1,\ t \geqslant 0 \Big\}.$$

[6] Semilattice technologies are either $\mathbb{B}$-convex or $\mathbb{B}^{-1}$-convex technologies, introduced respectively by Briec and Horvath (2009) and Briec and Liang (2011).

The $\mathbb{B}^{-1}$-convex hull of a set A is given by:

$$\mathbb{B}^{-1}(A) = \left\{ \bigwedge_{k\in\mathcal{S}} s_k z^k,\ \min_{k=1,\dots,|\mathcal{S}|} s_k = 1,\ s \geqslant 0 \right\}.$$

The $\mathbb{B}$-convex and $\mathbb{B}^{-1}$-convex technologies are given by:

$$T^{\mathcal{S}}_{\max} = \Big\{ (x,y) \in \mathbb{R}^{m+n}_+ : x \geqslant \bigvee_{k\in\mathcal{S}} t_k x^k, y \leqslant \bigvee_{k\in\mathcal{S}} t_k y^k, \max_{k\in\mathcal{S}} t_k = 1,\ t \geqslant 0 \Big\},$$

$$T^{\mathcal{S}}_{\min} = \Big\{ (x,y) \in \mathbb{R}^{m+n}_+ : x \geqslant \bigwedge_{k\in\mathcal{S}} s_k\, x_k, y \leqslant \bigwedge_{k\in\mathcal{S}} s_k\, y_k, \min_{k\in\mathcal{S}} s_k = 1,\ s \geqslant 0 \Big\}.$$

$\mathbb{B}$-convex technologies belong to the class of Kohli technologies. These technologies exhibit output weak complementarity in the production. Let us remark that the free disposal assumption can be represented thanks to the free disposal cone $K = \mathbb{R}^m_+ \times (-\mathbb{R}^n_+)$. In this respect any technology respecting the free disposal assumption may be rewritten as: $T = (A + K) \cap \mathbb{R}^{m+n}_+$. As a consequence, $\mathbb{B}$-convex and $\mathbb{B}^{-1}$-convex technologies are given by:

$$T_{\max} = (\mathbb{B}(A) + K) \cap \mathbb{R}^{m+n}_+ \ ;\ T_{\min} = \big(\mathbb{B}^{-1}(A) + K\big) \cap \mathbb{R}^{n+m}_+.$$

Inverse $\mathbb{B}$-convex technologies are related to Leontief production functions and imply input weak complementarity of production factors. They are, however, defined in a multi-output context. These are represented in Figures 1a and 1b.

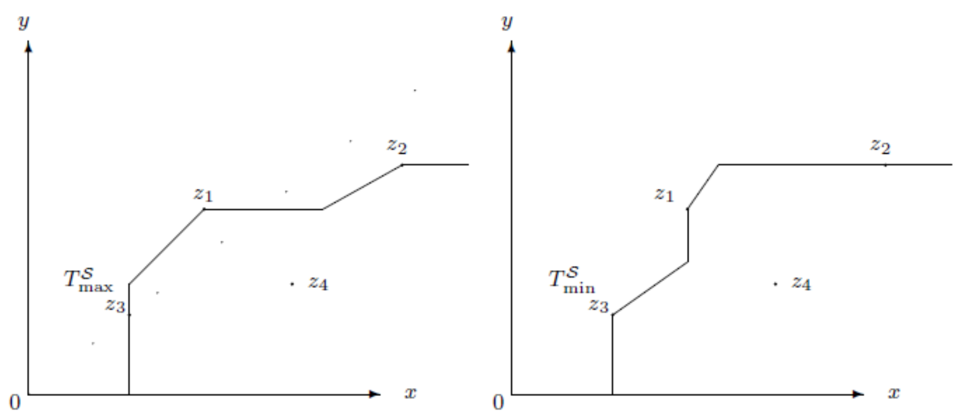

Figure 1.a $\mathbb{B}$-*convex Technology* **Figure 1.b** *Inverse* $\mathbb{B}$-*convex Technology*

In what follows, the properties of aggregated semilattice technologies are analyzed:

$$\begin{aligned} \textstyle\sum^{\Phi_\alpha}_{k\in\mathcal{S}} T^k &= \textstyle\bigvee_{k\in\mathcal{S}} T^k \quad &\text{if } \alpha = +\infty \\ \textstyle\sum^{\Phi_\alpha}_{k\in\mathcal{S}} T^k &= \textstyle\bigwedge_{k\in\mathcal{S}} T^k \quad &\text{if } \alpha = -\infty. \end{aligned} \tag{4.3}$$

Suppose that for all $k \in \mathcal{S}$, T^k is an upper [lower] semilattice. We prove that the aggregated technology $\bigvee_{k\in\mathcal{S}} T^k$ [$\bigwedge_{k\in\mathcal{S}} T^k$] satisfies assumptions (T1)-(T5).

Proposition 4.1
(i) *If for all $k \in \mathcal{S}$, T^k satisfies the no free lunch assumption* (T1)*, then $\bigvee_{k\in\mathcal{S}} T^k$ [respectively $\bigwedge_{k\in\mathcal{S}} T^k$] satisfies* (T1)*.*
(ii) *If for all $k \in \mathcal{S}$, T^k satisfies* (T2)*, then $\bigvee_{k\in\mathcal{S}} T^k$ [respectively $\bigwedge_{k\in\mathcal{S}} T^k$] satisfies* (T2)*.*
(iii) *For all $k \in \mathcal{S}$, if T^k satisfies the closedness assumption* (T3)*, then $\bigvee_{k\in\mathcal{S}} T^k$ [respectively $\bigwedge_{k\in\mathcal{S}} T^k$] satisfies* (T3)*.*
(iv) *If for all $k \in \mathcal{S}$, T^k satisfies a free disposal assumption* (T4)*, then $\bigvee_{k\in\mathcal{S}} T^k$ [respectively $\bigwedge_{k\in\mathcal{S}} T^k$] satisfies* (T4)*.*
(v) *If for all $k \in \mathcal{S}$, T^k is an upper [lower] semilattice and satisfies the constant returns to scale assumption* (T5)*, then $\bigvee_{k\in\mathcal{S}} T^k$ [respectively $\bigwedge_{k\in\mathcal{S}} T^k$] is an upper [lower] semilattice respecting* (T5)*.*

