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The multi-resonant Lugiato-Lefever model: a new paradigm for cavity nonlinear optics

Matteo Conforti[1,*] **and Fabio Biancalana**[2]

[1] Univ. Lille, CNRS, UMR 8523-PhLAM-Physique des Lasers Atomes et Molećules, F-59000 Lille, France
[2] School of Engineering and Physical Sciences, Heriot-Watt University, EH14 4AS Edinburgh, UK
[*] Corresponding author: matteo.conforti@univ-lille1.fr

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We introduce a new model, which extends the Lugiato-Lefever equation to the description of multiple resonances in Kerr optical cavities. It perfectly agrees quantitatively (in both stationary and dynamical regimes) with the exact Ikeda map even when using a small number of resonances. Our model predicts the onset of complex phenomena such as the recently observed super-cavity solitons, and the coexistence of multiple nonlinear states. It will be of crucial importance for the analytical understanding of new nonlinear phenomena in Kerr cavities when the intensities or the nonlinearities are high enough to be able to excite more than one cavity resonance.

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Optical resonators featuring Kerr media display a wealth of phenomena, encompassing frequency combs [1], cavity solitons [2, 3] and instabilities [4–9], which are being intensively studied in view of their high impact applications [10]. Given the complexity and the diversity of the physical phenomena, deriving simple, accurate and efficient models is of paramount importance. The workhorse for the description of nonlinear cavity dynamics is the celebrated Lugiato-Lefever equation (LLE) [11–13], which allows for deep theoretical insight and fast and accurate numerical modelling [14]. Despite the fact that the LLE holds valid well beyond the mean-field approximation under which it has been historically derived [15], it is not capable of modelling all the phenomena of interest. Indeed, LLE can model the evolution of only one Fabry-Pérot mode, that corresponds to a *single resonance*. Phenomena not captured by LLE range from the recently discovered "super cavity solitons" (SCSs) [16] to the coexistence of stable modulational instability (MI) patterns and solitons, observed in [17]. The complete and exact dynamical scenario can be reproduced by the famous Ikeda map [18], but this model, albeit exact, does not give any reasonable physical insight owing to its complex mathematical structure. A model sharing the accuracy of the Ikeda map and the simplicity of LLE will be of paramount importance in the design of the next-generation of resonators [19]. A reliable model capable to reproduce quantitatively the results of the Ikeda map and the latest experiments [17] is still missing. In this Letter, we derive a multi-resonant LLE system, which agrees *quantitatively* with the Ikeda map. Each Fabry-Pérot resonance is described by an LLE-type equation, which is coupled to the others in a non-trivial way. An arbitrary number of resonances can be treated, making the model highly flexible and scalable to any situation of experimental interest. For definiteness, we consider a fiber ring cavity, but the method can be of course straightforwardly applied to micro-resonators by an appropriate scaling of the parameters.

We start from the Ikeda map in dimensional units:

$$E^{(n+1)}(Z=0,T) = \theta E_{in} + \rho e^{i\phi_0} E^{(n)}(Z=L,T), \quad (1)$$

$$i\frac{\partial E^{(n)}}{\partial Z} - \frac{\beta_2}{2}\frac{\partial^2 E^{(n)}}{\partial T^2} + \gamma |E^{(n)}|^2 E^{(n)} = 0, \quad 0 < Z < L. \quad (2)$$

$E^{(n)}$ is the electric field envelope at the n-th round-trip (measured in $\sqrt{W}$), $P_{in} = |E_{in}|^2$ is the input pump power, ρ^2, θ^2 are respectively the power reflection and transmission coefficients of the coupler, and $\phi_0 = \beta_0 L$ is the linear cavity round-trip phase shift. For simplicity we lump all the losses in the boundary condition, with $1-\rho^2$ describing the total power lost per round-trip. Z measures the propagation distance inside the fiber of length L, and T is time in a reference frame traveling at the group velocity of the pulse.

