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PROJECTIVE STRUCTURE ON RIEMANN SURFACE AND
NATURAL DIFFERENTIAL OPERATORS

INDRANIL BISWAS AND SORIN DUMITRESCU

Abstract. We investigate the holomorphic differential operators on a Riemann surface $M$. This is done by endowing $M$ with a projective structure. Let $\mathcal{L}$ be a theta characteristic on $M$. We explicitly describe the jet bundle $J^k(E \otimes \mathcal{L}^\otimes n)$, where $E$ is a holomorphic vector bundle on $M$ equipped with a holomorphic connection, for all $k$ and $n$. This provides a description of holomorphic differential operators from $E \otimes \mathcal{L}^\otimes n$ to another holomorphic vector bundle $F$ using the natural isomorphism $\text{Diff}^k(E \otimes \mathcal{L}^\otimes n, F) = F \otimes (J^k(E \otimes \mathcal{L}^\otimes n))^\ast$.

1. Introduction

The study of natural differential operators between natural bundles is a major topic in global analysis on manifolds (see, for instance, [Ni], [Te] and [KMS]).

Our aim here is to study natural holomorphic differential operators between natural holomorphic vector bundles on a Riemann surface $M$. To be able to formulate their description, we make use of the existence of compatible projective structures on $M$ (see [Gu]).

We recall that a projective structure on $M$ is given by a holomorphic coordinate atlas such that all the transition functions are restrictions of Möbius transformations. This approach to holomorphic differential operators on $M$ entails a detailed understanding of the action of $\text{SL}(2, \mathbb{C})$ on the jet bundles on $\mathbb{CP}^1$.

Let $\mathbb{V}$ be a complex vector space of dimension two. For notational convenience, denote by $X$ the space $\mathbb{P}(\mathbb{V})$ of complex lines in $\mathbb{V}$. Let $L$ be the tautological line bundle of degree one on $X$. The trivial holomorphic vector bundle $X \times \mathbb{V}$ on $X$ with fiber $\mathbb{V}$ will be denoted by $\mathbb{V}_1$, while the symmetric product $\text{Sym}^i(\mathbb{V}_1)$ will be denoted by $\mathbb{V}_j$. The holomorphically trivial line bundle $X \times \mathbb{C}$ on $X$ will be denoted by $\mathbb{V}_0$.

We prove the following (see Theorem 3.5):

**Theorem 1.1.** If $n < 0$ or $n \geq k$, then

$$J^k(L^n) = L^{n-k} \otimes \mathbb{V}_k.$$ 

If $k > n \geq 0$, then

$$J^k(L^n) = \mathbb{V}_n \oplus (L^{-(k+1)} \otimes \mathbb{V}_{k-n-1}).$$

Both the isomorphisms are $\text{SL}(\mathbb{V})$-equivariant.

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The natural forgetful homomorphism \( J^k(L^n) \rightarrow J^{k-1}(L^n) \) of holomorphic vector bundles with action of \( \text{SL}(\mathbb{V}) \) is described in Proposition 3.1, Proposition 3.2 and Proposition 3.4.

Since all the above mentioned isomorphisms and homomorphisms are \( \text{SL}(\mathbb{V}) \)-equivariant, we are able to translate them into the set-up of a Riemann surface equipped with a projective structure (see Section 4 to see that any projective structure admits an underlying atlas for which the transition maps are restrictions of elements in \( \text{SL}(\mathbb{V}) \)).

Let \( M \) be a compact Riemann surface equipped with a projective structure \( P \). Using \( P \), the above tautological line bundle \( L \) on \( X \) produces a holomorphic line bundle \( L \) on \( M \) with the property that \( L \otimes^2 = TM \). Moreover, using \( P \), the trivial vector bundle \( V_1 \) produces a holomorphic vector bundle \( V_1 \) of rank two over \( M \) that fits in the following short exact sequence of holomorphic vector bundles on \( M \):

\[
0 \rightarrow L^* \rightarrow V_1 \rightarrow L \rightarrow 0.
\]

This involves fixing a nonzero element of \( \bigwedge^2 \mathbb{V} \). The symmetric power \( \text{Sym}^j(V_1) \) will be denoted by \( V_j \), while the trivial line bundle \( \mathcal{O}_M \) will be denoted by \( V_0 \).

Let us now describe our results about natural differential holomorphic operators on a Riemann surface endowed with a projective structure.

The following is deduced using Theorem 1.1 (see Corollary 4.2):

**Corollary 1.2.** If \( n < 0 \) or \( n \geq k \), then there is a canonical isomorphism

\[
J^k(L^n) = L^{n-k} \otimes V_k.
\]

If \( k > n \geq 0 \), then there is a canonical isomorphism

\[
J^k(L^n) = V_n \oplus (L^{-(k+1)} \otimes V_{k-n-1}).
\]

This enables us to deduce the following result (see Lemma 4.4):

**Lemma 1.3.** Let \( M \) be a compact connected Riemann surface endowed with a projective structure \( P \). Consider \( E \) a holomorphic vector bundle on \( M \) equipped with a holomorphic connection.

If \( n < 0 \) or \( n \geq k \), then there is a canonical isomorphism

\[
J^k(E \otimes L^n) = E \otimes L^{n-k} \otimes V_k.
\]

If \( k > n \geq 0 \), then there is a canonical isomorphism

\[
J^k(E \otimes L^n) = E \otimes (V_n \oplus (L^{-(k+1)} \otimes V_{k-n-1})).
\]

Let \( E \) be a holomorphic vector bundle on \( M \) equipped with a holomorphic connection, as in the previous Lemma. Given a holomorphic vector bundle \( F \) on \( M \), denote by \( \text{Diff}_M^k(E \otimes L^n, F) \) the sheaf of holomorphic differential operators of order \( k \) from \( E \otimes L^n \) to \( F \).

The associated symbol homomorphism

\[
\text{Diff}_M^k(E \otimes L^n, F) \rightarrow \text{Hom}(E \otimes L^n, F) \otimes (TM)^{\otimes k}
\]

is described in Corollary 4.7 and Corollary 4.11.

Fix integers \( k, n \) and \( l \) such that at least one of the following two conditions is valid:
Also fix a section
\[ \theta_0 \in H^0(M, \text{End}(E) \otimes L^l). \]
Using \( \theta_0 \), and a projective structure \( P \) on \( M \), we construct a holomorphic differential operator
\[ \mathcal{T}_\theta \in H^0(M, \text{Diff}^k(E \otimes L^n, E \otimes L^{l+n-2k})). \]
The following theorem shows that \( \mathcal{T}_\theta \) is a lift of the symbol \( \theta_0 \) to a holomorphic differential operator (see Theorem 5.1):

**Theorem 1.4.** The symbol of the above holomorphic differential operator \( \mathcal{T}_\theta \) is the section \( \theta_0 \).

2. Preliminaries

2.1. Jet bundles and differential operators. Let \( X \) be a connected Riemann surface. The holomorphic cotangent bundle of \( X \) will be denoted by \( K_X \). Consider the Cartesian product \( X \times X \). For \( i = 1, 2 \), let
\[ p_i : X \times X \longrightarrow X \]
be the projection to the \( i \)-th factor. Let
\[ \Delta := \{ (x, x) \mid x \in X \} \subset X \times X \]
be the diagonal divisor. At times we shall identify \( \Delta \) with \( X \) using the mapping \( x \mapsto (x, x) \). For any nonnegative integer \( k \), consider the natural exact sequence
\[ \begin{align*}
0 & \longrightarrow \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow \mathcal{O}_{X \times X}(-k\Delta) \rightarrow \mathcal{O}_{X \times X}(-k\Delta)/\mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow 0.
\end{align*} \]
Using the Poincaré adjunction formula, we have
\[ \mathcal{O}_{X \times X}(-k\Delta)/\mathcal{O}_{X \times X}(-(k+1)\Delta) = K_X^{\otimes k}; \]
here \( X \) is identified with \( \Delta \). Let \( z \) be a holomorphic coordinate function on an open subset \( U \subset X \). The isomorphism in (2.3) over \((U \times U) \cap \Delta\) sends the section \((z \circ p_2 - z \circ p_1)^k\) of \( \mathcal{O}_{X \times X}(-k\Delta)/\mathcal{O}_{X \times X}(-(k+1)\Delta) \) over \((U \times U) \cap \Delta\) to the section \((dz)^{\otimes k}\) of \( K_X^{\otimes k}|_U \) (after identifying \( U \) with \((U \times U) \cap \Delta\)).

Let \( W \) be a holomorphic vector bundle on \( X \) of rank \( r_W \). For any nonnegative integer \( k \), consider the subsheaf
\[ (p_2^*W) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \subset p_2^*W. \]
The \( k \)-th order jet bundle \( J^k(W) \) is defined to be the direct image
\[ J^k(W) := p_1*((p_2^*W) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta))) \longrightarrow X, \]
which is a holomorphic vector bundle on \( X \) of rank \((k+1)r_W \).