Proof: (i) Straightforward.
(ii) From (T2) each set $A(x^k)$ is assumed to be bounded. Hence, $A(\bigvee_{k\in\mathcal{S}} x^k)$ is also bounded, and so, $\bigvee_{k\in\mathcal{S}} T^k$ respects (T2). The same holds true for $A(\bigwedge_{k\in\mathcal{S}} x^k)$.
(iii) Let $(w, z) = (\bigvee_{k\in\mathcal{S}} w^k, \bigvee_{k\in\mathcal{S}} z^k)$. Assume by contradiction that $\bigvee_{k\in\mathcal{S}} T^k$ is open. Hence, there is some $w, z \in \mathbb{R}_+^{n+m}$ such that $(w, z) \notin \bigvee_{k\in\mathcal{S}} T^k$. If T^k is an upper semilattice, then $\bigvee_{k\in\mathcal{S}} T^k$ is an upper semilattice. Then, $\bigvee_{k\in\mathcal{S}}(w^k \vee z^k) \notin \bigvee_{k\in\mathcal{S}} T^k$. Hence, the map $w^k \mapsto \bigvee_{k\in\mathcal{S}} w_k$ is not defined and not continuous on some intervals, that is, it exists some $k \in \mathcal{S}$ and at least one $i \in \{1, \ldots, d\}$ such that $\max_{k\in\mathcal{S}} w_i^k$ is not defined. Then, for $w_k = (x^k, y^k)$ it exists $x_i^k \in [\bar{x}_i^k, \bar{x}_i^k + \epsilon]$ or $y_i^k \in [\bar{y}_i^k, \bar{y}_i^k + \epsilon]$ with $\epsilon > 0$ such that $\max_{k\in\mathcal{S}} w_i^k$ is not defined. In such a case, $x^k \vee y^k \notin T^k$, and T^k is open on some intervals $[\bar{x}_i^k, \bar{x}_i^k + \epsilon]$ or $[\bar{y}_i^k, \bar{y}_i^k + \epsilon]$, which concludes the proof. The proof is similar for $\bigwedge_{k\in\mathcal{S}} T^k$.
(iv) Suppose that $z = (x, y) \in \bigvee_{k\in\mathcal{S}} T^k$ and let $z' = (x', y') \in \mathbb{R}_+^{n+m}$ such that $x' \geqslant x$ and $y' \leqslant y$. We need to prove that $z' \in \bigvee_{k\in\mathcal{S}} T^k$. By hypothesis one can find $(z^1, \ldots, z^{|\mathcal{S}|}) \in \bigvee_{k\in\mathcal{S}} T^k$ such that $z = \bigvee_{k\in\mathcal{S}} z^k$. If $y' \leqslant y$, then there is some $v \in \mathbb{R}_+^m$ such that $y' = y - v$. Moreover $y - v = (\bigvee_{k\in\mathcal{S}} y^k) - v = \bigvee_{k\in\mathcal{S}}(y^k - v)$. However, $y - v \geqslant 0_m$, thus $y - v = (y - v) \vee 0_m$. Consequently,

$$y - v = \left(\bigvee_{k\in\mathcal{S}} (y^k - v)\right) \vee 0_m = \bigvee_{k\in\mathcal{S}} \left[(y^k - v) \vee 0_m\right].$$

For all k, since $y^k \geqslant 0_m$, $(y^k - v) \vee 0_m \leqslant y^k \vee 0_m = y^k$. Similarly, if $x' \geqslant x$, then there is some $u \in \mathbb{R}_+^n$ such that $x' = x + u = (\bigvee_{k\in\mathcal{S}} x^k) + u = \bigvee_{k\in\mathcal{S}}(x^k + u)$.

However, since each T^k satisfies a free disposal assumption the inequalities $(y^k - v) \vee 0_m \leqslant y^k$ and $x^k + u \geqslant x^k$ implies that $(x^k + u, (y^k - v) \vee 0_m) \in T^k$. Hence $z' = (x', y') \in \bigvee_{k \in \mathcal{S}} T^k$, which ends the proof. The proof is similar for $\bigwedge_{k \in \mathcal{S}} T^k$ except that one should use the distributivity of the operation $\vee$ on $\wedge$. Suppose that $z = (x, y) \in \bigwedge_{k \in \mathcal{S}} T^k$ and let $z' = (x', y') \in \mathbb{R}_+^{n+m}$ such that $x' \geqslant x$ and $y' \leqslant y$. By hypothesis one can find $(z^1, \dots, z^{|\mathcal{S}|}) \in \bigwedge_{k \in \mathcal{S}} T^k$ such that $z = \bigwedge_{k \in \mathcal{S}} z^k$. Paralleling the proof above, if $y' \leqslant y$, then there is some $v \in \mathbb{R}_+^m$ such that $y' = y - v$. Moreover $y - v = (\bigwedge_{k \in \mathcal{S}} y^k) - v = \bigwedge_{k \in \mathcal{S}} (y^k - v)$. Since, $y - v \geqslant 0_m$, $y - v = (y - v) \vee 0$. Therefore,

$$y - v = \left(\bigwedge_{k \in \mathcal{S}} (y^k - v) \right) \vee 0_m = \bigwedge_{k \in \mathcal{S}} \left[(y^k - v) \vee 0_m \right].$$

For all k, since $y^k \geqslant 0_m$ one has $(y^k - v) \vee 0_m \leqslant y^k \vee 0_m = y^k$. Moreover, if $x' \geqslant x$, then there is some $u \in \mathbb{R}_+^n$ such that $x' = x + u = (\bigwedge_{k \in \mathcal{S}} x^k) + u = \bigwedge_{k \in \mathcal{S}} (x^k + u)$. However, since each T^k satisfies a free disposal assumption $(y^k - v) \vee 0_m \leqslant y^k$ and $x^k + u \geqslant x^k$ implies that $z' = (x', y') \in \bigwedge_{k \in \mathcal{S}} T^k$, which ends the proof.
(v) Suppose that $z, w \in \bigvee_{k \in \mathcal{S}} T^k$. We have to prove that $\beta z \vee \beta w \in \bigvee_{k \in \mathcal{S}} T^k$ for some $\beta \geq 0$. For all $k \in \mathcal{S}$ choose some $z^k, w^k \in T^k$ such that $z = \bigvee_{k \in \mathcal{S}} z^k$ and $w = \bigvee_{k \in \mathcal{S}} w^k$. Therefore,

$$\beta z \vee \beta w = (\bigvee_{k \in \mathcal{S}} \beta z^k) \vee (\bigvee_{k \in \mathcal{S}} \beta w^k) = \bigvee_{k \in \mathcal{S}} (\beta z^k \vee \beta w^k).$$

By (T5), since T^k is an upper semilattice for all $k \in \mathcal{S}$, we get that $\beta z^k \vee \beta w^k \in T^k$. Thus $\beta z \vee \beta w \in \bigvee_{k \in \mathcal{S}} T^k$, and so $z \vee w \in \bigvee_{k \in \mathcal{S}} T^k$. Consequently, $\bigvee_{k \in \mathcal{S}} T^k$ is an upper semilattice satisfying (T5), which ends the proof. For $\bigwedge_{k \in \mathcal{S}} T^k$, the proof is similar. ■

A last property of equal technology will be useful in the firm games designed below. Inside a coalition, when the technologies of the firms are identical, the aggregated technology of the coalition inherits the same technology.

Proposition 4.2 *The two following properties hold:*
(i) *Suppose that for all $k \in \mathcal{S}$, $T^k = T$ is an upper semilattice. Then $\bigvee_{k \in \mathcal{S}} T^k = T$.*
(ii) *Suppose that for all $k \in \mathcal{S}$, $T^k = T$ is a lower semilattice. Then $\bigwedge_{k \in \mathcal{S}} T^k = T$.*

Proof: (i) By hypothesis $(0_n, 0_m) \in T^k$ for all k. Therefore, for all k, $T^k = T \subset \bigvee_{k \in \mathcal{S}} T^k$. Let us show the converse. Assume that $z \in T$. It follows that for all k there are some $z^k \in T^k$, such that $z = \bigvee_{k \in \mathcal{S}} z^k$. Since

$T = T^k$, $z^k \in T$ for all k. Since T is an upper semilattice it follows that $\bigvee_{k\in\mathcal{S}} z^k \in T$. Therefore $\bigvee_{k\in\mathcal{S}} T^k \subset T$. Consequently, $\bigvee_{k\in\mathcal{S}} T^k = T$.
(ii) Assume that $z \in T$. It follows that for all k there are some $z^k \in T^k$, such that $z = \bigwedge_{k\in\mathcal{S}} z^k$. Since $T = T^k$, $z^k \in T$ for all k. Since T is a lower semilattice it follows that $\bigwedge_{k\in\mathcal{S}} z^k \in T$. Therefore $\bigwedge_{k\in\mathcal{S}} T^k \subset T$. Let us prove the converse inclusion, Suppose that, $z \in T$. Since $T = T^k$ for all k, $z \in T^k$. Obviously $z = \bigwedge_{k\in\mathcal{S}} z^k$. Hence $z \in \bigwedge_{k\in\mathcal{S}} T^k$ and it follows that $T \subset \bigwedge_{k\in\mathcal{S}} T^k$ which proves the converse inclusion. ■

Finally, these properties indicate that aggregated semilattice technologies respect the traditional assumptions (T1)-(T5) of the literature (we will show however in the next section that (T5) is not necessary to obtain negative technical bias). Also, these technologies exhibit multi-input and multi-output weak complementarity.