To proceed, we note that the Ikeda map Eqs. (1-2) can be written – without loss of generality – in terms of a single equation [20] where the boundary conditions are explicitly incorporated in an NLSE-type equation of an "unfolded cavity". Specifically, using the Dirac delta comb to model the periodic application of the boundary conditions, and using the so-called "Poisson resummation identity" $\sum_{n=-\infty}^{+\infty} \delta(Z-nL) = \frac{1}{L}\sum_{n=-\infty}^{+\infty} e^{inkZ}$ (where $k = 2\pi/L$) and letting $Z \in [0,+\infty)$ ("unfolded" cavity), we arrive at the equation:

$$i\frac{\partial E}{\partial Z} - \frac{\beta_2}{2}\frac{\partial^2 E}{\partial T^2} + \gamma|E|^2 E = \frac{i\theta}{L}E_{in}\sum_n e^{i(nk-\beta_0)Z} + i\frac{\rho-1}{L}E\sum_n e^{inkZ}. \quad \mathbf{(3)}$$

Equation (3) is the NLSE forced by two combs with equal wave-number spacing k and a relative shift β_0. A conceptually different model was derived very recently following a single-equation approach [19], but crucially it is not based on the exact Ikeda map and it is still far too complicated to allow a deep physical understanding (e.g. stationary solutions like CSs cannot be found even numerically). The solution of Eq. (3) can be written as a sum of slowly-varying envelopes, which modulate the longitudinal Fabry-Pérot modes of the cavity. We assume that $N = N_R + N_L + 1$ Fabry-Pérot resonances are efficiently excited: $E(Z,T) = \sum_{n=-N_R}^{N_L} E_n(Z,T)e^{iknZ}$, where N_L (or N_R), are the number of modes corresponding to a resonance to the left towards smaller detuning (or to the right towards bigger detuning), of the central resonance denoted $n = 0$. By collecting exponentials oscillating with the same wave-number, we arrive at the following compact and general expression, consisting of N coupled LLEs (CLLEs):

$$i\frac{\partial U_n}{\partial Z} - \frac{\delta_n}{L}U_n - \frac{\beta_2}{2}\frac{\partial^2 U_n}{\partial T^2} + \gamma\sum_{p=-N_R}^{N_L}\sum_{q=q_{min}}^{q_{max}} U_p U_q U^*_{p-n+q} = i\frac{\theta}{L}E_{in} - i\frac{\alpha}{L}\sum_{p=-N_R}^{N_L} U_p, \quad (n=-N_R,\ldots,N_L) \quad \mathbf{(4)}$$

where $\alpha = 1-\rho$, $U_n = E_n \exp[i\delta_0 Z/L]$, $q_{min} = \max\{-N_R, n-p-N_R\}$, $q_{max} = \min\{N_L, n-p+N_L\}$, $\delta_n = \delta_0 + 2\pi n$. The conditions on the integers $q_{min,max}$ select only the correct nonlinear couplings. The cavity detuning from the central mode is defined as $\delta_0 = mk - \beta_0$, $m = \arg\min_n |nk-\beta_0|$, entailing $-\pi \le \delta_0 \le \pi$, which is consistent with the 2π periodicity of the Ikeda map. Any choice $|\delta_0| > \pi$ means that we have neglected the slowest oscillating term in favour of a rapid one.

If we assume that only one mode is excited in the cavity ($N_R = N_L = 0$) we recognize in Eq. (4) the standard, single-resonance LLE. It is worth noting that our CLLE (4) *do not use the mean-field approximation*. In the standard single-resonance LLE this approximation is valid when the field does not change much over one roundtrip. However in our formulation the mean-field approach is not used, since the field is allowed to change arbitrarily fast and oscillate strongly, due to the presence of a large number of resonances.