For any \( k \geq 1 \), using the natural inclusions
\[ (p_2^*W) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \subset (p_2^*W) \otimes \mathcal{O}_{X \times X}(-k\Delta) \subset p_2^*W. \]
together with (2.3), we get a short exact sequence of holomorphic vector bundles on \(X\)
\[
0 \rightarrow W \otimes K_X^\otimes k \overset{\iota}{\rightarrow} J^k(W) \overset{\partial_k}{\rightarrow} J^{k-1}(W) \rightarrow 0.
\]

(2.5)

Let \(V\) be a holomorphic vector bundle on \(X\). Let \(\text{Diff}^k(W, V)\) denote the sheaf of holomorphic differential operators of order \(k\) from \(W\) to \(V\). It is the sheaf of holomorphic sections of
\[
\text{Diff}^k(W, V) := \text{Hom}(J^k(W), V) = V \otimes J^k(W)^*.
\]

(2.6)

Consider the homomorphism
\[
\text{Diff}^k(W, V) = V \otimes J^k(W)^* \overset{\text{Id}_V \otimes \iota^*}{\longrightarrow} V \otimes (W \otimes K_X^\otimes k)^* = (TX)^\otimes k \otimes \text{Hom}(W, V),
\]

(2.7)

where \(\iota\) is the homomorphism in (2.5). This homomorphism in (2.7) is known as the symbol map.

### 2.2. Line bundles on the projective line.

Let \(V\) be a complex vector space of dimension two. Now set
\[
X := \mathbb{P}(V)
\]
to be the compact Riemann surface of genus zero parametrizing all the complex lines in \(V\). Let
\[
L := \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}(V)}(1) \rightarrow X
\]
be the tautological holomorphic line bundle of degree one whose fiber over the point corresponding to a line \(\xi \subset V\) is the quotient line \(V/\xi\). For any integer \(n\), by \(L^n\) we shall denote the holomorphic line bundle
- \(L^\otimes n\), if \(n > 0\),
- \((L^\otimes -n)^*\), if \(n < 0\), and
- \(\mathcal{O}_X\) (the trivial line bundle), if \(n = 0\).

A theorem of Grothendieck asserts that any holomorphic vector bundle \(F\) on \(X\) of rank \(r\) is of the form
\[
F = \bigoplus_{i=1}^{r} L^{a_i},
\]
where \(a_i \in \mathbb{Z}\) [Gr, p. 122, Théorème 1.1]; moreover, the above integers \(\{a_i\}\) are uniquely determined, up to a permutation, by \(F\) [Gr, p. 122, Théorème 1.1], [At1, p. 315, Theorem 2]. The holomorphic cotangent bundle \(K_X\) is holomorphically isomorphic to \(L^{-2}\).

Let \(L \rightarrow X\)

(2.9)

be the tautological holomorphic line bundle of degree \(-1\) whose fiber over the point corresponding to a line \(\xi \subset V\) is \(\xi\) itself. This \(L\) is holomorphically isomorphic to \(L^{-1}\). We note that there is no natural isomorphism between \(L^{-1}\) and \(L\).

Let \(\text{SL}(V)\) be the complex Lie group consisting of all linear automorphisms of \(V\) that act trivially on the determinant line \(\wedge^2 V\). This group \(\text{SL}(V)\) acts on \(\mathbb{P}(V)\) as follows: the action of any \(A \in \text{SL}(V)\) sends a line \(\xi \subset V\) to the line \(A(\xi)\). The corresponding action homomorphism
\[
\text{SL}(V) \rightarrow \text{Aut}(X)
\]
is holomorphic, and its kernel is \(\pm \text{Id}_V\). The action of \(\text{SL}(V)\) has a tautological lift to an action of \(\text{SL}(V)\) on \(K_X^\otimes m\) for every integer \(m\).
The above action of $\text{SL}(V)$ on $\mathbb{P}(V)$ clearly lifts to the line bundle $L$. Indeed, the action of any $A \in \text{SL}(V)$ on $V$ produces an isomorphism

$$V/\xi \sim \to V/A(\xi)$$

for every line $\xi \subset V$. The resulting action of $\text{SL}(V)$ on $L$ produces an action of $\text{SL}(V)$ on $L^n$ for every $n$.

The diagonal action of $\text{SL}(V)$ on $X \times X$ preserves the divisor $\Delta$ in (2.1). This action of $\text{SL}(V)$ on $X \times X$, and the action of $\text{SL}(V)$ on $L^n$, together produce an action of $\text{SL}(V)$ on $J^k(L^n)$ defined in (2.4). The homomorphism $\phi_k$ in (2.5) is in fact $\text{SL}(V)$–equivariant.

3. Description of Jet Bundles on the Projective Line

Consider the short exact sequence of coherent sheaves on $X = \mathbb{P}(V)$

$$0 \longrightarrow (p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta) \longrightarrow p_1^*L^n \longrightarrow (p_2^*L^n)|_{(k+1)\Delta} \longrightarrow 0. \quad (3.1)$$

From (2.4) we know that $J^k(L^n)$ is the direct image of $(p_2^*L^n)|_{(k+1)\Delta}$ to $X$ under the projection $p_1$. We would investigate $J^k(L^n)$ using the long exact sequence of direct images for the short exact sequence of sheaves in (3.1).

Let

$$0 \longrightarrow R^0p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow R^0p_{1*}p_2^*L^n \longrightarrow R^0p_{1*}((p_2^*L^n)|_{(k+1)\Delta})$$

$$\quad =: J^k(L^n) \longrightarrow R^1p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta)) \longrightarrow R^1p_{1*}p_2^*L^n \longrightarrow 0 \quad (3.2)$$

be the long exact sequence of direct images, under $p_1$, associated to the short exact sequence in (3.1). Note that

$$R^1p_{1*}((p_2^*L^n)|_{(k+1)\Delta}) = 0,$$

because the support of $(p_2^*L^n)|_{(k+1)\Delta}$ is finite over $X$ (for the projection $p_1$).

Since the diagonal action of $\text{SL}(V)$ on $X \times X$ preserves the divisor $\Delta$, each sheaf in (3.2) is equipped with an action of $\text{SL}(V)$ that lifts the action of $\text{SL}(V)$ on $X$. Moreover, all the homomorphisms in the exact sequence (3.2) are $\text{SL}(V)$–equivariant.

In (3.2), the direct image $R^1p_{1*}p_2^*L^n$ is the trivial holomorphic vector bundle

$$R^1p_{1*}p_2^*L^n = X \times H^i(X, L^n) \quad (3.3)$$

over $X$ with fiber $H^i(X, L^n)$ for $i = 0, 1$.

Next, to understand $R^1p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k+1)\Delta))$, we first note that the holomorphic line bundle $\mathcal{O}_{X \times X}(\Delta)$ is holomorphically isomorphic to $(p_1^*L) \otimes (p_2^*L)$; however, there is no natural isomorphism between these two isomorphic line bundles. To construct an explicit isomorphism between them, fix a nonzero element

$$0 \neq \omega \in \wedge^2 V. \quad (3.4)$$

Note that $\omega$ defines a symplectic structure on $V$, in particular, $\omega$ produces an isomorphism

$$V^* \sim \to V. \quad (3.5)$$
The action of \( \text{SL}(\mathbb{V}) \) on \( \bigwedge^2 \mathbb{V} \), given by the action of \( \text{SL}(\mathbb{V}) \) on \( \mathbb{V} \), is the trivial action, and hence this action fixes the element \( \omega \). We note that \( \omega \) produces a holomorphic isomorphism

\[
\Psi : L^{-1} \rightarrow \mathbb{L}, \quad (3.6)
\]

where \( \mathbb{L} \) was constructed in (2.9). Indeed, from the short exact sequence of holomorphic vector bundles

\[
0 \rightarrow \mathbb{L} \rightarrow X \times \mathbb{V} \rightarrow L \rightarrow 0 \quad (3.7)
\]

we conclude that \( \mathbb{L} \otimes L = X \times \bigwedge^2 \mathbb{V} \). Therefore, the element \( \omega \in \bigwedge^2 \mathbb{V} \) gives a nowhere vanishing holomorphic section of \( \mathbb{L} \otimes L \). This section produces the isomorphism \( \Psi \) in (3.6). Note that the diagonal action of \( \text{SL}(\mathbb{V}) \) on \( X \times \mathbb{V} \) preserves the subbundle \( \mathbb{L} \) in (3.7); more precisely, the homomorphisms in (3.7) are \( \text{SL}(\mathbb{V}) \)–equivariant. The isomorphism \( \Psi \) in (3.6) is evidently \( \text{SL}(\mathbb{V}) \)–equivariant. It should be mentioned that \( \Psi \) depends on \( \omega \).