4.3 Aggregated technologies and Complementarity

An intuition of the negative technical bias relying on semilattice technologies (studied in the next Section) is provided. Let us analyze the aggregated output correspondence $\bigvee_k P^k(x)$ and the aggregated input correspondence $\bigwedge_k L^k(y)$, with $\alpha = +\infty$ and $\alpha = -\infty$, respectively. These technologies are nonjoint. Nonjointness is characterized either by the intersection of output sets or input sets, see Chambers (1988) Chapter 7 and Kholi (1981, 1983). Although semilattice technologies depicted in Figures 2a/b include the intersection of output/input sets, they generalize the notion of nonjointness in the sense that they exhibit interdependencies between the firm production in a given coalition.

In the first case (Figure 2.a), given a fixed input amount x, the firm coalition (dashed line) may reach a better multi-output production (more y_1 and more y_2). This represents multi-output weak complementarity, in other terms, the cooperation between firms does not affect efficiency since more outputs may be produced with a same quantity of inputs.[7]

In the second case, given a fixed amount of output y (Figure 2.b), the firm coalition (dashed line) may reach a lower use of inputs (less x_1 and less x_2). This represents the multi-input weak complementarity. As can be seen in Figure 2.b, the lower semilattice technology is a coalitional Leontief technology.

[7]These cubic technologies are also relevant to what happens in different states of nature. In a state-contingent approach, when each dimension represents a state of nature, the cubic technology provides an evaluation of the production across all those states, see Chambers and Quiggin (2000).

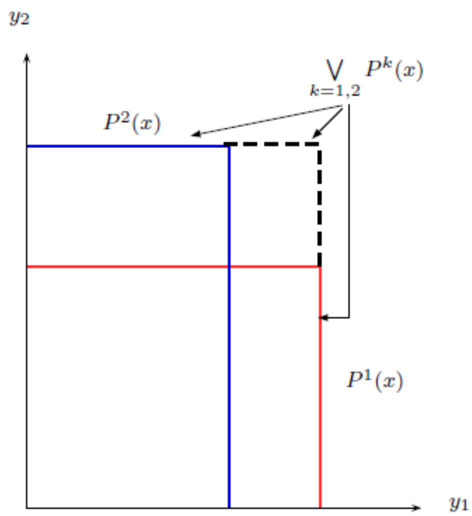

Figure 2.a $\alpha = +\infty$

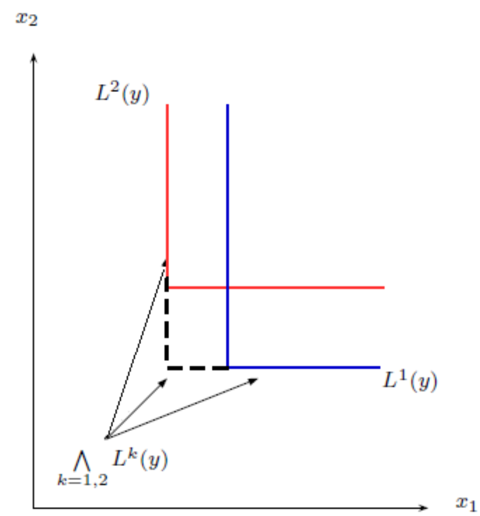

Figure 2.b $\alpha = -\infty$

It is noteworthy that the previous cases with $\alpha = \pm\infty$ are interesting because: first they characterize new aggregated (semilattice) technologies with weak complementarity, second this weak complementarity gives rise to negative biases, that is, improvement of technical efficiency of firm groups. This second point is studied in the next Section.

5 Negative Technical Bias: Firm Games

Input/output firm games are introduced in order to outline the conditions allowing for technical negative biases to be conceived with aggregated semilattice technologies.

A transferable utility game, *i.e.* a TU-game in the direction of g, is a quadruplet $(\mathcal{K}, v_\alpha, \Phi_\alpha, g)$, where v_α is defined as $v_\alpha : 2^{|\mathcal{K}|} \to \mathbb{R}_+$ such that $v_\alpha(\emptyset) := 0$. The set of all maps v_α is denoted Γ, such that $v_\alpha(\mathcal{S})$ provides the worth of coalition (group) $\mathcal{S} \subseteq \mathcal{K}$, that is, the directional distance function of coalition $\mathcal{S}$ based on the Φ_α-aggregator:

$$v_\alpha(\mathcal{S}) := D_{T_\alpha^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}}^{\Phi_\alpha} x^k, \sum_{k\in\mathcal{S}}^{\Phi_\alpha} y^k; g \right), \tag{5.1}$$

such that for any given coalition $\mathcal{S} \subseteq \mathcal{K}$, $D_{T_\alpha^{\mathcal{S}}} : \Phi_\alpha^{-1}\big(M_+^{n+m}\big) \times \Phi_\alpha^{-1}\big(M_+^{n+m}\big) \longrightarrow \mathbb{R}_+$. The valued solution $\varphi(v_\alpha)$ is the *pay-off* vector of the TU-game $(\mathcal{K}, v_\alpha, \Phi_\alpha, g)$, that is, a $|\mathcal{K}|$-dimensional real vector representing the expected improvement of technical efficiency issued from cooperation.

Some restrictions are imposed on inputs and outputs, for all firms of the group, in order to identify negative and positive technical biases.

Definition 5.1 – **Input/Output fixed firm games** – *Let the firm game be* $(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g)$ *for all* $\mathcal{S} \subseteq \mathcal{K}$.
(i) *An input fixed firm game is given by,*

$$\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_o, \bar{x}) := (\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g)\,, \forall \mathcal{S} \subseteq \mathcal{K},$$

where $x^k = \bar{x}$ *for all* $k \in \mathcal{S}$ *and* $\alpha = \pm\infty$.
(ii) *An output fixed firm game is given by,*

$$\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_i, \bar{y}) := (\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g)\,, \forall \mathcal{S} \subseteq \mathcal{K},$$

where $y^k = \bar{y}$ *for all* $k \in \mathcal{S}$ *and* $\alpha = \pm\infty$.

