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A Condition-Based Maintenance Model With Past-Dependent Imperfect Preventive Repairs for Continuously Deteriorating Systems

K.T. Huynh, A. Grall

tuan.huynh@utt.fr, antoine.grall@utt.fr

ICD, ROSAS, LM2S, Université de Technologie de Troyes, UMR 6281, CNRS, Troyes, France

Abstract

Most condition-based imperfect maintenance models developed over the last few decades are memoryless in the sense that maintenance efficiency is completely s -independent of previous interventions. However, many maintenance activities exhibit their past dependency in engineering practice, and this significant property should not be ignored in maintenance modeling. In this spirit, our aim is to develop a condition-based maintenance model for continuously deteriorating systems subject to a special kind of past-dependent imperfect repairs. Such a repair can put the system back to a deterioration level better than the one at just before the current repair, but worse than the one reached at the last repair. Besides, inspection and replacement are memoryless actions available for the system. They result in different effects on the system deterioration, and incur different costs. To achieve high economic performances in the long-term, these actions are coordinated into a control-limit deterioration-based maintenance policy. Its long-run maintenance cost rate is analytically evaluated using the semi-regenerative process theory. Numerous sensitivity studies to maintenance costs and to system characteristics give a thorough understanding about the policy behavior. Furthermore, comparisons with more classical policies justify the importance of incorporating the past dependency in maintenance modeling.

Index Terms

Condition-based maintenance, past-dependent imperfect repair, Gamma deterioration process, semi-regenerative process, long-run maintenance cost rate.

ACRONYMS

ARA ₁	arithmetic reduction of age with memory 1
ARD ₁	arithmetic reduction of deterioration with memory 1
AGAN	as-good-as-new
CBIM	condition-based imperfect maintenance
CR	corrective replacement(s)
PIR	preventive imperfect repair(s)
PPR	preventive perfect replacement(s)
pdf	probability density function(s)

NOTATIONS

X_t	system deterioration level at time t
α, β	shape, scale parameters of the homogeneous Gamma deterioration process $\{X_t\}_{t \geq 0}$
μ, σ^2	average rate, variance rate of $\{X_t\}_{t \geq 0}$
$f_{\alpha, t, \beta}, \bar{F}_{\alpha, t, \beta}$	probability density function, survival function of X_t
$\Gamma(\cdot), \Gamma(\cdot, \cdot)$	complete Gamma function, upper incomplete Gamma function
L, τ_L	system failure threshold, system failure time

R_j	j -th repair/replacement time,
$T_{j,k}$	k -th inspection time over the j -th repair/replacement cycle $[R_{j-1}, R_j)$
δ	inspection period
ζ, η, λ	preventive repair/replacement thresholds
$g(\cdot \cdot, \cdot)$	pdf of the system deterioration level after a past-dependent PIR
$p(\cdot \cdot, \cdot)$	pdf of the system deterioration level after a memoryless PIR
C_i, C_r, C_p, C_c, C_d	inspection cost, PIR cost, PPR cost, CR cost, downtime cost rate
$C(t), C_\infty$	cumulative maintenance cost up to time t , long-run maintenance cost rate
$N_i(t), N_i([0^+, R_1^+])$	number of inspections up to time t , and over $[0^+, R_1^+]$
$N_r(t), N_r([0^+, R_1^+])$	number of PIR up to time t , and over $[0^+, R_1^+]$
$N_c(t), N_c([0^+, R_1^+])$	number of CR up to time t , and over $[0^+, R_1^+]$
$W(t), W([0^+, R_1^+])$	cumulative downtime of the system up to time t , and over $[0^+, R_1^+]$
ΔR_1	length of the first Markov renewal cycle
$\{Y_j\}_{j \in \mathbb{N}}$	Markov chain describing the system deterioration at repair/replacement times ($Y_j = X_{R_j^+}$)
π	stationary law of $\{Y_j\}_{j \in \mathbb{N}}$
$P(\cdot, \cdot)$	transition kernel of $\{Y_j\}_{j \in \mathbb{N}}$
$E_\pi[\cdot]$	expectation with respect to the measure π