As a simple but important example of how to use Eq. (4), we write explicitly the equations for $N = 3$, when the main central cavity resonance is accompanied by two other resonances on its left (i.e. located at negative detuning from the main resonance). In this case we have obviously $N_L = 2$ and $N_R = 0$. The choice of privileging resonances at negative detunings stems from the fact that the nonlinearity tends to tilt resonaces towards positive detunings. It turns out that this configuration is extremely accurate in reproducing the CW as well as the short pulse predictions of the full Ikeda map in most cases. In this particularly important case, Eq. (4) become the following three coupled equations:

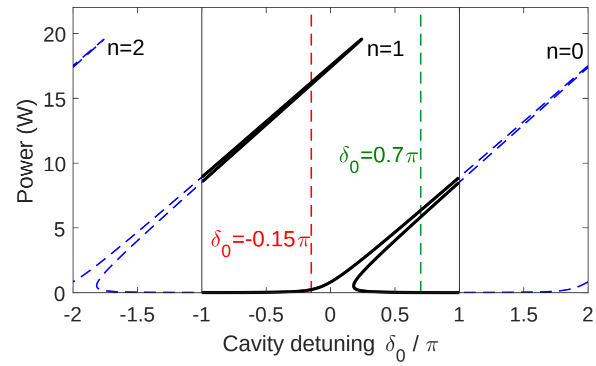

Fig. 1. Steady-state response of the cavity from Ikeda map (dashed blue) and CLLEs Eqs. (5-7) (solid black), corresponding to $N = 3$ resonances in Eqs. (4) ($N_L = 2, N_R = 0$). Parameters $L = 300$ m, $\alpha = 0.0619$, $\theta^2 = 0.05$, $\beta_2 = -22$ ps^2/km, $\gamma = 1.2$ W^{-1}km^{-1}, $P_{in} = |E_{in}|^2 = 1.5$ W.

$$i\frac{\partial U_0}{\partial Z} - \frac{\delta_0}{L}U_0 - \frac{\beta_2}{2}\frac{\partial^2 U_0}{\partial T^2} + \gamma(|U_0|^2 + 2|U_1|^2 + 2|U_2|^2)U_0 + \gamma U_1^2 U_2^* = i\frac{\theta}{L}E_{in}(T) - i\frac{\alpha}{L}(U_0+U_1+U_2), \quad \mathbf{(5)}$$

$$i\frac{\partial U_1}{\partial Z} - \frac{\delta_0+2\pi}{L}U_1 - \frac{\beta_2}{2}\frac{\partial^2 U_1}{\partial T^2} + \gamma(|U_1|^2 + 2|U_0|^2 + 2|U_2|^2)U_1 + 2\gamma U_0 U_1^* U_2 = i\frac{\theta}{L}E_{in}(T) - i\frac{\alpha}{L}(U_0+U_1+U_2), \quad \mathbf{(6)}$$

$$i\frac{\partial U_2}{\partial Z} - \frac{\delta_0+4\pi}{L}U_2 - \frac{\beta_2}{2}\frac{\partial^2 U_2}{\partial T^2} + \gamma(|U_2|^2 + 2|U_0|^2 + 2|U_1|^2)U_2 + \gamma U_0^* U_1^2 = i\frac{\theta}{L}E_{in}(T) - i\frac{\alpha}{L}(U_0+U_1+U_2), \quad \mathbf{(7)}$$