(i) In input fixed firm games $\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_o, \bar{x})$, coalitions of firms depend on the possible outputs to be produced whereas the amount of inputs is limited to $\bar{x}$ for all possible coalitions. This may arise when firms are constrained by the amount of their inputs, for instance when a maximum is imposed on the use of natural resources to preserve biodiversity. blueAs a consequence, everything happens as if the distance function is output oriented and depends on the direction g_o only. In this case, it is possible to interpret the directional distance as the firm physical output loss measured in terms of the numeraire, see Peyrache (2013).
(ii) In output fixed firm games $\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_i, \bar{y})$, coalitions depend on inputs only, whereas the amount of outputs for each coalition is constrained by $\bar{y}$, which may represent a production quota. In this case, everything happens as if the distance function is input oriented and depends on the direction g_i only. The directional distance is interpreted as the firm physical input waste.

Proposition 5.1 *If* T^k *is an upper semilattice (lower semilattice respectively) respecting* (T1)-(T4) *such that* $T^k = T$ *for all* $k \in \mathcal{S} \subseteq \mathcal{K}$, *then we have, respectively:*

(i) $[\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_o, \bar{x}) \wedge (\alpha = +\infty)] \implies [AB_{+\infty}(\mathcal{S}; g) \leq 0]$.
(ii) $[\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_o, \bar{x}) \wedge (\alpha = -\infty)] \implies [AB_{-\infty}(\mathcal{S}; g) \geq 0]$.

Proof: (i) If $\alpha = \infty$, then $\overset{\Phi_\infty}{\sum}_{k\in\mathcal{S}} T^k = \bigvee_{k\in\mathcal{S}} T^k$ is an upper semilattice whenever T^k is an upper semilattice. From Proposition 4.1 this set satisfies the free disposal assumption (T4), and moreover $T = \bigvee_{k\in\mathcal{S}} T^k$ (see Proposition 4.2). It follows that the directional distance function is weakly monotonic on T, that is, $(x, y), (u, v) \in T$ such that $u \leqslant y$ and $v \geqslant x$ imply that $D_T(u, v; g) \geq D_T(x, y; g)$. Since this is an input fixed firm game, we have $x^k = \bar{x}$, for all $k \in \mathcal{S}$. Hence, for all $\mathcal{S} \subseteq \mathcal{K}$, we have

$\bigvee_{k\in\mathcal{S}} x^k = \bar{x}$. Moreover, $\bigvee_{k\in\mathcal{S}} y^k \geqslant y^k$ for all k. From weak monotonicity, we have $D_T(\bar{x}, \bigvee_{k\in\mathcal{S}} y^k; g) \leq D_T(x^k, y^k; g) = D_T(\bar{x}, y^k; g)$. However, since $T^k = T$, we have for all $k \in \mathcal{S}$:

$$D_T\left(\bigvee_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) = D_T(\bar{x}, \bigvee_{k\in\mathcal{S}} y^k; g) \leq D_T(\bar{x}, y^k; g) = D_{T^k}(x^k, y^k; g).$$

This implies that $D_T(\bigvee_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g) \leq \max_{k\in\mathcal{S}} D_{T^k}(x^k, y^k; g)$ which proves (i).

(ii) If $\alpha = -\infty$, then $\overset{\Phi_{-\infty}}{\sum}_{k\in\mathcal{S}} T^k = \bigwedge_{k\in\mathcal{S}} T^k$ is a lower semilattice. This set satisfies the free disposal assumption (T4) from Proposition 4.1. It follows that the directional distance function is weakly monotonic on $\bigwedge_{k\in\mathcal{S}} T^k$, with $\bigwedge_{k\in\mathcal{S}} T^k = T$ by Proposition 4.2. Then $(x, y), (u, v) \in T$, such that $u \leqslant y$ and $v \geqslant x$ imply that $D_T(u, v; g) \geq D_T(x, y; g)$. Since this is an input fixed game, we have $x^k = \bar{x}$, for all $k \in \mathcal{S}$. Hence, for all $\mathcal{S} \subseteq \mathcal{K}$, we have $\bigwedge_{k\in\mathcal{S}} x^k = \bar{x}$. Moreover, $\bigwedge_{k\in\mathcal{S}} y^k \leqslant y^k$ for all k. From weak monotonicity, we have $D_T(\bar{x}, \bigwedge_{k\in\mathcal{S}} y^k; g) \leq D_T(x^k, y^k; g) = D_T(\bar{x}, y^k; g)$. Moreover, since $T^k = T$, we have for all $k \in \mathcal{S}$:

$$D_T\left(\bigwedge_{k\in\mathcal{S}} x^k, \bigwedge_{k\in\mathcal{S}} y^k; g\right) = D_T(\bar{x}, \bigwedge_{k\in\mathcal{S}} y^k; g) \geq D_T(\bar{x}, y^k; g) = D_{T^k}(x^k, y^k; g).$$

This implies that $D_T(\bigwedge_{k\in\mathcal{S}} x^k, \bigwedge_{k\in\mathcal{S}} y^k; g) \geq \min_{k\in\mathcal{S}} D_{T^k}(x^k, y^k; g)$ which proves (ii). ■

The technical biases inherent to the output fixed firm games are the following.

Proposition 5.2 *If T^k is an upper semilattice (lower semilattice respectively) respecting* (T1)-(T4) *such that $T^k = T$ for all $k \in \mathcal{S} \subseteq \mathcal{K}$, then we have, respectively:*

(i) $[\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_i, \bar{y}) \wedge (\alpha = +\infty)] \implies [AB_{+\infty}(\mathcal{S}; g) \geq 0]$

(ii) $[\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_i, \bar{y}) \wedge (\alpha = -\infty)] \implies [AB_{-\infty}(\mathcal{S}; g) \leq 0]$.

Proof: The proof is similar to that of Proposition 5.1. ■

The previous results indicate that the aggregator Φ_∞ (respectively $\Phi_{-\infty}$) is relevant with a negative technical bias that embodies an improvement of technical efficiency in the input fixed firm game (respectively in the output fixed firm game). Let us investigate whether this result is relevant to the core of the firm game. The core is defined as:

$$\mathcal{C}_\alpha := \left\{ \varphi \in M^{|\mathcal{K}|} : \sum_{k\in\mathcal{S}} \varphi_k \leq v_\alpha(\mathcal{S}), \forall \mathcal{S} \subset \mathcal{K} \right\} \bigcap \left\{ \sum_{k\in\mathcal{K}} \varphi_k = v_\alpha(\mathcal{K}) \right\}. \quad (5.2)$$

The negative bias is not sufficient to avoid the non-vacuity of the core. For that purpose, submodular games are investigated. Solutions inside the core are characterized, so that the core is partitioned either with negative biases or with positive ones.

The respect of the following standard axioms is necessary.

Linearity: $\varphi(\gamma_1 v_{\alpha,1} + \gamma_2 v_{\alpha,2}) = \gamma_1 \varphi(v_{\alpha,1}) + \gamma_2 \varphi(v_{\alpha,2})$, for all maps $v_{\alpha,1}, v_{\alpha,2} \in \Gamma$ and $\gamma_1, \gamma_2 \in \mathbb{R}$.

Symmetry: for any given pay-off vector $\varphi = (\varphi_1, \ldots, \varphi_k, \ldots, \varphi_{|\mathcal{K}|})$, then $\varphi_k(v_\alpha) = \varphi_{\pi(k)}(v_\alpha)$ for all permutations, where a permutation is given by $v_\alpha(\pi(\mathcal{S})) = v_\alpha(\mathcal{S})$ for all $\mathcal{S} \subseteq \mathcal{K}$ and $v_\alpha \in \Gamma$.[8]

Efficiency: $\sum_{k\in\mathcal{K}} \varphi_k(v_\alpha) = v_\alpha(\mathcal{K})$, for all $v_\alpha \in \Gamma$.