I. INTRODUCTION

With usage and age, most industrial systems suffer gradual deterioration leading eventually to random failure. Maintenance policies are thus vital for keeping their long-term operation at low costs. Among existing maintenance policies (see e.g., [1] for a recent overview), the condition-based imperfect maintenance (CBIM) proved to be highly relevant by two main reasons. Firstly, the policy utilizes the advance of condition monitoring technologies to assess the health state of a system, and thence carries out adequate and timely maintenance actions [2]. This allows avoiding inopportune interventions and saving maintenance costs, especially when compared with more classical policies such as run-to-failure maintenance and time-based maintenance [3]. Secondly, imperfect maintenance, which restores a badly deteriorated system to a condition between as-good-as-new (AGAN) and as-bad-as-old [4], can characterize a large kind of realistic actions whose imperfectness may be caused by various factors such as human errors, spare parts quality, the lack of materials, lack of maintenance time, etc.

Over the last few decades, a great deal of effort has been put into the modeling of CBIM for continuously deteriorating systems (see e.g., [5]–[8] among others). Most of these models are memoryless in the sense that maintenance efficiency is completely s -independent of previous interventions. This property leads to some simplifications in the mathematical development of CBIM policies. For instance, the classical renewal theorem (see e.g., [9]) or the dynamic programming (see e.g., [10]) could be used to derive analytical maintenance cost models. Nevertheless, many maintenance activities exhibit their past dependency in engineering practice [11]. Memoryless CBIM models are therefore no longer suitable for such maintenance activities.

Motivated by this practical need, we consider in the present paper a special past dependency characterized by the phenomenon that *the improvement due to an imperfect repair just can bring a system back into a deterioration level worse than the one returned by the last repair*. Such a phenomenon can be found in the deterioration paths of draught fans and of gyroscopes provided in [12] and in [13], [14] respectively. Our aim is to model this kind of past-dependent imperfect repairs in the context of condition-based maintenance applied to continuously deteriorating systems. To the best of our knowledge, three main modeling approaches have been used to deal with this problem in the literature.

- 1) *Repairs number-based modeling*. The first approach considers that the system residual damage after each imperfect repair exhibits an increasing trend with the sequence of repairs. Since the number of repairs increases over time until the next

perfect replacement, their ability to improve the system deterioration weakens. As a result, the dependency between past and current repairs can be modeled via the repairs number. Based on this approach, Liao *et al.* proposed in [15] a so-called condition-based availability limit model for continuously monitored systems subject to gamma deterioration process. Guo *et al.* developed a similar model in [16] for a mission-oriented system based on a Wiener deterioration process. More recently, the repairs number-based modeling approach has been also applied in estimating the remaining useful life of condition-based maintained systems (see e.g., [13], [14]).

- 2) *Virtual age-based modeling.* The second approach links the virtual age of a system to its deterioration level. When an imperfect repair removes a portion of virtual age accumulated since the last repair, it also puts the system back to a deterioration level where it was some time before. In this spirit, Ahmadi used the Kijima's type I model [17] to develop CBIM policies for periodically and non-periodically inspected deteriorating systems in [18] and [19] respectively. The well-known renewal reward theorem was applied to compute their long-run maintenance cost rate. Meanwhile, based on the arithmetic reduction of age with memory 1 (ARA_1) model [20], Mercier and Castro proposed in [21] a deterioration-based maintenance policy for a continuously monitored deteriorating system. The reliability and availability functions, as well as the expected maintenance cost of the maintained system were evaluated in the short-term using the Markov renewal theory.
- 3) *Deterioration level-based modeling.* Unlike the two above approaches, the third one enables a connection to the past by assuming that each imperfect repair can directly reduce a part of the deterioration accumulated by the system from the last repair. To describe such a past dependency, Ponchet *et al.* mimicked the ARA_1 model to build in [22] a so-called arithmetic reduction of deterioration with memory 1 (ARD_1) model. The model was further used to minimize the average maintenance cost of a maintained system operating over a finite time span. Also relied on the ARD_1 model, Castro and Mercier described in [23] the behavior of a deteriorating system subject to imperfect and delayed repairs. The interval reliability of the system was defined as a performance measure, and was evaluated by the Markov renewal theory.