We have \( H^0(X, L) = \mathbb{V} \).

\[
H^0(X \times X, (p_1^*L) \otimes (p_2^*L)) = H^0(X, L) \otimes H^0(X, L) = \mathbb{V} \otimes \mathbb{V},
\]

and hence \( \omega \in \bigwedge^2 \mathbb{V} \subset \mathbb{V} \otimes \mathbb{V} \) in (3.4) defines a holomorphic section

\[
\hat{\omega} \in H^0(X \times X, (p_1^*L) \otimes (p_2^*L)). \quad (3.8)
\]

This section \( \hat{\omega} \) vanishes exactly on the diagonal \( \Delta \subset X \times X \). Consequently, there is a unique holomorphic isomorphism

\[
\Phi : (p_1^*L) \otimes (p_2^*L) \sim \to O_{X \times X}(\Delta)
\]

such that the section \( \Phi(\hat{\omega}) \), where \( \hat{\omega} \) is constructed in (3.8), coincides with the holomorphic section of \( O_{X \times X}(\Delta) \) given by the constant function 1 on \( X \times X \). Therefore, combining \( \Phi \) with \( \Psi \) in (3.6), we have

\[
(p_1^*\mathbb{L}) \otimes (p_2^*\mathbb{L}) = (p_1^*L)^{-1} \otimes (p_2^*L)^{-1} = (p_1^*\mathbb{L}) \otimes (p_2^*\mathbb{L}) = O_{X \times X}(-\Delta). \quad (3.9)
\]

Using the identification of \( X \) with \( \Delta \), the Poincaré adjunction formula gives that \( O_{X \times X}(\Delta)|_\Delta = TX \), where \( TX \) is the holomorphic tangent bundle of \( X \); it is the dual of the isomorphism in (2.3) for \( k = 1 \). Therefore, restricting the isomorphisms in (3.9) to \( \Delta \subset X \times X \), we have a holomorphic isomorphism

\[
\Phi_0 : L^2 = (\mathbb{L}^*)^\otimes 2 \sim \to TX. \quad (3.10)
\]

Using the isomorphisms in (3.9), we have

\[
(p_1^*L^n) \otimes O_{X \times X}(-(k + 1)\Delta) = (p_1^*L^{-k-1}) \otimes (p_2^*L^{n-k-1}) = (p_1^*\mathbb{L}^{k+1}) \otimes (p_2^*L^{n-k-1}).
\]

Hence by the projection formula, [Ha, p. 426, A4],

\[
R^i p_1\ast((p_2^*L^n) \otimes O_{X \times X}(-(k + 1)\Delta)) = R^i p_1\ast((p_1^*\mathbb{L}^{k+1}) \otimes (p_2^*L^{n-k-1})) = \mathbb{L}^{k+1} \otimes R^i p_1\ast(p_2^*L^{n-k-1}) \quad (3.11)
\]

for \( i = 0, 1 \).

We shall break \((n, k)\) into three cases.
3.1. The case of $n < 0$. In (3.2), we have $R^0p_1^*p_2^*L^n = 0$, because $n < 0$. Consequently, the short exact sequence in (3.2) becomes
\[ 0 \longrightarrow J^k(L^n) \longrightarrow R^1p_1^*(p_2^*L^n) \otimes O_{X \times X}(-(k + 1)\Delta)) \longrightarrow R^1p_1^*p_2^*L^n \longrightarrow 0. \] (3.12)

Now using (3.11), the exact sequence in (3.12) becomes the short exact sequence
\[ 0 \longrightarrow J^k(L^n) \longrightarrow \mathbb{L}^{k+1} \otimes R^1p_1^*(p_2^*L^{-k-1}) \longrightarrow R^1p_1^*p_2^*L^n \longrightarrow 0. \] (3.13)

Invoking Serre duality and using the isomorphism $\Phi_0$ in (3.10),
\[ H^1(X, L^n) = H^0(X, L^{-n-2})^* = \text{Sym}^{-n-2}(\mathbb{V})^*, \]
where $\text{Sym}^{-n-2}(\mathbb{V})$ denotes:
- the $(-n-2)$-th symmetric power of $\mathbb{V}$, if $-n-2 > 0$,
- $\mathbb{C}$, if $-n = 2$, and
- $0$, if $-n = 1$.

Therefore, $R^1p_1^*p_2^*L^n$ in (3.13) is the trivial holomorphic vector bundle
\[ X \times \text{Sym}^{-n-2}(\mathbb{V})^* \longrightarrow X \]
(see (3.3)).

For notational convenience, the trivial holomorphic vector bundle $X \times \text{Sym}^j(\mathbb{V})$ over $X$ with fiber $\text{Sym}^j(\mathbb{V})$ will be denoted by $\mathbb{V}_j$. The trivial line bundle $O_X$ will be denoted by $\mathbb{V}_0$. So we have
\[ R^1p_1^*p_2^*L^n = (\mathbb{V}_{-n-2})^* = \mathbb{V}_{-n-2}. \] (3.14)

Similarly, we have
\[ R^1p_1^*(p_2^*L^{-k-1}) = X \times \text{Sym}^k(\mathbb{V})^* = \mathbb{V}_{-n-1} \longrightarrow X. \] (3.15)

Hence $\mathbb{L}^{k+1} \otimes R^1p_1^*(p_2^*L^{-k-1})$ in (3.13) has the following description:
\[ \mathbb{L}^{k+1} \otimes R^1p_1^*(p_2^*L^{-k-1}) = \mathbb{L}^{k+1} \otimes \mathbb{V}_{-n-1}. \] (3.16)

Using (3.14) and (3.16), the homomorphism $\gamma_1$ in (3.13) is a homomorphism
\[ \gamma_1 : \mathbb{L}^{k+1} \otimes \mathbb{V}_{-n-1} \longrightarrow \mathbb{V}_{-n-2}. \] (3.17)

The homomorphism $\gamma_1$ in (3.17) is simply the contraction of elements of $\mathbb{V}$ by $\mathbb{V}^*$ (recall that $\mathbb{L} \subset X \times \mathbb{V}$). More precisely, consider the contraction homomorphism
\[ \mathbb{V} \otimes (\mathbb{V}^*)^\otimes j \longrightarrow (\mathbb{V}^*)^\otimes (j-1). \]

Since $\mathbb{L} \subset \mathbb{V}_1$ (see (3.7)), this contraction produces a holomorphic homomorphism of vector bundles
\[ \mathbb{L} \otimes \mathbb{V}_j^* \longrightarrow \mathbb{V}_{j-1}^*. \]

The homomorphism $\gamma_1$ in (3.17) is $(k+1)$–times iteration of this homomorphism starting with $j = k - n - 1$.

We shall compute the kernel of the homomorphism $\gamma_1$ in (3.17).

Taking dual of the short exact sequence in (3.7), we have
\[ L^* \subset X \times \mathbb{V}^* = \mathbb{V}_1^*. \]
Hence, using the isomorphism $\Psi$ in (3.6), we have
\[ L = L^* \subset V_1^*. \] (3.18)

For the natural duality paring $V_1 \otimes V_1^* \rightarrow O_X$, the annihilator of the subbundle $L \subset V_1$ in (3.7) is the subbundle $L \subset V_1^*$ in (3.18). Therefore, from the above description of the homomorphism $\gamma$ in (3.7) we conclude that the subbundle
\[ L^{k+1} \otimes (V^*_{k-n-1}) \supset L^{k+1} \otimes (L^{-n-1} \otimes V_k^*) = L^{k-n} \otimes V_k^* \]
is contained in the kernel of the surjective homomorphism $\gamma$; note that $L^{-n-1} \otimes V_k^*$ is realized as a subbundle of $V_{k-n-1}^*$ using the inclusion $L \subset V_1^*$ in (3.18). From (3.13) we know that $\gamma$ is surjective. Since
\[ \text{rank}(L^{k-n} \otimes V_k^*) = k + 1 = (k - n) - (-n - 1) = \text{rank}(L^{k+1} \otimes V_{k-n-1}^*) - \text{rank}(V_{n-2}^*), \]
we now conclude that
\[ L^{k-n} \otimes V_k^* = \text{kernel}(\gamma). \]

Therefore, from (3.13) it follows that
\[ J^k(L^n) = \text{kernel}(\gamma) = L^{k-n} \otimes V_k^*. \] (3.19)

Now using the isomorphism between $V_k$ and $V_k^*$ given by the isomorphism in (3.5), from (3.19) we deduce that
\[ J^k(L^n) = L^{k-n} \otimes V_k. \]

Moreover, using the isomorphism $\Psi$ in (3.6) we have
\[ J^k(L^n) = L^{n-k} \otimes V_k. \] (3.20)

Recall that all the homomorphisms in (3.2) are $\text{SL}(V)$-equivariant. From the construction of the isomorphism in (3.19) between $J^k(L^n)$ and $L^{k-n} \otimes V_k^*$ it is evident that it is $\text{SL}(V)$-equivariant. Therefore, the isomorphism in (3.20) is $\text{SL}(V)$-equivariant.