where U_0 is the field propagating in the central resonance, and $U_{1,2}$ are those propagating in the two extra resonances on negative detuning. In order to test our model, we simulate the fiber ring resonator described in [17], which has been exploited to explore regimes where nonlinearly tilted cavity resonances overlap with one another. It is made of a 300 m loop of standard single mode fiber (SMF-28) with group-velocity $\beta_2 = -22$ ps^2/km, and nonlienear coefficient $\gamma = 1.2$ W^{-1}km^{-1} (at 1550 nm). It makes use of a 95/5 fiber coupler ($\theta^2 = 0.05$) and the total roundrip losses are incorporated in the parameter $\alpha = 0.0619$. We found that the $N = 3$ resonances ($N_L = 2, N_R = 0$) model Eqs. (5-7) reported above perfectly describe the full range of detuning for the input power used in the experiment, $P_{in} = 1.5$ W. Figure 1 shows the tilted cavity resonances obtained by solving the stationary ($\partial/\partial Z = \partial/\partial T = 0$) CLLEs Eqs. (5-7) via a Newton-Raphson method (solid black curve). The agreement with the Ikeda map (dashed blue curve) is *perfect* over the full range $-\pi \le \delta_0 \le \pi$.

At a detuning $\delta_0 = -0.15\pi$ (dashed red vertical line in Fig. 1), the central resonance $n = 0$ is affected by MI, so we expect the

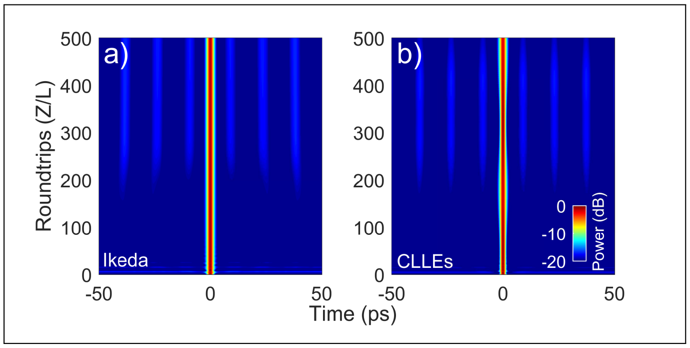

Fig. 2. Coexistence between CS and MI pattern at $\delta_0 = -0.15\pi$ (vertical dashed red line in Fig. 1). Input for Ikeda map $|E^{(0)}(Z=0)|^2 = P_S \mathrm{sech}\,(T/T_S)^2$; for CLLEs $|U_1(Z=0)|^2 = P_S \mathrm{sech}\,(T/T_S)^2$, $U_{0,2}(Z=0) = 0$. Small white noise is added to the initial condition. Parameters: $P_S = 2(\delta_0 + 2\pi)/(\gamma L)$, $T_S^2 = |\beta_2| L/(2(\delta_0 + 2\pi))$ [16]

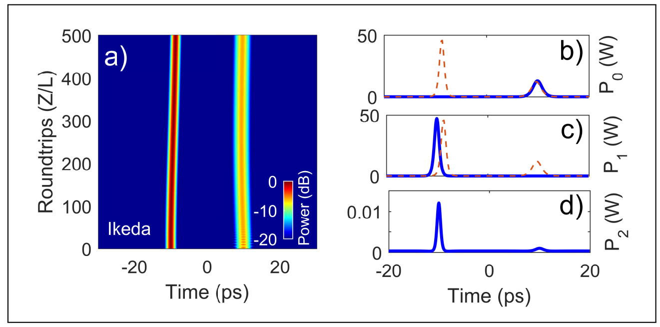

Fig. 4. Coexistence between CS and SCS at $\delta_0 = 0.7\pi$ (dashed red line in Fig. 1). (a): Intracavity power from Ikeda map with approximated input CS+SCS [16] separated by 20 ps. (b,c,d): Power profiles ($P_n = |U_n|^2$) from solution of steady CLLEs (blue solid lines). Red dashed line is the output from Ikeda map.