Since the higher the repairs number or the higher the virtual age, the more the system is deteriorated, the above approaches can be connected to each other. For a suitable choice among them, some comparative works have been done. For instance, Mercier and Castro [24] performed stochastic comparisons between the ARA_1 and ARD_1 models under the assumption of a Gamma deteriorating system. Based on a system subject to the Wiener deterioration process, Kahle [25] recently compared Kijima's type models [17] applied for both the system virtual age and the system deterioration.

Considering a single-unit system subject to a gamma deterioration process, this paper applies the third approach to take into account directly the system deterioration levels revealed by periodic inspections in past-dependent imperfect repair modeling. An imperfect repair is done on a working system as soon as the system deterioration level exceeds a given preventive threshold. Meanwhile, a replacement is carried out to restore a failed system to an AGAN condition. The economic performance of the maintained system is assessed via its long-run maintenance cost rate. Such a CBIM model differs from similar existing works (see e.g., [22]–[24]) at two major points. Firstly, to express the dependency of the repair efficiency on the past, we just rely on a truncated probability distribution. After a repair, the restarting deterioration level of the system is sampled from a probability distribution truncated by the deterioration levels just before a current repair and just after the last repair/replacement. Unlike arithmetic reduction type and Kijima's type models, this simple model allows breaking the memory assumption: *the system after a repair is put back to an exact deterioration level where it was in the past*, which is not easily verified in practice due to the stochastic nature of deterioration process. Secondly, the long-run cost rate of the considered CBIM model is analytically derived using the semi-regenerative theory. Even though this approach has now become rather classical in reliability literature [26], its development in the context of past dependency is still very meaningful, especially in terms of numerical computation and Monte Carlo simulation.

The remainder of the paper is structured as follows. Section II describes the characteristics of the condition-based maintained system. Section III is devoted to the development and validation of mathematical cost model for the system evaluation and

optimization. Numerous sensitivity studies to interventions costs and system deterioration characteristics are provided in Section IV. In Section V, comparisons with more classical benchmark policies are done. Finally, the paper ends with some conclusions and perspectives in Section VI.

II. DESCRIPTION OF CONDITION-BASED MAINTAINED SYSTEM

We describe in this section a gradually deteriorating system subject to past-dependent imperfect repairs and memoryless actions such as inspections and replacements. A model characterizing the system deterioration and the system failure is first introduced. Next, a control-limit deterioration-based imperfect maintenance policy is implemented to coordinate the available maintenance actions. Its performance is assessed via a long-run maintenance cost rate. Finally, the practicality of the maintained system is illustrated by a real-world example.

A. Stochastic Deterioration Process-Based Failure Model

Let consider a system which starts working at time $t = 0$ and is subject to a stochastic deterioration phenomenon leading eventually to random failures. From the maintenance point of view, a system, even if multiple components, can be considered as a single-unit consisting of one critical component or one group of connected components [27]. Therefore, the system deterioration state at time $t \geq 0$ can be summarized by a scalar random variable $X_t \geq 0$. We assume that the system is initially new (i.e., $X_0 = 0$), then evolves following an underlying deterioration process $\{X_t\}_{t \geq 0}$ toward the failure. Between two successive repair/replacement actions, the system deterioration increases continuously, monotonically and stochastically over time. Such a deterioration path can be approximated by a sequence of infinite number of random and positive tiny increments. This property leads us to apply a Gamma stochastic process to the system deterioration evolution $\{X_t\}_{t \geq 0}$. In reality, the relevance of Gamma process to deterioration modeling has been justified by diverse practical applications [28] and considered appropriate by experts [29]. In the present paper, a homogeneous version of Gamma process is chosen. This choice facilitates mathematical developments while satisfying a wide rank of applications (e.g., pressure vessel corrosion [30], actuator performance loss [31], stress corrosion cracking [32], etc.). As such, without any repair or replacement action, $\{X_t\}_{t \geq 0}$ is assimilated by a homogeneous Gamma process with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$. The random deterioration increment $X_t - X_s$, for all $0 \leq s < t$, is Gamma distributed with probability density function (pdf)