Fix $k \geq 1$ an integer. Using (3.20), the homomorphism $\phi_k$ in (2.5) is a projection
\[ \phi_k : L^{n-k} \otimes V_k = J^k(L^n) \rightarrow J^{k-1}(L^n) = L^{n-k+1} \otimes V_{k-1}. \] (3.21)

We shall describe the homomorphism in (3.21).

Recall that $V_1 = \text{Sym}^2(V_1) = \text{Sym}^2(X \times V)$, and the line bundle $L$ is a quotient of $V_1$ (see (3.7)). Therefore, we have a natural projection
\[ \varpi^0_k : V_k \rightarrow L \otimes V_{k-1}. \] (3.22)

To describe this $\varpi^0_k$ explicitly, let $q_L : V_1 \rightarrow L$ be the quotient map in (3.7). Consider the homomorphism
\[ q_L \otimes \text{Id}_{\otimes^2(k-1)} : V_1^{\otimes k} \rightarrow L \otimes V_1^{\otimes(k-1)}. \]

Note that $q_L \otimes \text{Id}_{\otimes^2(k-1)}$ sends the subbundle
\[ \text{Sym}^k(V_1) \subset V_1^{\otimes k} \]
to the subbundle $L \otimes V_{k-1} \subset L \otimes V_1^{\otimes(k-1)}$. The homomorphism $\varpi^0_k$ in (3.22) is this restriction of $q_L \otimes \text{Id}_{\otimes^2(k-1)}$.

**Proposition 3.1.** The homomorphism $\phi_k$ in (3.21) coincides with $\text{Id}_{L^{n-k}} \otimes \varpi^0_k$, where $\varpi^0_k$ is the homomorphism in (3.22).
Proof. Consider the short exact sequence in (3.12). We have the commutative diagram of homomorphisms

\[
\begin{array}{c}
0 \\
L^n \otimes K_X^{\otimes k} \\
\downarrow \\
J^k(L^n) \\
\downarrow \\
0 \\
0 \\
\end{array} \quad \begin{array}{c}
0 \\
L^n \otimes K_X^{\otimes k} \\
\downarrow \\
R^1p_1^*((p_2^*L^n) \otimes O_{X \times X}(-(k+1)\Delta)) \\
\downarrow \mu \\
R^1p_1^*p_2^*L^n \\
0 \\
\end{array}
\]

(3.23)

where the bottom exact sequence is the one in (3.12) with \( k \) substituted by \( k-1 \), the top exact sequence is the one in (3.12), the left-vertical exact sequence is the one in (2.5) for \( W = L^n \), and the other vertical exact sequence is the long exact sequence of direct images, under \( p_1 \), for the short exact sequence

\[
0 \longrightarrow (p_2^*L^n) \otimes O_{X \times X}(-(k+1)\Delta) \longrightarrow (p_2^*L^n) \otimes O_{X \times X}(-k\Delta) \longrightarrow (L^n \otimes K_X^{\otimes k})|_\Delta \longrightarrow 0
\]

of sheaves on \( X \times X \) obtained by tensoring with \( p_2^*L^n \) the short exact sequence given by (2.2).

Using (3.11) and (3.16) we have

\[
R^1p_1^*((p_2^*L^n) \otimes O_{X \times X}(-(k+1)\Delta)) = \mathbb{L}^{k+1} \otimes V_{k-n-1}^*.
\]

Replacing \( k \) by \( k-1 \), we have

\[
R^1p_1^*((p_2^*L^n) \otimes O_{X \times X}(-k\Delta)) = \mathbb{L}^k \otimes V_{k-n-2}^*.
\]

So the homomorphism \( \mu \) in (3.23) becomes a homomorphism

\[
\mu : \mathbb{L}^{k+1} \otimes V_{k-n-1}^* \longrightarrow \mathbb{L}^k \otimes V_{k-n-2}^*.
\]

This homomorphism is simply the contraction of elements of \( V^* \) by elements of \( \mathbb{L} \subset V \).

The proposition follows from this and the commutative diagram in (3.23). \( \square \)

3.2. The case of \( n \geq k \). Note that \( n \geq 0 \), because \( n \geq k \). Since now we have \( n-k-1 \geq -1 \), it follows that

\[
H^1(X, L^{n-k-1}) = 0,
\]

and hence using (3.11) and (3.3) it is deduced that

\[
R^1p_1^*((p_2^*L^n) \otimes O_{X \times X}(-(k+1)\Delta)) = 0.
\]

Consequently, the exact sequence in (3.2) becomes the short exact sequence

\[
0 \longrightarrow R^0p_1^*((p_2^*L^n) \otimes O_{X \times X}(-(k+1)\Delta)) \longrightarrow R^0p_1^*p_2^*L^n \longrightarrow J^k(L^n) \longrightarrow 0. \quad (3.24)
\]

Therefore, using (3.11) and (3.3), the exact sequence in (3.24) becomes the short exact sequence

\[
0 \longrightarrow \mathbb{L}^{k+1} \otimes R^0p_1^*(p_2^*L^{n-k-1}) = \mathbb{L}^{k+1} \otimes V_{n-k-1} \overset{\gamma_2}{\longrightarrow} X \times \text{Sym}^n(V) \quad (3.25)
\]
Consider the homomorphism given by the composition of the natural homomorphisms
\[ V \otimes V_j \longrightarrow X \times V \otimes (j+1) \longrightarrow X \times \text{Sym}^{j+1}(V) = V_{j+1}. \]
Since \( L \subset V \) (see (3.7)), this restricts to a homomorphism
\[ L \otimes V_j \longrightarrow V_{j+1}. \]
The homomorphism \( \gamma_2 \) in (3.25) is \((k+1)\)-fold iteration of this homomorphism starting with \( j = n - k - 1 \).

To compute the cokernel of this homomorphism \( \gamma_2 \), consider the surjective homomorphism \( V_1 \longrightarrow L \) in (3.7). It produces a surjective homomorphism
\[ \theta_2 : V_n \longrightarrow L_n \otimes \mathbb{V}_k. \]
It is straightforward to check that \( \theta_2 \circ \gamma_2 = 0 \). Consequently, \( \theta_2 \) produces a surjective homomorphism from cokernel(\( \gamma_2 \)) to \( L_n \otimes \mathbb{V}_k \). On the other hand,
\[ \text{rank}(V_n) - \text{rank}(L_{n-k} \otimes V_{n-k-1}) = k + 1 = \text{rank}(L_n \otimes \mathbb{V}_k). \]
Hence we conclude that the above surjection from cokernel(\( \gamma_2 \)) to \( L_n \otimes \mathbb{V}_k \) is also injective.

Therefore, from (3.25) it follows that
\[ J^k(L^n) = L_n^{n-k} \otimes \mathbb{V}_k. \]

Proposition 3.2. The homomorphism \( \phi_k \) in (3.27) coincides with \( \text{Id}_{L_n^{n-k}} \otimes \varpi^0_k \), where \( \varpi^0_k \) is the homomorphism in (3.22).

Proof. Consider the short exact sequence of sheaves in (3.24). We have the commutative diagram of homomorphism

\[
\begin{array}{ccccccccc}
0 & \to & R^0 p_1^\ast((p_2^2 L^n) \otimes \mathcal{O}_{X \times X}(-k \Delta)) & \to & R^0 p_1^\ast p_2^2 L^n & \to & J^k(L^n) & \to & 0 \\
& & \downarrow \mu & & \| & & \downarrow & & \\
0 & \to & R^0 p_1^\ast((p_2^2 L^n) \otimes \mathcal{O}_{X \times X}(-k \Delta)) & \to & R^0 p_1^\ast p_2^2 L^n & \to & J^{k-1}(L^n) & \to & 0 \\
& & \downarrow L_n \otimes K_X^{\otimes k} & & \downarrow & & \downarrow & & \\
0 & \to & 0 & & \downarrow & & \downarrow & & \\
& & \downarrow L_n \otimes K_X^{\otimes k} & & \downarrow & & \downarrow & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & & \\
\end{array}
\]

(3.28)
where the bottom exact row is (3.24) with $k - 1$ substituted in place of $k$, the top exact row is the one in (3.24), the right vertical exact sequence is the one in (2.5) and the left vertical exact sequence is a part of the long exact sequence of direct images, under $p_1$, for the short exact sequence

$$0 \to (p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k + 1)\Delta) \to (p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta) \to (L^n \otimes K_X^\otimes)^{|\Delta|} \to 0$$

of sheaves on $X \times X$ obtained by tensoring (2.3) with $p_2^*L^n$. Using (3.11), in (3.25) we saw that

$$R^0p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k + 1)\Delta)) = L^{k+1} \otimes \mathcal{V}_{n-k-1},$$

So replacing $k$ by $k - 1$, we have

$$R^0p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-k\Delta)) = L^k \otimes \mathcal{V}_{n-k}.$$  

Using these isomorphisms, the injective homomorphism

$$\mu : L^{k+1} \otimes \mathcal{V}_{n-k-1} = L^{k+1} \otimes Sym^{n-k-1}(\mathcal{V}_1) \to L^k \otimes Sym^{n-k}(\mathcal{V}_1)$$

in (3.28) is the natural homomorphism induced by the inclusion of $L$ in $\mathcal{V}_1$. Now the proposition follows from (3.28). \hfill \Box

The remaining case will now be considered.