These axioms provide some well-known values such as, among others, the Shapley value and the solidarity value. The core of the firm game is non void whenever the game is submodular (see Shapley, 1972).[9]

Definition 5.2 – **Submodularity** – *For all firm games $(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g)$, such that $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{K}$ with $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$, the game is submodular (or concave) if:*

$$v_\alpha(\mathcal{S}_1 \cup \mathcal{S}_2) \leq v_\alpha(\mathcal{S}_1) + v_\alpha(\mathcal{S}_2) - v_\alpha(\mathcal{S}_1 \cap \mathcal{S}_2).$$

In the same manner, the submodularity of the aggregation bias is, for $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{K}$ such that $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$,

$$AB_\alpha(\mathcal{S}_1 \cup \mathcal{S}_2; g) \leq AB_\alpha(\mathcal{S}_1; g) + AB_\alpha(\mathcal{S}_2; g) - AB_\alpha(\mathcal{S}_1 \cap \mathcal{S}_2; g). \quad (5.3)$$

Following Briec and Mussard (2014), the submodularity of the aggregation bias displays the following interpretation: the loss of technical efficiency due to the cooperation between two coalitions is no higher than the aggregated loss of $\mathcal{S}_1$ and $\mathcal{S}_2$ $[AB_\alpha(\mathcal{S}_1; g) + AB_\alpha(\mathcal{S}_2; g)]$, by taking into account in addition the loss of their cooperation $AB_\alpha(\mathcal{S}_1 \cap \mathcal{S}_2; g)$. It is shown below that the submodularity of the aggregation bias is closely related to that of the game $v_\alpha(\cdot)$.

[8] π is an automorphism, so that the value of the game v_α does not depend on how the players are ordered inside a coalition $\mathcal{S}$, $v_\alpha(\pi(S)) = v_\alpha(S)$. In the same manner, the imputation of firm k does not depend on its label, that is, $\varphi_k(v_\alpha) = \varphi_{\pi(k)}(v_\alpha)$.

[9] Although the concept of Nucleolus is not studied in the paper, it could be investigated and compared with the Shapley value in the case where the core of the firm game is non void.

Proposition 5.3 *If T^k is an upper semilattice (lower semilattice respectively) respecting* (T1)-(T4) *such that $T^k = T$ for all $k \in \mathcal{S} \subseteq \mathcal{K}$, then we have, respectively:*

(i) $[(AB_{+\infty}$ is submodular$)] \Longrightarrow [v_{+\infty}$ is submodular$] \Longleftrightarrow [\mathring{\mathcal{C}}_{\infty} \neq \emptyset]$.

(ii) *If* $\min_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} v_{-\infty}(\{k\}) = \min_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} v_{-\infty}(\{k\})$ *then:*

$[(AB_{-\infty}$ is submodular$)] \Longrightarrow [v_{-\infty}$ is submodular$] \Longleftrightarrow [\mathring{\mathcal{C}}_{-\infty} \neq \emptyset]$.

Proof: (i) Let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{K}$ such that $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$. Since $T^k = T$ for all $k \in \mathcal{S}$, then:

$$T = T^{\mathcal{S}_1} = T^{\mathcal{S}_2} = T^{\mathcal{S}_1 \cap \mathcal{S}_2} = T^{\mathcal{S}_1 \cup \mathcal{S}_2}.$$

Let $z^k = (x^k, y^k) \in \mathbb{R}_+^{n+m}$, the submodularity of the technical bias (5.3) entails,

$$D_T\left(\bigvee_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} z^k; g\right) - \bigvee_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} D_T(z^k; g) + D_T\left(\bigvee_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} z^k; g\right) - \bigvee_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} D_T(z^k; g)$$

$$\leq D_T\left(\bigvee_{k \in \mathcal{S}_1} z^k; g\right) - \bigvee_{k \in \mathcal{S}_1} D_T(z^k; g) + D_T\left(\bigvee_{k \in \mathcal{S}_2} z^k; g\right) - \bigvee_{k \in \mathcal{S}_2} D_T(z^k; g).$$

Note that for $\mathcal{R} = \mathcal{S}_1, \mathcal{S}_2$:

$$\max_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} D_T(z^k; g) \leq \max_{k \in \mathcal{R}} D_T(z^k; g) \leq \max_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} D_T(z^k; g).$$

Hence, the game $v_\alpha(\cdot)$ represented by the characteristic function $D_T(\cdot)$ is concave, that is, the distance function is submodular:

$$D_T\left(\bigvee_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} z^k; g\right) \leq D_T\left(\bigvee_{k \in \mathcal{S}_1} z^k; g\right) + D_T\left(\bigvee_{k \in \mathcal{S}_2} z^k; g\right) - D_T\left(\bigvee_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} z^k; g\right).$$

To find the previous relation, note that three cases have to be considered: either the maximum distance $v_\alpha(\{k\})$ is such that $k \in \{\mathcal{S}_1 \setminus \mathcal{S}_2\}$, or $k \in \{\mathcal{S}_2 \setminus \mathcal{S}_1\}$ or finally $k \in \mathcal{S}_1 \cap \mathcal{S}_2$. The submodularity of the game (distance function) ensures that the core interior is non empty (Shapley, 1972).

(ii) In the lower semilattice case, we have for $\mathcal{R} = \mathcal{S}_1, \mathcal{S}_2$:

$$\min_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} D_T(z^k; g) \geq \min_{k \in \mathcal{R}} D_T(z^k; g) \geq \min_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} D_T(z^k; g).$$

As a consequence, it is easy to show that the previous condition is not sufficient to ensure the submodularity of $v_{-\infty}$. However, if $\min_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} v_{-\infty}(\{k\}) = \min_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} v_{-\infty}(\{k\})$ then the submodularity of $v_{-\infty}$ follows,

$$D_T\left(\bigwedge_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} z^k; g\right) \leq D_T\left(\bigwedge_{k \in \mathcal{S}_1} z^k; g\right) + D_T\left(\bigwedge_{k \in \mathcal{S}_2} z^k; g\right) - D_T\left(\bigwedge_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} z^k; g\right).$$

Therefore, the game $v_{-\infty}(\cdot)$ is concave (submodular), then in the same manner than (i), the core interior $\overset{\circ}{\mathcal{C}}_{-\infty}$ is non void. ■

The last proposition is interesting since it allows the non vacuity of the core to be designed with a simple sufficient condition. Indeed, in order to find coalitions with efficient operation points, characterized by a better aggregate technical efficiency, it is sufficient to test for the submodularity of the aggregation bias. Moreover, if this submodularity is proven for all possible coalitions, then the core of the firm game is non empty. However, it does not tell us the whole story about the sign of the aggregation bias. Indeed, from the earlier result, it is clear that the core of the firm game is non void and that the aggregation bias may be positive or negative (Propositions 5.1 and 5.2). To get a clear result about the non vacuity of the core and the sign of the technical bias, a partition of the core is introduced. The first core displays the set of imputations φ related to a positive bias, which corresponds to an improvement of allocative efficiency (see Briec and Mussard, 2014, in the case where $\alpha = 1$):

$$\mathcal{C}_{AB_\alpha \geq 0} := \left\{ \varphi \in M^{|\mathcal{K}|} \middle| \begin{array}{ll} \sum_{k\in\mathcal{S}} \varphi_k & \leq v_\alpha(\mathcal{S}), \forall \mathcal{S} \subset \mathcal{K} \\ AB_\alpha(\mathcal{S}; g) & \geq 0,\ \forall \mathcal{S} \subset \mathcal{K} \end{array} \right\} \bigcap \left\{ \sum_{k\in\mathcal{K}} \varphi_k = v_\alpha(\mathcal{K}) \right\}.$$