$$f_{\alpha \cdot (t-s), \beta}(x) = \frac{1}{\Gamma(\alpha \cdot (t-s))} \beta^{\alpha \cdot (t-s)} x^{\alpha \cdot (t-s)-1} e^{-\beta x} 1_{\{x \geq 0\}}, \quad (\text{II.1})$$

and survival function

$$\bar{F}_{\alpha \cdot (t-s), \beta}(x) = P(X_t - X_s \geq x) = \frac{\Gamma(\alpha \cdot (t-s), \beta x)}{\Gamma(\alpha \cdot (t-s))}, \quad (\text{II.2})$$

where $1_{\{\cdot\}}$ stands for the indicator function which equals 1 if the argument is true and 0 otherwise, $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz$ and $\Gamma(\alpha, x) = \int_x^\infty z^{\alpha-1} e^{-z} dz$ denote respectively the complete and upper incomplete Gamma functions. The couple of parameters (α, β) can be estimated from deterioration data by statistical methods [28]. Different values of the couple (α, β) give different kinds and different variabilities of deterioration behaviors with average deterioration rate $\mu = \frac{\alpha}{\beta}$ and associated variance $\sigma^2 = \frac{\alpha}{\beta^2}$.

In engineering practice, a critically deteriorating system is generally unacceptable due to economic reasons (e.g., high consumption of raw material, poor products quality, etc.) or safety reasons (e.g., high risk of hazardous breakdowns) [33]. This is why a system is usually declared as “failed” as soon as its deterioration level exceeds a fixed critical threshold L , even if it is still functioning. The system failure is thus non-self-announcing and cannot be detected without an inspection. Its random failure time τ_L is the first hitting time of the failure threshold L by the deterioration process $\{X_t\}_{t \geq 0}$

$$\tau_L = \inf \{t \in \mathbb{R}^+, X_t \geq L\}. \quad (\text{II.3})$$

The value of L can be provided by experts [34], or estimated from lifetime data by inverse first passage transform [35]. Statistical characteristics of τ_L can be found in [36, pages 97-109].

B. Control-Limit Deterioration-Based Maintenance Policy

Inspection, perfect replacement and imperfect repair are the three maintenance actions available for the system. The inspection and replacement are assumed memoryless, while the imperfect repair is past-dependent via the deterioration level given at the last repair. Since these actions are costly, a control-limit deterioration-based maintenance policy with an inspection period δ and a preventive maintenance threshold ζ has been proposed to organize them in a proper manner. Let $R_0, R_1, \dots, R_{j-1}, R_j, \dots$, with $R_0 = 0$, be the successive repair/replacement times of the system, the j -th repair/replacement cycle, $j \in \mathbb{N}^*$, is thus the time interval $[R_{j-1}, R_j)$. If $X_{R_{j-1}^+} = 0$, the system has been perfectly replaced at R_{j-1} ; otherwise, it has been partially repaired. Under the considered maintenance policy, the deterioration evolution of the maintained system on the cycle $[R_{j-1}, R_j)$ is as follows.

- 1) The system is periodically inspected at times $T_{j,k} = R_{j-1} + k \cdot \delta$, with $k = 1, 2, \dots$, until $X_{T_{j,k}} \geq \zeta$ with a constant unit cost $C_i > 0$. The inspection is assumed perfect in the sense that it takes negligible time, reveals the exact deterioration level of the system without impacting on its deterioration behavior. Such an assumption is widely used in the literature (see e.g., [6], [32]). We also note that the inspection may be imperfect due to e.g., the measurement noise [37], the detection errors [38], etc. However, the inspection imperfectness is omitted here because it is out of the paper scope.
- 2) At an inspection time $T_{j,k}$, a control-limit rule based on the detected deterioration level $X_{T_{j,k}}^-$ is adopted.
 - a) If $X_{T_{j,k}}^- \geq L$, a corrective replacement (CR) with constant cost unit cost C_c is immediately carried out on the failed system. It takes negligible time, and brings the system back to an AGAN condition (i.e., $X_{T_{j,k}}^+ = 0$). Thus, the next repair/replacement cycle begins at R_j with initial deterioration level $X_{R_j^+} = 0$. Furthermore, before the CR starts at $T_{j,k}$, a failure has been occurred in the time interval $(T_{j,k-1}, T_{j,k}]$, that makes the system unavailable until $T_{j,k}$. Such a system downtime incurs a constant cost rate $C_d > 0$.
 - b) If $\zeta \leq X_{T_{j,k}}^- < L$, a preventive imperfect repair (PIR) with constant unit cost $C_r \in (C_i, C_c)$ is immediately carried out on the repairable badly deteriorated system. Just after an instantaneous PIR, the system deterioration is put back to a level $X_{T_{j,k}}^+ \in [X_{R_{j-1}^+}, X_{T_{j,k}}^-]$ sampled from a pdf truncated by $X_{R_{j-1}^+}$ and $X_{T_{j,k}}^-$ (see Figure II.1)