### 3.3. The case of $k > n \geq 0$. Since $n \geq 0$, it follows that $H^1(X, L^n) = 0$. Hence in (3.2) we have

$$R^1p_1*p_2^*L^n = 0;$$

see (3.3). Since $n - k - 1 < 0$, we have $H^0(X, L^{n-k-1}) = 0$. Therefore, from (3.11) and (3.3) we conclude that

$$R^0p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k + 1)\Delta)) = 0.$$  

Consequently, the exact sequence in (3.2) becomes the short exact sequence

$$0 \to R^0p_{1*}p_2^*L^n \to J^k(L^n) \to R^1p_{1*}((p_2^*L^n) \otimes \mathcal{O}_{X \times X}(-(k + 1)\Delta)) \to 0. \quad (3.29)$$

Now using (3.11) and (3.3) together with the projection formula, the exact sequence in (3.29) becomes the exact sequence

$$0 \to \mathcal{V}_n \xrightarrow{\gamma_3} J^k(L^n) \xrightarrow{\theta_3} L^{k+1} \otimes R^1p_{1*}(p_2^*L^{n-k-1}) \to 0. \quad (3.30)$$

The homomorphisms $\gamma_3$ and $\theta_3$ in (3.30) are clearly $SL(\mathcal{V})$--equivariant.

**Lemma 3.3.** The short exact sequence in (3.30) has a canonical $SL(\mathcal{V})$--equivariant holomorphic splitting.

**Proof.** Since $k > n$, there is a natural projection

$$J^k(L^n) \to J^n(L^n)$$

(see (2.5)). Now $J^n(L^n) = \mathcal{V}_n$ by (3.26), so this projection defines a projection

$$\beta : J^k(L^n) \to \mathcal{V}_n. \quad (3.31)$$

It is straight-forward to check that $\beta \circ \gamma_3 = \text{Id}_{\mathcal{V}_n}$, where $\gamma_3$ and $\beta$ are constructed in (3.30) and (3.31) respectively. This produces an $SL(\mathcal{V})$--equivariant holomorphic splitting

$$J^k(L^n) = \text{image}(\gamma_3) \oplus \text{kernel}(\beta)$$

of the short exact sequence in (3.30). \hfill \Box
Lemma 3.3 gives a holomorphic decomposition
\[ J^k(L^n) = V_n \oplus (L^k \otimes R^1 p_* (p_2^* L^{n-k-1})) \] (3.32)
which is compatible with the actions of SL(\(\mathcal{V}\)). Now, \(R^1 p_* (p_2^* L^{n-k-1}) = \mathcal{V}_{k-n-1}^*\) (see (3.15)), and \(\mathcal{V}_{k-n-1}^* = \mathcal{V}_{k-n-1}\) which is constructed using the isomorphism in (3.5). Therefore, the decomposition in (3.32) can be reformulated as
\[ J^k(L^n) = V_n \oplus (L^k \otimes \mathcal{V}_{k-n-1}) = V_n \oplus (L^{-(k+1)} \otimes \mathcal{V}_{k-n-1}) \] (3.33)
using \(\Psi\) in (3.6). As before, the isomorphisms in (3.33) are compatible with the actions of SL(\(\mathcal{V}\)).

If \(k-1 > n \geq 0\), then from (3.33) we know that
\[ J^{k-1}(L^n) = V_n \oplus (L^k \otimes \mathcal{V}_{k-n-2}) = V_n \oplus (L^{-(k+1)} \otimes \mathcal{V}_{k-n-2}) \]
and if \(k-1 = n \geq 0\), then from (3.26) we have
\[ J^{k-1}(L^n) = \mathcal{V}_{k-1} \].

The proof of the following proposition is similar to the proofs of Proposition 3.1 and Proposition 3.2.

**Proposition 3.4.** If \(k-1 > n \geq 0\), then the projection \(J^k(L^n) \rightarrow J^{k-1}(L^n)\) in (2.5) is the homomorphism
\[ V_n \oplus (L^{-(k+1)} \otimes \mathcal{V}_{k-n-1}) \rightarrow V_n \oplus (L^{-k} \otimes \mathcal{V}_{k-n-2}) \]
which is the direct sum of the identity map of \(V_n\) and the natural contraction map \(L^{-(k+1)} \otimes \mathcal{V}_{k-n-1} \rightarrow L^{-k} \otimes \mathcal{V}_{k-n-2}\).

If \(k-1 = n \geq 0\), then the projection \(J^k(L^n) \rightarrow J^{k-1}(L^n)\) in (2.5) is the homomorphism
\[ V_n \oplus (L^{-(k+1)} \otimes \mathcal{V}_{k-n-1}) \rightarrow V_n \]
which is the direct sum of the identity map of \(V_n\) and the zero homomorphism of \(L^{-(k+1)} \otimes \mathcal{V}_{k-n-1} = L^{-(k+1)}\) (recall that \(V_0 = \mathcal{O}_X\)).

### 3.4. The jet bundle.

Combining (3.20), (3.26) and (3.33), we have the following:

**Theorem 3.5.** If \(n < 0\) or \(n \geq k\), then
\[ J^k(L^n) = L^{n-k} \otimes \mathcal{V}_k \].

If \(k > n \geq 0\), then
\[ J^k(L^n) = V_n \oplus (L^{-k} \otimes \mathcal{V}_{k-n-1}) \].

Both the isomorphisms are SL(\(\mathcal{V}\))–equivariant.

### 4. Holomorphic connections on vector bundles

In this section we construct and study natural differential holomorphic operators between natural holomorphic vector bundles on a Riemann surface equipped with a projective structure.
4.1. Jets of Riemann surfaces with a projective structure. Let $M$ be a compact connected Riemann surface of genus $g$. A holomorphic coordinate chart on $M$ is a pair of the form $(U, \varphi)$, where $U \subset M$ is an open subset and $\varphi : U \to \mathbb{P}(\mathcal{V}) = X$ is a holomorphic embedding. A holomorphic coordinate atlas on $M$ is a collection of coordinate charts $\{(U_i, \varphi_i)\}_{i \in I}$ such that $M = \bigcup_{i \in I} U_i$. A projective structure on $M$ is given by a holomorphic coordinate atlas $\{(U_i, \varphi_i)\}_{i \in I}$ such that for all pairs $(i, j) \in I \times I$ for which $U_i \cap U_j \neq \emptyset$, there exists an element $\tau_{j,i} \in \text{Aut}(\mathbb{P}(\mathcal{V})) = \text{PGL}(\mathcal{V})$ such that $\varphi_j \circ \varphi_i^{-1}$ is the restriction of $\tau_{j,i}$ to $\varphi_i(U_i \cap U_j)$. A holomorphic coordinate atlas satisfying this condition is called a projective atlas. Two such projective atlases $\{(U_i, \varphi_i)\}_{i \in I}$ and $\{(U_i', \varphi_i')\}_{i \in J}$ are called equivalent if their union $\{(U_i, \varphi_i)\}_{i \in I \cup J}$ is again a projective atlas. A projective structure on $M$ is an equivalence class of projective atlases (see [Gu]).

Note that the collection $\{\tau_{j,i}\}$ in the definition of a projective atlas forms a 1-cocycle with values in $\text{PGL}(\mathcal{V})$. The corresponding cohomology class in $H^1(M, \text{PGL}(\mathcal{V}))$ depends only on the projective structure and can be lifted to $H^1(M, \text{SL}(\mathcal{V}))$. Moreover the space of such lifts is in bijection with the theta characteristics of $M$ (see [Gu]). We recall that the theta characteristic on $M$ is a holomorphic line bundle $K_M^{1/2}$ on $M$ together with a holomorphic isomorphism of $K_M^{1/2} \otimes K_M^{1/2}$ with the holomorphic cotangent bundle $K_M$.