The second core yields the set of pay-off vectors φ inherent to a negative bias, which corresponds to an improvement of technical efficiency:

$$\mathcal{C}_{AB_\alpha \leq 0} := \left\{ \varphi \in M^{|\mathcal{K}|} \middle| \begin{array}{ll} \sum_{k\in\mathcal{S}} \varphi_k & \leq v_\alpha(\mathcal{S}), \forall \mathcal{S} \subset \mathcal{K} \\ AB_\alpha(\mathcal{S}; g) & \leq 0,\ \forall \mathcal{S} \subset \mathcal{K} \end{array} \right\} \bigcap \left\{ \sum_{k\in\mathcal{K}} \varphi_k = v_\alpha(\mathcal{K}) \right\}.$$

It is obvious that $\mathcal{C}_\alpha = \mathcal{C}_{AB_\alpha \leq 0} \cup \mathcal{C}_{AB_\alpha \geq 0}$. Consequently, either in the input or the output fixed firm game, whenever the game v_α is submodular, it is always possible to find a solution in the core interior either with improvement of technical efficiency or which improvement of allocative efficiency.

Corollary 5.1 *If T^k is an upper semilattice (lower semilattice respectively) respecting* (T1)-(T4) *such that $T^k = T$ for all $k \in \mathcal{S} \subseteq \mathcal{K}$, then we have, respectively:*
(i) $\left[\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_o, \bar{x}) \wedge (\alpha = +\infty) \wedge (v_\alpha \text{ is submodular})\right] \Longrightarrow [\overset{\circ}{\mathcal{C}}_{AB_{+\infty} \leq 0} \neq \emptyset]$.
(ii) $\left[\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_o, \bar{x}) \wedge (\alpha = -\infty) \wedge (v_\alpha \text{ is submodular})\right] \Longrightarrow [\overset{\circ}{\mathcal{C}}_{AB_{-\infty} \geq 0} \neq \emptyset]$.

Corollary 5.2 *If T^k is an upper semilattice (lower semilattice respectively) respecting* (T1)-(T4) *such that $T^k = T$ for all $k \in \mathcal{S} \subseteq \mathcal{K}$, then we have, respectively:*

(i) $[\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_i, \bar{y}) \wedge (\alpha = +\infty) \wedge (v_\alpha \text{ is submodular})] \Longrightarrow [\mathring{\mathcal{C}}_{AB_{+\infty} \geq 0} \neq \emptyset]$.

(ii) $[\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \Phi_\alpha, g_i, \bar{y}) \wedge (\alpha = -\infty) \wedge (v_\alpha \text{ is submodular})] \Longrightarrow [\mathring{\mathcal{C}}_{AB_{-\infty} \leq 0} \neq \emptyset]$.

Proof: Both corollaries are deduced from Propositions 5.1, 5.2 and 5.3. ■

The results above about input and output fixed firm games yield the conditions to obtain either positive or negative technical biases, and also non empty cores. However, the constraints imposed on the input-output quantity $(\bar{x}, \bar{y})$ available in any given sector of the economy may be seen as restrictive assumptions. Relaxing these conditions, as depicted in Section 6, it is still possible to show that negative technical biases and non empty core exist.

6 Firm Games and Negative Technical Bias: An Illustration

An example of firm game is proposed. It is shown that without imposing the conditions inherent to the input (output) fixed firm games, it is still possible to obtain negative technical biases and a non empty core. Let us assume four firms with the following technologies:

$T^1 = \{(x, y) \in \mathbb{R}^2_+ : y \leqslant x\}$,
$T^2 = \{(x, y) \in \mathbb{R}^2_+ : y \leqslant 2x\}$,
$T^3 = T^2$,
$T^4 = \{(x, y) \in \mathbb{R}^2_+ : y \leqslant 3x\}$.

The input-output vectors are $(x^1, y^1) = (2, 1)$, $(x^2, y^2) = (1, 1)$, $(x^3, y^3) = (1, 2)$ and $(x^4, y^4) = (2, 0)$. Taking $g = (0, 1)$ we have $D_{T^1}(x^1, y^1; g) = 1$, $D_{T^2}(x^2, y^2; g) = 1$, $D_{T^3}(x^3, y^3; g) = 0$, $D_{T^4}(x^4, y^4; g) = 6$.

• If the size of the coalition is $|\mathcal{S}| = 1$, then $D_{T^k}(x^k, y^k; g) = D_{T^k_\infty}(x^k, y^k; g)$ for all $k = 1, \ldots, 4$.

• If the size of the coalition is $|\mathcal{S}| = 2$, the coalitions are:

$\mathcal{S} = \{1, 2\}$: $T^{\mathcal{S}} = T^1 \vee T^2 = T^2$, $D_{T^{\mathcal{S}}}\left(\bigvee_{k \in \mathcal{S}} x^k, \bigvee_{k \in \mathcal{S}} y^k; g\right) = 3$ and $AB_\infty(\mathcal{S}; g) = 2$.

$\mathcal{S} = \{1, 3\}$: $T^{\mathcal{S}} = T^1 \vee T^3 = T^3$, $D_{T^{\mathcal{S}}}\left(\bigvee_{k \in \mathcal{S}} x^k, \bigvee_{k \in \mathcal{S}} y^k; g\right) = 2$ and $AB_\infty(\mathcal{S}; g) = 1$.

$\mathcal{S} = \{1, 4\}$: $T^{\mathcal{S}} = T^1 \vee T^4 = T^4$, $D_{T^{\mathcal{S}}}\left(\bigvee_{k \in \mathcal{S}} x^k, \bigvee_{k \in \mathcal{S}} y^k; g\right) = 5$ and $AB_\infty(\mathcal{S}; g) = -1$.

$\mathcal{S} = \{2, 3\}$: $T^{\mathcal{S}} = T^2 \vee T^3 = T^2$, $D_{T^{\mathcal{S}}}\left(\bigvee_{k \in \mathcal{S}} x^k, \bigvee_{k \in \mathcal{S}} y^k; g\right) = 0$ and $AB_\alpha(\mathcal{S}; g) = 0$.

$\mathcal{S} = \{2,4\}$: $T^{\mathcal{S}} = T^2 \vee T^4 = T^4$, $D_{T^{\mathcal{S}}}\left(\bigvee_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) = 5$ and $AB_\infty(\mathcal{S}; g) = -1$.

$\mathcal{S} = \{3,4\}$: $T^{\mathcal{S}} = T^3 \vee T^4 = T^4$, $D_{T^{\mathcal{S}}}\left(\bigvee_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) = 4$ and $AB_\infty(\mathcal{S}; g) = -2$.

• If the size of the coalition is $|\mathcal{S}| = 3$, the coalitions are:

$\mathcal{S} = \{1,2,3\}$: $T^{\mathcal{S}} = T^1 \vee T^2 \vee T^3 = T^3$, $D_{T^{\mathcal{S}}}\left(\bigvee_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) = 2$ and $AB_\infty(\mathcal{S}; g) = 1$.