$$X_{T_{j,k}}^+ \sim g(y | x, z), \quad (\text{II.4})$$

where x is a realization of the deterioration level $X_{R_{j-1}^+}$ returned by the last repair/replacement at time R_{j-1} , z is a realization of the deterioration level $X_{T_{j,k}}^-$ at just before the current repair, and $y \in [x, z]$ is a realization of $X_{R_j^+}$. The next repair/replacement cycle begins at R_j with initial deterioration level $X_{R_j^+} = X_{T_{j,k}}^+$.

Figure II.1: Illustration of past-dependent imperfect repair model

- c) If $X_{T_{j,k}^-} < \zeta$, no further intervention is needed at $T_{j,k}$, so the system deterioration is let unchanged (i.e. $X_{T_{j,k}^+} = X_{T_{j,k}^-}$). The decision is postponed to the next inspection at $T_{j,k+1} = T_{j,k} + \delta$.

Figure II.2 illustrates the deterioration evolution of the maintained system over three first repair/replacement cycles. For this

Figure II.2: Schematic evolution of the maintained system state under the (δ, ζ) policy

maintenance policy, the maintenance cost C_i , C_r , C_c and C_d are input data. The function $g(\cdot | \cdot, \cdot)$ is the pdf to determined from deterioration and maintenance data. Although this problem has not been dealt with in this paper due to the data missing, we still believe that the following two-steps procedure could be applied. For some conjectured parametric forms of $g(\cdot | \cdot, \cdot)$ (see e.g., [39], [40]), classical methods (e.g., maximum likelihood method, method of moment, etc.) are used to estimate the model parameters from deterioration and maintenance data. Next, we perform goodness-of-fit tests to find the best fit of the data. For this estimation-testing procedure, the deterioration and maintenance data are obviously prerequisite. This is why building such a data-set is recognized as a key perspective of the paper. Finally, the inspection period δ , and the PIR threshold ζ are decision variables to be jointly optimized. To highlight the importance of these two variables, we call the policy (δ, ζ) .

C. Cost-Based Performance Criterion

This paper chooses the well-known long-run maintenance cost rate as a criterion to assess the economic performance of the (δ, ζ) policy. As argued by Wagner in [41, chapter 11], this choice is well adequate because of two main reasons. First, in making repeated investment decisions, it is better to employ an unbounded horizon model than to simply ignore the future. Second, the mathematical models are less complex while providing reasonable answers in practice. Mathematically, the long-run maintenance cost rate is defined as

$$C_\infty(\delta, \zeta) = \lim_{t \rightarrow \infty} \frac{C(t)}{t}, \quad (\text{II.5})$$

where $C(t)$ stands for the cumulative maintenance cost including the downtime cost up to time t . Under the (δ, ζ) policy, $C(t)$ is expressed as

$$C(t) = C_i \cdot N_i(t) + C_r \cdot N_r(t) + C_c \cdot N_c(t) + C_d \cdot W(t), \quad (\text{II.6})$$

where $N_i(t)$, $N_r(t)$ and $N_c(t)$ are the number of inspections, the number of PIR and the number of CR in the time interval $[0, t]$ respectively, and $W(t)$ denotes the system downtime interval in $[0, t]$. Optimizing the (δ, ζ) policy returns to find the couple of decision variables $(\delta_{opt}, \zeta_{opt})$ that minimizes $C_\infty(\delta, \zeta)$

$$C_\infty(\delta_{opt}, \zeta_{opt}) = \min_{(\delta, \zeta)} \{C_\infty(\delta, \zeta), \delta \geq 0, 0 \leq \zeta \leq L\}. \quad (\text{II.7})$$

The optimization procedure is performed through an analytical evaluation of $C_\infty(\delta, \zeta)$, which is the aim of Section III.