We fix a theta characteristic on $M$. So a projective structure on $M$ is given by a coordinate atlas $\{(U_i, \varphi_i)\}_{i \in I}$ together with an element

$$
\tau_{j,i} \in \text{SL}(\mathcal{V})
$$

for every ordered pair $(i, j) \in I \times I$ such that $U_i \cap U_j \neq \emptyset$ such that the following four conditions hold:

1. $\varphi_j \circ \varphi_i^{-1}$ is the restriction to $\varphi_i(U_i \cap U_j)$ of the automorphism of $X$ given by $\tau_{j,i}$,
2. $\tau_{i,k} \tau_{k,j} \tau_{j,i} = \text{Id}_X$ for all $i, j, k \in I$ such that $U_i \cap U_j \cap U_k \neq \emptyset$,
3. $\tau_{i,i} = \text{Id}_X$ for all $i \in I$, and
4. the theta characteristic corresponding to the data is the given one.

Two such collections of triples $\{(U_i, \varphi_i)\}_{i \in I}, \{\tau_{i,j}\}$ and $\{(U_i', \varphi_i')\}_{i \in J}, \{\tau_{i,j}'\}$ are called equivalent if their union $\{(U_i, \varphi_i)\}_{i \in I \cup J}, \{\tau_{i,j}\}$ is again a part of a collection of triples satisfying the above conditions. A $\text{SL}(\mathcal{V})$-projective structure on $M$ is an equivalence class of collection of triples satisfying the above conditions.

Let $P = \{(U_i, \varphi_i)_{i \in I}, \{\tau_{i,j}\}\}$ be a $\text{SL}(\mathcal{V})$-projective structure on $M$. Recall that the collection $\{\tau_{i,j}\}$ forms a 1-cocycle on $M$ with values in the group $\text{SL}(\mathcal{V})$. Hence $\{\tau_{i,j}\}$ produce a holomorphic rank two bundle $\mathcal{V}$ on $M$ equipped with a holomorphic connection $D$ (see [At2] for holomorphic connections). To construct the pair $(\mathcal{V}, D)$ explicitly, for each $i \in I$, consider the trivial holomorphic vector bundle $U_i \times \mathcal{V} \to U_i$ equipped with the trivial connection. For any ordered pair $(i, j) \in I \times I$ with $U_i \cap U_j \neq \emptyset$, glue $U_i \times \mathcal{V}$ and $U_j \times \mathcal{V}$ over $U_i \cap U_j$ using the automorphism of $(U_i \cap U_j) \times \mathcal{V}$ that sends any

$$(y, v) \in (U_i \cap U_j) \times \mathcal{V} \subset U_j \times \mathcal{V}$$

to $(y, \tau_{i,j}(v)) \in (U_i \cap U_j) \times \mathcal{V} \subset U_i \times \mathcal{V}$. This automorphism of $(U_i \cap U_j) \times \mathcal{V}$ is holomorphic, and it evidently preserves the trivial connection. Therefore, this gluing operation produces a holomorphic vector bundle on $M$, which we shall denote by $\mathcal{V}$, and a holomorphic connection on $\mathcal{V}$, which we shall denote by $D$. 

The diagonal action of SL($\mathcal{V}$) on $X \times \mathcal{V}$ preserves the subbundle $\mathbb{L}$ in (3.7). Therefore, the subbundles $\varphi_i(U_i) \times \mathbb{L} \subset \varphi(U_i) \times \mathcal{V}$ patch together compatibly to produce a holomorphic line subbundle of the vector bundle $\mathcal{V}$ on $M$ constructed above. This line subbundle of $\mathcal{V}$ will be denoted by $\mathcal{L}$. Let
\[ L := \mathcal{V}/\mathcal{L} \] (4.1)
be the quotient bundle.

Recall that the isomorphism $\Psi$ in (3.6) and the isomorphism $\Phi_0$ in (3.10) are both $\text{SL}(\mathcal{V})$–equivariant. However, it should be clarified that a nonzero element $\omega \in \bigwedge^2 \mathcal{V}$ (see (3.4)) was used in their construction. Since $\Psi$ and $\Phi_0$ are $\text{SL}(\mathcal{V})$–equivariant, they produce isomorphisms
\[ L = \mathcal{L}^* \quad \text{and} \quad L^{\otimes 2} = (\mathcal{L}^*)^{\otimes 2} = TM \] (4.2)
respectively. Recall that we fixed a theta characteristic on $\mathcal{V}$ coincides with the chosen one.

For any $j \geq 1$, the vector bundle $\text{Sym}^j(\mathcal{V})$ will be denoted by $\mathcal{V}_j$. Also, $\mathcal{V}_0$ will denote the trivial line bundle $\mathcal{O}_M$. The holomorphic connection $D$ on $\mathcal{V}$ induces a holomorphic connection on $\mathcal{V}_j$ for every $j \geq 1$; this connection on $\mathcal{V}_j$ will be denoted by $D_j$. The trivial connection on $\mathcal{V}_0 = \mathcal{O}_M$ will be denoted by $D_0$. The connections $D_m$ and $D_n$ on $\mathcal{V}_m$ and $\mathcal{V}_n$ respectively together produce a holomorphic connection on $\mathcal{V}_m \otimes \mathcal{V}_n$.

**Proposition 4.1.** Let $M$ be a compact connected Riemann surface equipped with a projective structure $P$. Take integers $m > n \geq 0$. Then $\mathcal{V}_m \otimes \mathcal{V}_n$ has a canonical decomposition
\[ \mathcal{V}_m \otimes \mathcal{V}_n = \bigoplus_{i=0}^{n} \mathcal{V}_{m+n-2i}. \]
The above isomorphism takes the connection on $\mathcal{V}_m \otimes \mathcal{V}_n$ induced by $D_m$ and $D_n$ to the connection $\bigoplus_{i=0}^{n} D_{m+n-2i}$ on $\bigoplus_{i=0}^{n} \mathcal{V}_{m+n-2i}$.

**Proof.** First consider the case of $X = \mathbb{P}(\mathcal{V})$ investigated in Section 2. The $\text{SL}(\mathcal{V})$–module $\text{Sym}^j(\mathcal{V})$ is irreducible for every $j \geq 0$ [FH, p. 150], and furthermore, for $m > n \geq 0$, the $\text{SL}(\mathcal{V})$–module $\text{Sym}^m(\mathcal{V}) \otimes \text{Sym}^n(\mathcal{V})$ decomposes as
\[ \text{Sym}^m(\mathcal{V}) \otimes \text{Sym}^n(\mathcal{V}) = \bigoplus_{i=0}^{n} \text{Sym}^{m+n-2i}(\mathcal{V}) \] (4.3)
[FH, p. 151, Ex. 11.11].

Choose data $\{(U_i, \varphi_i)\}_{i \in I}, \{\tau_{i,j}\}$ representing a $\text{SL}(\mathcal{V})$–projective structure associated to $P$ and to a fixed theta characteristic on $M$. On each $U_i$ we have the holomorphic vector bundles $\varphi_i^* \mathcal{V}_j = U_i \times \text{Sym}^j(\mathcal{V})$ equipped with the trivial connection. Patching these together using $\{\tau_{i,j}\}$ as the transition functions we get $\mathcal{V}_j$ equipped with the connection $D_j$ for every $j \geq 0$. Now it can be shown that the isomorphism in (4.3) over each $U_i$ patch together compatibly to give a global isomorphism over $M$. Indeed, this is an immediate consequence of the fact that the isomorphism in (4.3) intertwines the actions of $\text{SL}(\mathcal{V})$. This completes the proof. $\square$

Henceforth we assume that $\text{genus}(M) = g \geq 1$. Since all the isomorphisms in Theorem 3.5 are $\text{SL}(\mathcal{V})$–equivariant, the following corollary is deduced from Theorem 3.5.
Corollary 4.2. Let $M$ be a connected Riemann surface equipped with a projective structure $P$. If $n < 0$ or $n \geq k$, then there is a canonical isomorphism
\[ J^k(\mathbb{L}^n) = \mathbb{L}^{n-k} \otimes \mathcal{V}_k. \]
If $k > n \geq 0$, then there is a canonical isomorphism
\[ J^k(\mathbb{L}^n) = \mathcal{V}_n \oplus (\mathbb{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1}). \]

Remark 4.3. Note that Proposition 4.1, Corollary 4.2 and (4.2) together give a description of any jet bundles of the form $J^k(\mathbb{L}^a) \otimes J^k(\mathbb{L}^b)$ as a direct sum of vector bundles of the form $\mathcal{V}^p \otimes \mathbb{L}^q$.