$\mathcal{S} = \{1,2,4\}$: $T^{\mathcal{S}} = T^1 \vee T^2 \vee T^4 = T^4$, $D_{T^{\mathcal{S}}}\left(\bigvee_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) = 5$ and $AB_\infty(\mathcal{S}; g) = -1$.

$\mathcal{S} = \{1,3,4\}$: $T^{\mathcal{S}} = T^1 \vee T^3 \vee T^4 = T^4$, $D_{T^{\mathcal{S}}}\left(\bigvee_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) = 4$ and $AB_\infty(\mathcal{S}; g) = -2$.

$\mathcal{S} = \{2,3,4\}$: $T^{\mathcal{S}} = T^2 \vee T^3 \vee T^4 = T^4$, $D_{T^{\mathcal{S}}}\left(\bigvee_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) = 4$ and $AB_\infty(\mathcal{S}; g) = -2$.

• If the size of the coalition is $|\mathcal{S}| = 4$, the grand coalitions is obtained:

$\mathcal{K} = \{1,2,3,4\}$: $T^{\mathcal{K}} = T^1 \vee T^2 \vee T^3 \vee T^4 = T^4$, $D_{T^{\mathcal{K}}}\left(\bigvee_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) = 4$ and $AB_\infty(\mathcal{K}; g) = -2$.

This example shows that many coalitions exhibit some improvement of technical efficiency (negative bias). Now, using a concept solution of cooperative game, the Shapley value, it is shown that the solution lies in the core of the game:

$$Sh_k := \sum_{\mathcal{S}\subseteq\mathcal{K}\setminus\{k\}} \frac{(|\mathcal{K}| - 1 - |\mathcal{S}|)!|\mathcal{S}|!}{|\mathcal{K}|!} \times \left[D_{T^{\mathcal{S}\cup\{k\}}}\left(\bigvee_{j\in\mathcal{S}\cup\{k\}} x^j, \bigvee_{j\in\mathcal{S}\cup\{k\}} y^j; g\right) - D_{T^{\mathcal{S}}}\left(\bigvee_{j\in\mathcal{S}} x^j, \bigvee_{j\in\mathcal{S}} y^j; g\right)\right].$$

The Shapley value yields the contribution of each firm to the overall efficiency of the grand coalition $\mathcal{K}$:

$Sh_1 = 0.6667$,

$Sh_2 = 0.3333$,

$Sh_3 = -0.6667$,

$Sh_4 = 3.6667$.

It can be checked that (efficiency rule):

$$\sum_{k\in\mathcal{K}} Sh_k = D_{T^{\mathcal{K}}}\left(\bigvee_{k\in\mathcal{S}} x^k, \bigvee_{k\in\mathcal{S}} y^k; g\right) = 4.$$

Note that the negative score of firm 3 ($Sh_3 = -0.6667$) outlines its efficiency *i.e.* its expected efficiency in joining any given coalition, in other words its ability to improve technical efficiency when merging with other firms. This is relevant since firm 3 is the unique firm being efficient $D_{T^3}((x^3, y^3); g) = 0$. The solution lies in the core, indeed individual and collective rationalities are respected. Individual rationality is respected: $Sh_k \leq D_{T^k}(x^k, y^k; g)$ for all $k = 1, \ldots, 4$. Also, collective rationality is matched since $\sum_{k \in \mathcal{S}} Sh_k \leq D_{T^{\mathcal{S}}}(\cdot)$ for all $\mathcal{S} \subset \mathcal{K}$.

This example allows one to learn more about cooperation. For instance group $\mathcal{S} = \{1, 2\}$ is not group-efficient in the sense that the technical bias is positive $AB_\infty(\{1, 2\}; g) = 2$. Firm 1 is capable to produce 1 with 2 inputs, whereas firm 3 produces 2 with 1 input. Then, it makes sense to predict that moving from coalition $\{1, 2\}$ to coalition $\{1, 2, 3\}$ will improve technical efficiency. This is exactly the case since the bias is lower: $AB_\infty(\{1, 2, 3\}; g) = 1 \leq AB_\infty(\{1, 2\}; g) = 2$. However, in the case of the standard sum ($\alpha = 1$) the result is not intuitive because the bias remains constant $AB_1(\{1, 2, 3\}; g) = AB_1(\{1, 2\}; g) = 2$.

As pointed out by a referee, the previous approach combines two types of effects. It mixes the effect of the firm's relative efficiency with that of the firm's technology after merging. As a consequence, the Shapley contributions computed above, *i.e.* the contributions of the expected efficiency of each firm to the overall amount of efficiency of the group, must be cautiously interpreted. Indeed, a firm may be very inefficient but may have a comparatively advanced technology or it may be relatively efficient but with a comparatively inferior technology. Then, after merging with others, firm coalitions produce effects that depend on the aggregate technology as well as on the firm's relative efficiency. In order to disentangle these two components, it is possible to take recourse to the well-known Luenberger productivity indicator.[10] The Luenberger indicator based on the directional distance function is given by:

$$\begin{aligned} L^{t,t+1}(x_t, y_t, x_{t+1}, y_{t+1}; g) :=& \frac{1}{2}\left[D_T^t(x_t, y_t; g) - D_T^t(x_{t+1}, y_{t+1}; g)\right] \\ &+ \frac{1}{2}\left[D_T^{t+1}(x_t, y_t; g) - D_T^{t+1}(x_{t+1}, y_{t+1}; g)\right], \end{aligned} \tag{6.1}$$

where t and $t+1$ denote time t and $t+1$, respectively. The first term in bracket yields the improvement (decline) of productivity inherent to technology at time t when it is positive (negative). The second term in bracket yields the improvement (decline) of productivity due to technology at time $t+1$ when it is positive (negative). Assuming that the gap between time t and $t+1$ corresponds to the time necessary to join a coalition, then it is possible to measure the components of efficiency change and technical change inherent to the cooperation. For instance, if we assume that firm 4 is alone at time t

[10]See Caves *et al.* (1982) for an analogous decomposition of the Malmquist index.

(with technology T^4) and that it joins the grand coalition at time $t+1$ (with technology $T^{\mathcal{K}}$), then the Luenberger indicator of firm 4 is given by:

$$L_4^{t,t+1}(x_t^4, y_t^4, x_{t+1}^4, y_{t+1}^4; g) = \frac{1}{2}\Big[D_{T^4}(x_t^4, y_t^4; g) - D_{T^4}\big(\bigvee_{j\in\mathcal{K}} x_{t+1}^j, \bigvee_{j\in\mathcal{K}} y_{t+1}^j; g\big)\Big] + \frac{1}{2}\Big[D_{T^{\mathcal{K}}}(x_t^4, y_t^4; g) - D_{T^{\mathcal{K}}}\big(\bigvee_{j\in\mathcal{K}} x_{t+1}^j, \bigvee_{j\in\mathcal{K}} y_{t+1}^j; g\big)\Big].$$