D. Practicality of The Proposed Condition-Based Maintained System

To illustrate the practicality of the proposed condition-based maintained system, let us introduce the gyroscope equipment represented by Hu *et al.* in [13], [14]. Gyroscope is a core component in inertial navigation systems. Due to the wear of rotor spin axis and the friction of gimbal bearings, the gyroscopic drift increases over time, and hence degrades the gyroscope performance. Therefore, the drift can be seen as a deterioration index of the gyroscope. In the experiment provided in [13], [14], the gyroscope is periodically inspected for $\delta = 2.5\text{h}$ each time. Whenever the gyroscopic drift revealed by an inspection exceeds a threshold $L = 0.37^\circ/\text{h}$, the gyroscope is considered as failed and must be replaced. If the drift value is still less than $L = 0.37^\circ/\text{h}$ but greater than $\zeta = 0.30^\circ/\text{h}$, the current in the torque coil of the gyroscope is adjusted to compensate the drift value. Such an adjustment is a imperfect repair action on the gyroscope. The evolution of the drift data of two maintained gyroscopes are plotted in Figure II.3. Obviously, such a system behavior can be completely described by our CBIM model.

Figure II.3: Drift data of two maintained gyroscopes adapted from [13], [14]

We also note that the works of Hu *et al.* differ from ours at two points. Firstly, to model the efficiency of past-dependent imperfect repairs, they have based on the repair number-based approach rather than on the deterioration-level based approach (see also Section I). Secondly, the main aim of [13], [14] is to estimate the remaining useful life of condition-based maintained systems, while our aim is to evaluate and optimize the CBIM model.

III. MAINTENANCE COST EVALUATION AND OPTIMIZATION

This section aims at analytically evaluating the cost rate $C_\infty(\delta, \zeta)$, thence finding the couple of decision parameters $(\delta_{opt}, \zeta_{opt})$ that optimizes the (δ, ζ) policy. We find that after each PIR or CR at time R_j , the evolution of the system depends only on the deterioration level $X_{R_j^+}$ at time R_j^+ . The deterioration process $\{X_t\}_{t \geq 0}$ is therefore a semi-regenerative process with semi-regeneration times $R_j, j \in \mathbb{N}$. The length between two successive semi-regeneration times is called semi-regenerative cycle. Embedded in $\{X_t\}_{t \geq 0}$, there exists a discrete-time random process $\{Y_j\}_{j \in \mathbb{N}}$, where $Y_j = X_{R_j^+}$, describing the system state at just after PIR or CR. $\{Y_j\}_{j \in \mathbb{N}}$ is a Markov chain with continuous state space $[0, L)$ and with stationary law π . As such, the semi-regenerative cycle is also known as Markov renewal cycle (see Figure II.2). The study of asymptotic behavior of $\{X_t\}_{t \geq 0}$ can be restricted to a Markov renewal cycle, and we can use this property to compute $C_\infty(\delta, \zeta)$ [26]. Let define $\Delta R_1 = R_1$ as the length of the first “artificial” Markov renewal cycle $[0^+, R_1^+]$ under the (δ, ζ) policy. The notation 0^+ here is not simply the initial time at which the system starts working, but rather indicates the beginning of a semi-renewal cycle. That is why the system deterioration level X_{0^+} is not always equal to 0, but $X_{0^+} = x$ in general, where $0 \leq x < L$.