4.2. Jets of vector bundles with a holomorphic connection. Let $M$ be a compact connected Riemann surface. Let $E$ be a holomorphic vector bundle on $M$ equipped with a holomorphic connection $D_E$. Take a holomorphic vector bundle $W$ on $M$. We will construct a holomorphic homomorphism from $J^k(E \otimes W)$ to $E \otimes J^k(W)$ for all $k \geq 0$. To construct this homomorphism, let $f_1, f_2 : M \times M \rightarrow M$ be the projections to the first and second factors respectively. The diagonal divisor
\[ \{(x, x) \mid x \in M\} \subset M \times M \]
will be denoted by $\Delta_M$. Recall from (2.4) that
\[ J^k(E \otimes W) := f_{1*}(((f_1^* E \otimes f_2^* W))/((f_1^* E \otimes f_2^* W) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))) \quad (4.4) \]
\[ = f_{1*}(((f_1^* E) \otimes (f_2^* W))/((f_1^* E) \otimes (f_2^* W) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))) \rightarrow M. \]
Consider the pulled back connection $f_2^* D_E$ on $f_2^* E$. Since the sheaf of holomorphic two-forms on $M$ is the zero sheaf, the connection $D_E$ is flat (the curvature vanishes identically). Hence the connection $f_2^* D_E$ is also flat. Therefore, on any analytic neighborhood $U$ of $\Delta_M \subset M \times M$ that admits a deformation retraction to $\Delta_M$, the two vector bundles $(p_1^* E)|_U$ and $(p_2^* E)|_U$ are identified using parallel translation. Invoking this isomorphism between $(p_1^* E)|_U$ and $(p_2^* E)|_U$, from (4.4) we have
\[ J^k(E \otimes W) = f_{1*}(((f_1^* E) \otimes (f_2^* W))/((f_1^* E) \otimes (f_2^* W) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))). \]
Hence the projection formula gives that
\[ J^k(E \otimes W) = E \otimes f_{1*}((f_2^* W)/((f_2^* W) \otimes \mathcal{O}_{M \times M}(-(k+1)\Delta_M))) = E \otimes J^k(W). \quad (4.5) \]

Lemma 4.4. Let $M$ be a connected Riemann surface equipped with a projective structure $P$. Let $E$ be a holomorphic vector bundle on $M$ equipped with a holomorphic connection $D_E$. If $n < 0$ or $n \geq k$, then there is a canonical isomorphism
\[ J^k(E \otimes \mathbb{L}^n) = E \otimes \mathbb{L}^{n-k} \otimes \mathcal{V}_k. \]
If $k > n \geq 0$, then there is a canonical isomorphism
\[ J^k(E \otimes \mathbb{L}^n) = E \otimes (\mathcal{V}_n \oplus (\mathbb{L}^{-(k+1)} \otimes \mathcal{V}_{k-n-1})). \]

Proof. This follows from (4.5) and Corollary 4.2. We omit the details.
4.3. **The symbol map.** As before, $M$ is a compact connected Riemann surface equipped with a projective structure $P$, and $E$ is a holomorphic vector bundle over $M$ equipped with a holomorphic connection $D_E$. Consider the projection

$$
\phi_k : J^k(E \otimes L^n) \longrightarrow J^{k-1}(E \otimes L^n)
$$

(4.6)

in (2.5). We will describe $\phi_k$ in terms of the isomorphisms in Lemma 4.4.

4.3.1. **Case where either $n < 0$ or $n \geq k$.** Assume that at least one of the following two conditions is valid:

(1) $n < 0$, or

(2) $n \geq k$.

In view of Lemma 4.4, this implies that

$$
J^k(E \otimes L^n) = E \otimes L^{n-k} \otimes V_k 
$$

and

$$
J^{k-1}(E \otimes L^n) = E \otimes L^{n-k+1} \otimes V_{k-1}
$$

when $k \geq 1$ (note that in (4.6) we have $k \geq 1$). Using (4.7), the projection $\phi_k$ in (4.6) is a surjective homomorphism

$$
\phi_k : E \otimes L^{n-k} \otimes V_k \longrightarrow E \otimes L^{n-k+1} \otimes V_{k-1}.
$$

(4.8)

Recall that $V_j = \text{Sym}^j(V)$, and the line bundle $L$ is a quotient of $V$ (see (4.1)). Therefore, we have a natural projection

$$
\varpi_k : V_k \longrightarrow L \otimes V_{k-1}.
$$

(4.9)

To describe $\varpi_k$ explicitly, let $q_L : V \longrightarrow L$ be the quotient map. Consider the homomorphism

$$
q_L \otimes \text{Id}_{V^\otimes(k-1)} : V^\otimes k \longrightarrow L \otimes V^\otimes(k-1).
$$

Note that $q_L \otimes \text{Id}_{V^\otimes(k-1)}$ sends the subbundle $\text{Sym}^k(V) \subset V^\otimes k$ to the subbundle $L \otimes V_{k-1} \subset L \otimes V^\otimes(k-1)$. The homomorphism $\varpi_k$ in (4.9) is this restriction of the homomorphism $q_L \otimes \text{Id}_{V^\otimes(k-1)}$.

**Proposition 4.5.** The homomorphism $\phi_k$ in (4.8) coincides with $\text{Id} \otimes \varpi_k$, where $\text{Id}$ denotes the identity map of $E \otimes L^{n-k}$, and $\varpi_k$ is the homomorphism in (4.9).

**Proof.** This follows from Proposition 3.1 and Proposition 3.2. \hfill \Box

Consider the inclusion homomorphism $\iota_L : \mathcal{L} \hookrightarrow \mathcal{V}$ (see (4.1)). We have the subbundle

$$
\iota' := \text{Id}_{E} \otimes \iota_L^\otimes k : E \otimes \mathcal{L}^\otimes k \hookrightarrow E \otimes \text{Sym}^k(V) = E \otimes V_k.
$$

Using the isomorphism $L = \mathcal{L}^*$ in (4.2), the above homomorphism $\iota'$ produces a homomorphism

$$
\tilde{\iota} := \iota' \otimes \text{Id}_{L^{n-k}} : E \otimes L^{n-2k} = E \otimes \mathcal{L}^\otimes k \otimes L^{n-k} \hookrightarrow E \otimes V_k \otimes L^{n-k}.
$$

(4.10)

The following is an immediate consequence of Proposition 4.5.

**Corollary 4.6.** The kernel of the surjective homomorphism $\phi_k$ in (4.8) coincides with the subbundle $E \otimes L^{n-2k} \hookrightarrow E \otimes V_k \otimes L^{n-k}$ given by the image of the homomorphism $\tilde{\iota}$ in (4.10).
Let $F$ be any holomorphic vector bundle on $M$. Using (2.6) and (4.7), we have
\[
\text{Diff}_M^k(E \otimes L^n, F) = F \otimes (E \otimes L^{n-k} \otimes V_k)^* = F \otimes L^{k-n} \otimes (E \otimes V_k)^*. \tag{4.11}
\]
Using the homomorphism $\tilde{\iota}$ in (4.10) construct the homomorphism
\[
\text{Id}_F \otimes \iota^* : F \otimes (E \otimes V_k \otimes L^{n-k})^* \longrightarrow F \otimes (E \otimes L^{n-2k})^* = \text{Hom}(E, F) \otimes L^{2k-n}. \tag{4.12}
\]
Combining (4.11) and (4.12), we have a surjective homomorphism
\[
\sigma_k : \text{Diff}_M^k(E \otimes L^n, F) \longrightarrow \text{Hom}(E, F) \otimes L^{2k-n}.
\]

The following is an immediate consequence of Corollary 4.6.

**Corollary 4.7.** The symbol homomorphism
\[
\text{Diff}_M^k(E \otimes L^n, F) \longrightarrow \text{Hom}(E \otimes L^n, F) \otimes (TM)^@k
\]
in (2.7) coincides with the above homomorphism $\sigma_k$ after TM is identified with $L^@2$ as in (4.2).