It is decomposable into two components as follows:

$$L_4^{t,t+1}(x_t^4, y_t^4, x_{t+1}^4, y_{t+1}^4; g) = \Big[D_{T^4}(x_t^4, y_t^4; g) - D_{T^{\mathcal{K}}}\big(\bigvee_{j\in\mathcal{K}} x_{t+1}^j, \bigvee_{j\in\mathcal{K}} y_{t+1}^j; g\big)\Big] + \frac{1}{2}\Big[D_{T^{\mathcal{K}}}\big(\bigvee_{j\in\mathcal{K}} x_{t+1}^j, \bigvee_{j\in\mathcal{K}} y_{t+1}^j; g\big) - D_{T^4}\big(\bigvee_{j\in\mathcal{K}} x_{t+1}^j, \bigvee_{j\in\mathcal{K}} y_{t+1}^j; g\big) + D_{T^{\mathcal{K}}}(x_t^4, y_t^4; g) - D_{T^4}(x_t^4, y_t^4; g)\Big].$$

The first term in bracket yields the *efficiency change* and the second one the *technical change*. The result of the productivity decomposition of firm 4 is:

$$L_4^{t,t+1}(x_t^4, y_t^4, x_{t+1}^4, y_{t+1}^4; g) = [6-4] + \frac{1}{2}[4-4+6-6] = 2.$$

Firm 4 is the least efficient with $D_{T^4}(x_t^4, y_t^4; g) = 6$. Its Shapley value is 3.6667, which indicates that firm 4 may improve its technical efficiency when it contemplates doing all possible coalitions with the other firms. When we simulate that firm 4 joins the grand coalition only, the indicator is positive (2), consequently there is an improvement of productivity. The Luenberger indicator shows that the efficiency change amounts to 2 whereas the technical change is valued to be 0. This means that the overall productivity indicator of 2 is only due to the efficiency change. There is no effect of technical change because, in this illustration, it is apparent that $T^4 = T^{\mathcal{K}}$. As a consequence, it is always possible to know whether the cooperation between firms is due to efficiency change or technical change. The productivity decomposition of firm 1 is given by:

$$L_1^{t,t+1}(x_t^1, y_t^1, x_{t+1}^1, y_{t+1}^1; g) = [1-4] + \frac{1}{2}[4-0+5-1] = 1.$$

Its efficiency change is negative showing that when joining the grand coalition firm 1 loses some efficiency (-3), however it takes benefit from the technology of the grand coalition so that its technical change is positive (4). For firm 2, we have an overall decrease of productivity:

$$L_2^{t,t+1}(x_t^2, y_t^2, x_{t+1}^2, y_{t+1}^2; g) = [1-4] + \frac{1}{2}[4-2+2-1] = -1.5.$$

There is a negative efficiency change (-3) with an increase of technical change (1.5). For firm 3, a decrease of productivity is also recorded:

$$L_3^{t,t+1}(x_t^3, y_t^3, x_{t+1}^3, y_{t+1}^3; g) = [0-4] + \frac{1}{2}[4-2+1-0] = -2.5.$$

The examples above show that the gain/loss of cooperation can be decomposed to explain the increase/decrease of productivity when any firm joins one given coalition. It is noteworthy that this decomposition is possible whenever infeasibility does not occur, see Briec and Kerstens (2009).

Another possibility would be to apply the Shapley value on the Luenberger indicator as the characteristic function of the game. In this case, it is possible to take into account the fact that one firm may join all possible coalitions. For firm k, we would obtain:

$$Sh_k(L_k^{t,t+1}) := \sum_{\mathcal{S}\subseteq\mathcal{K}\setminus\{k\}} \frac{(|\mathcal{K}|-1-|\mathcal{S}|)!|\mathcal{S}|!}{|\mathcal{K}|!} \times$$
$$\Big[L_k^{t,t+1}\big(x_t^k, y_t^k, \bigvee_{j\in\mathcal{S}\cup\{k\}} x_{t+1}^j, \bigvee_{j\in\mathcal{S}\cup\{k\}} y_{t+1}^j; g\big) - L_k^{t,t+1}\big(x_t^k, y_t^k, \bigvee_{j\in\mathcal{S}} x_{t+1}^j, \bigvee_{j\in\mathcal{S}} y_{t+1}^j; g\big)\Big].$$

This would provide, by the use of the decomposability property of the Luenberger indicator into two terms, the expected contribution of each firm of the efficiency change and that of the technical change to the overall efficiency of the group.[11]

If the Shapley value is applied to the aggregation bias as the characteristic function of the game, this would yield the ability to interpret the result as costs or savings games,

$$Sh_k(AB_\alpha) := \sum_{\mathcal{S}\subseteq\mathcal{K}\setminus\{k\}} \frac{(|\mathcal{K}|-1-|\mathcal{S}|)!|\mathcal{S}|!}{|\mathcal{K}|!} \cdot [AB_\alpha(\mathcal{S}\cup\{k\}; g) - AB_\alpha(\mathcal{S}; g)],$$

in which it is measured either the expected savings of technical efficiency of each firm $Sh_k(AB_\alpha) \leq 0$ or the expected cost of technical efficiency of each firm $Sh_k(AB_\alpha) \geq 0$.

Finally, theoretical findings must be associated with real world applications. For this purpose, it is important to show the relevance of our approach to DEA frameworks. After aggregating inputs and outputs over each group, it is possible to compute the directional distance function with standard linear programming. Another option is to use the non-linear distance function

[11] See also Mussard *et al.* (2006) for the application of the Shapley value to linearly decompose the Malmquist index into two components.

introduced by Ravelojaona (2019) for CES-CET technologies. With a pre-assigned vector $g = (g_i, g_o) \in \mathbb{R}_+^{n+m}$ and for (semilattice) aggregated inputs and outputs vectors $(x_{\mathcal{S}}, y_{\mathcal{S}}) \in \mathbb{R}_+^{n+m}$, it is given by:

$$D_{T^{\mathcal{S}}}(x_{\mathcal{S}}, y_{\mathcal{S}}; g) = \sup_{\delta} \left\{ \delta \in \mathbb{R} : (x_{\mathcal{S}} \overset{\alpha}{+} (-\delta g_i), y_{\mathcal{S}} \overset{\alpha}{+} \delta g_o) \in T^{\mathcal{S}} \right\}.$$

With linear programming, it is estimated as follows:

$$\begin{aligned} D_{T^{\mathcal{S}}}(x_{\mathcal{S}}, y_{\mathcal{S}}; g) = \max \delta & \\ \text{s.t. } x^{\alpha}_{\mathcal{S},i0} - \delta^{\alpha} g^{\alpha}_i &\geq \sum_k \theta_k x^{\alpha}_{\mathcal{S},ik} \\ y^{\alpha}_{\mathcal{S},j0} + \delta^{\alpha} g^{\alpha}_o &\leq \sum_k \theta_k y^{\alpha}_{\mathcal{S},jk} \\ \sum_k \theta_k = 1 \; &; \; \delta, \theta_k \geq 0. \end{aligned}$$

7 Conclusion

The improvement of (the aggregated) technical efficiency of a group of firms has been shown to be possible for a suitable class of of aggregators. The aggregation bias may be negative and the result belongs to the core interior of the firm game. This approach generalizes the firm game, introduced by Briec and Mussard (2014), which was defined for the standard sum of technology sets. The aggregation bias, issued from the Φ_α-aggregator, takes different values: positive, negative or zero. Then, the cooperation between firms entails all possible cases, especially with aggregated semilattice technologies that respect the desirable assumptions (T1)-(T5).

Our result can be extended to DEA frameworks to check the convergence rate of the Φ_α-aggregator. It is possible to show that the aggregation bias may be negative for technology sets associated with values of α that do not necessarily tend to infinity.

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