generation a periodic pattern as predicted by LLE [12]. However, the picture gived by LLE is not sufficient, because the resonance $n = 1$ is excited at the same time, and can support a (super) cavity soliton [16]. Overall, we expect the coexistence of a stable MI pattern and a SCS. This scenario, observed first in [17], is confirmed by numerical solution of the Ikeda map, as reportednin Fig. 2(a), taking as initial intracavity field a SCS perturbed by a small noise. The resulting spatiotemporal evolution agrees quantitatively with the one obtained by numerical solution of CLLEs (5-7), as illustated in Fig. 2(b). One of the main advantage of our coupled mode description Eqs. (5-7) with respect to the Ikeda map is that it permits to disentangle the different phenomena and to associate the different nonlinear structures to the different cavity modes. Indeed, Fig. 3 shows that the MI pattern is generated by the mode $n = 0$ [Fig. 3(a,d)] and the soliton by the mode $n = 1$ [Fig. 3(b,e)]. The power carried by the mode $n = 2$ [Fig. 3(c,f)] is rather small and could have in principle been neglected from the beginning. It is however convenient to keep it in the model because it improves the numerical accuracy without increasing too much the complexity of the simulations.

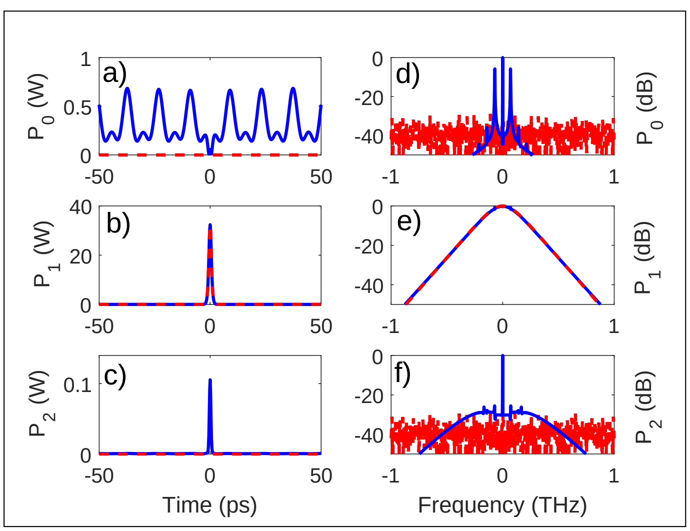

Fig. 3. Coexistence between CS and MI pattern at $\delta_0 = -0.15\pi$ (vertical dashed red line in Fig. 1). Temporal (a-c) and spectral (d-f) output profile (solid blue curves) calculated from CLLEs Eqs. (5-7) corresponding to evolution showed in Fig. 2. Dashed red curves are the input field.

A different scenario appears at a detuning $\delta_0 = 0.7\pi$ (dashed green vertical line in Fig. 1). The SCS associated to $n = 1$ resonance still exists, with a higher power proportional to $\delta_0 + 2\pi$. Moreover, a conventional CS of lower power proportional to δ_0 is supported by the fundamental $n = 0$ cavity resonance. This coexistence is confirmed by numerical solution of the Ikeda map reported in Fig. 4(a), corresponding to an initial field made of a simple sum of the approximated CS and SCS separated by 20 ps in time. A similar picture (not shown) is obtained from CLLEs. Another asset of CLLEs is that they permits to find stationary structure ($\partial/\partial Z = 0$) such as solitons or even multi-soliton complexes. This property is well illustrated in Fig. 4(b-d), where the a stationary solution of CLLEs (obtained by a Newton-Raphson method) made of a CS and sSCS describes remarkably well the output of the Ikeda map. Again we can recognize different nonlinear wave generated by the different cavity modes: The CS has its power concentrated in the component $n = 0$ [Fig. 4(b)], whereas the SCS resides predominantly in the component $n = 1$ [Fig. 4(c)].

In conclusion, we have derived a model based on coupled LLEs which allows for the accurate description of Kerr optical cavities when several Fabry-Pérot resonances interact. The coupling between the different fields is simple but non-trivial. Our new model is extremely accurate even when including a small number of modes, reproducing in a quantitative way the results of the Ikeda map. The model will allow researchers belonging to the fiber cavity and the microresonator communities to acquire great physical insight thanks to its simplicity and analytical tractability.

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