4.3.2. Case where $k > n \geq 0$. We first assume that $k-1 > n \geq 0$. Therefore, from Lemma 4.4 we have $J^k(E \otimes L^n) = E \otimes (V_n \oplus (L^{-(k+1)} \otimes V_{k-n-1}))$ and
\[
J^{k-1}(E \otimes L^n) = E \otimes (V_n \oplus (L^{-k} \otimes V_{k-n-2})).
\]
Let
\[
\phi_k : E \otimes (V_n \oplus (L^{-(k+1)} \otimes V_{k-n-1})) = J^k(E \otimes L^n) \tag{4.13}
\]
\[
\longrightarrow J^{k-1}(E \otimes L^n) = E \otimes (V_n \oplus (L^{-k} \otimes V_{k-n-2}))
\]
be the projection in (2.5). Just as the homomorphism $\text{Id} \otimes \varpi_k$ in Proposition 4.5, we have a homomorphism
\[
\widehat{\varpi}_k : E \otimes L^{-(k+1)} \otimes V_{k-n-1} \longrightarrow E \otimes L^{-k} \otimes V_{k-n-2}.
\]

**Lemma 4.8.** The homomorphism $\phi_k$ in (4.13) coincides with the homomorphism
\[
\text{Id}_{E \otimes V_n} \oplus \widehat{\varpi}_k : E \otimes (V_n \oplus (L^{-(k+1)} \otimes V_{k-n-1})) \longrightarrow E \otimes (V_n \oplus (L^{-k} \otimes V_{k-n-2})).
\]

**Proof.** This follows from Proposition 3.4. \qed

Note that using (4.2),
\[
\ker(\widehat{\varpi}_k) = E \otimes L^{-(k+1)} \otimes L^{\oplus(k-n-1)} = E \otimes K_M^{\oplus k} L^n.
\]

The following is an immediate consequence of Lemma 4.8.

**Corollary 4.9.** Let $F$ be any holomorphic vector bundle on $M$. The symbol homomorphism
\[
\text{Diff}_M^k(E \otimes L^n, F) \longrightarrow \text{Hom}(E \otimes L^n, F) \otimes (TM)^@k
\]
coincides with $\text{Id}_F \otimes \iota^*$, where $\iota : \ker(\widehat{\varpi}_k) \hookrightarrow E \otimes L^{-(k+1)} \otimes V_{k-n-1}$ is the inclusion map.
We now assume that \( k - 1 = n \geq 0 \).

Under this assumption, from Lemma 4.4 we have
\[
J^k(E \otimes L^n) = E \otimes (V_n \oplus L^{-(k+1)}) \quad \text{and} \quad J^{k-1}(E \otimes L^n) = E \otimes V_n.
\]

Let
\[
\phi_k : E \otimes (V_n \oplus L^{-(k+1)}) = J^k(E \otimes L^n) \longrightarrow J^{k-1}(E \otimes L^n) = E \otimes V_n \tag{4.14}
\]
be the projection in (2.5).

**Lemma 4.10.** The homomorphism \( \phi_k \) in (4.14) coincides with the projection
\[
E \otimes (V_n \oplus L^{-(k+1)}) \longrightarrow E \otimes V_n
\]
to the first factor.

**Proof.** This follows from Proposition 3.4. \( \square \)

The following is an immediate consequence of Lemma 4.10.

**Corollary 4.11.** Let \( F \) be any holomorphic vector bundle on \( M \). The symbol homomorphism
\[
\operatorname{Diff}_M^k(E \otimes L^n, F) \longrightarrow \operatorname{Hom}(E \otimes L^n, F) \otimes (TM)^{\otimes k}
\]
coincides with \( \operatorname{Id}_F \otimes p' \), where
\[
p' : E^* \otimes (V_n \oplus L^{-(k+1)})^* \longrightarrow E^* \otimes (L^{-(k+1)})^*
\]
is the natural projection.

## 5. Lifting of Symbol

As before, \( M \) is a connected Riemann surface equipped with a projective structure \( P \), and \( E \) is a holomorphic vector bundle on \( M \) equipped with a holomorphic connection \( D_E \). The connection \( D_E \) on \( E \) induces a holomorphic connection on \( \operatorname{End}(E) \). This induced connection on \( \operatorname{End}(E) \) will be denoted by \( \mathcal{D}_E \).

Take integers \( k, n \) and \( l \) such that at least one of the following two conditions is valid:

1. \( n < 0 \), or
2. \( n, l \geq k \).

Let
\[
\theta_0 \in H^0(M, \operatorname{End}(E) \otimes L^l) \tag{5.1}
\]
be a holomorphic section. Note that \( \theta_0 \) defines a section
\[
\theta \in H^0(M, J^k(\operatorname{End}(E) \otimes L^l)).
\]
Recall that \( D_E \) induces a holomorphic connection \( \mathcal{D}_E \) on \( \operatorname{End}(E) \). In view of this connection, Lemma 4.4 says that \( J^k(\operatorname{End}(E) \otimes L^l) = \operatorname{End}(E) \otimes L^{l-k} \otimes V_k \). So, we have
\[
\theta \in H^0(M, \operatorname{End}(E) \otimes L^{l-k} \otimes V_k). \tag{5.2}
\]

Recall from Lemma 4.4 that \( J^k(E \otimes L^n) = E \otimes L^{n-k} \otimes V_k \). Let
\[
\mathcal{T}_\theta^0 : J^k(E \otimes L^n) = E \otimes L^{n-k} \otimes V_k \longrightarrow E \otimes L^{l-n-2k} \otimes V_k \otimes V_k
\]
be the homomorphism defined by $a \otimes b \mapsto \sum_i (a'_i \otimes b \otimes b'_i)$, where $a$ and $b$ are local sections of $E$ and $L^{n-k} \otimes V_k$ respectively, while $\theta$ in (5.2) is locally expressed as $\sum_i a'_i \otimes b'_i$ with $a'_i$ and $b'_i$ being local sections of $\text{End}(E)$ and $L^{i-k} \otimes V_k$ respectively. Let

$$p_0 : V_k \otimes V_k \rightarrow V_0 = \mathcal{O}_M$$

be the projection constructed using the decomposition in Proposition 4.1. Now define the homomorphism

$$\mathcal{T}_\theta := (\text{Id} \otimes p_0) \circ \mathcal{T}_\theta^0 : J^k(E \otimes L^n) = E \otimes L^{n-k} \otimes V_k \rightarrow E \otimes L^{l+n-2k}, \quad (5.3)$$

where Id denotes the identity map of $E \otimes L^{l+n-2k}$. Therefore, we have

$$\mathcal{T}_\theta \in H^0(M, \text{Diff}^k(E \otimes L^n, E \otimes L^{l+n-2k})) \quad (5.4)$$

(see (2.6)). Let

$$\hat{\sigma} : \text{Diff}^k(E \otimes L^n, E \otimes L^{l+n-2k}) \rightarrow \text{End}(E) \otimes L^{l-2k} \otimes (TM)^{\otimes k} = \text{End}(E) \otimes L^l$$

be the symbol homomorphism in (2.7); in (4.2) it was shown that $TM = L^{\otimes 2}$.

**Theorem 5.1.** The symbol $\hat{\sigma}(\mathcal{T}_\theta)$ of the differential operator $\mathcal{T}_\theta$ in (5.4) is the section $\theta_0$ in (5.1).

**Proof.** This theorem can be derived from Corollary 4.7 that describes the symbol. The following is needed to derive the theorem from Corollary 4.7.

Consider the decomposition of the $\text{SL}(V)$–module

$$\text{Sym}^m(V) \otimes \text{Sym}^m(V) = \bigoplus_{i=0}^m \text{Sym}^{2(m-i)}(V)$$

in (4.3). Let

$$f : \text{Sym}^m(V) \otimes \text{Sym}^m(V) \rightarrow \text{Sym}^0(V) = \mathbb{C}$$

be the projection constructed using this decomposition (by setting $i = m$ in the decomposition). Then for any line $L_0 \subset V$, we have

$$f((L_0^{\otimes m}) \otimes (L_0 \otimes \text{Sym}^{m-1}(V))) = 0,$$

so $f$ produces a homomorphism

$$\hat{f} : (L_0^{\otimes m}) \otimes ((\text{Sym}^m(V))/(L_0 \otimes \text{Sym}^{m-1}(V))) \rightarrow (L_0^{\otimes m}) \otimes (V/L_0)^{\otimes m} \rightarrow \mathbb{C}.$$

This homomorphism $\hat{f}$ sends $v^{\otimes m} \otimes u^{\otimes m}$, where $v \in L_0$ and $u \in V/L_0$, to $\lambda^m \in \mathbb{C}$ that satisfies the equation

$$v \wedge u = \lambda \cdot \omega,$$

where $\omega$ is the fixed element in (3.4); note that $v \wedge u$ is a well-defined element of $\wedge^2 V$. The theorem follows from this observation and Corollary 4.7. $\square$

**Remark 5.2.** It may be mentioned that symbols do not always lift to a holomorphic differential operator. For such an example, set the symbol to be the constant function 1. It does not lift to a first order holomorphic differential operator from $\mathcal{L}^n$ to $\mathcal{L}^n \otimes \hat{K}_M$ if $g \geq 2$ and $n \neq 0$. Indeed, a first order holomorphic differential operator from $\mathcal{L}^n$ to $\mathcal{L}^n \otimes \hat{K}_M$ with symbol 1 is a holomorphic connection on $\mathcal{L}^n$. But $\mathcal{L}^n$ does not admit a holomorphic connection because its degree is nonzero.
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REFERENCES


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