Using the result of semi-regenerative process, we can express (II.5) as

$$C_\infty(\delta, \zeta) = \lim_{t \rightarrow \infty} \frac{C(t)}{t} = \frac{E_\pi[C([0^+, R_1^+])]}{E_\pi[\Delta R_1]} = C_i \cdot \frac{E_\pi[N_i([0^+, R_1^+])]}{E_\pi[\Delta R_1]} + C_r \cdot \frac{E_\pi[N_r(0^+, R_1^+)]}{E_\pi[\Delta R_1]} + C_c \cdot \frac{E_\pi[N_c([0^+, R_1^+])]}{E_\pi[\Delta R_1]} + C_d \cdot \frac{E_\pi[W([0^+, R_1^+])]}{E_\pi[\Delta R_1]}. \quad (\text{III.1})$$

where $E_\pi[\cdot]$ denotes the expectation with respect to the stationary π . The proof of (III.1) is omitted here; interested readers are invited to refer [27], [42] for more details. Over the Markov renewal cycle $[0^+, R_1^+]$, there is one and only one maintenance action (either a CR or a PIR at R_1); moreover, $\Delta R_1 = \delta \cdot N_i([0^+, R_1^+])$. This is why, (III.1) is rewritten by

$$C_\infty(\delta, \zeta) = \frac{1}{\delta} \left(C_i + \frac{C_r + (C_c - C_r) E_\pi[N_c([0^+, R_1^+])] + C_d \cdot E_\pi[W([0^+, R_1^+])]}{E_\pi[N_i([0^+, R_1^+])]} \right). \quad (\text{III.2})$$

Hereinafter, we focus on the mathematical formulation of π , $E_\pi[N_i([0^+, R_1^+])]$, $E_\pi[N_c([0^+, R_1^+])]$ and $E_\pi[W([0^+, R_1^+])]$. The exactness of the formulation is justified by comparing the results obtained by numerical computation and by Monte Carlo simulation. Finally, the existence of optimal (δ, ζ) policy is proved through numerical experiments, and a derivative free algorithm are used to find the associated cost rate. Numerical examples in this section are experimented for two different configurations of maintained system:

- 1) *small rate of deterioration variance*: $\alpha = 5$, $\beta = 5$ ($\mu = 1$, $\sigma^2 = 0.2$),
- 2) *high rate of deterioration variance*: $\alpha = 0.2$, $\beta = 0.2$ ($\mu = 1$, $\sigma^2 = 5$).

The function $g(\cdot | \cdot, \cdot)$ is a continuous uniform pdf, the failure threshold is $L = 15$, and maintenance costs are $C_i = 5$, $C_r = 20$, $C_c = 100$ and $C_d = 25$.

A. Stationary Law of The Markov Chain $\{Y_j\}_{j \in \mathbb{N}}$

As aforementioned, $\{Y_j\}_{j \in \mathbb{N}}$ is a Markov chain with continuous state space $[0, L)$ and $Y_0 = 0$. As $\{Y_j\}_{j \in \mathbb{N}}$ comes back to 0 (a regeneration set) almost surely, there exists a stationary measure π on $[0, L)$ for $\{Y_j\}_{j \in \mathbb{N}}$ which is the solution of the following invariance equation

$$\pi(dy) = \int_{[0, L)} P(x, dy) \pi(dx), \quad (\text{III.3})$$

where $P(x, dy)$ denotes the transition kernel of $\{Y_j\}_{j \in \mathbb{N}}$ from $X_{R_j^+} = x$ to $X_{R_{j+1}^+} = y$. Solving the integral equation (III.3) needs to know analytical expression of $P(x, dy)$ and a numerical method for Volterra integral equations of the second kind.

1) *Expression of Transition Kernel $P(x, dy)$* : As shown in Appendix A, the transition kernel $P(x, dy)$ can be expressed as follows

$$P(x, dy) = ((\rho_1(x) + \rho_2(x)) \cdot \delta_0(dy) + (p_1(y|x) + p_3(y|x)) \cdot 1_{\{y \neq 0, x \leq y < L\}} \cdot dy) \cdot 1_{\{0 \leq x < \zeta\}} + (\rho_1(x) \cdot \delta_0(dy) + p_2(y|x) \cdot 1_{\{y \neq 0, x \leq y < L\}} \cdot dy) \cdot 1_{\{\zeta \leq x < L\}}, \quad (\text{III.4})$$

where $\delta_0(\cdot)$ stands for the dirac delta function concentrated at 0, and

- for $0 \leq x < \zeta$,
 - the condition probability of a CR after one inspection period since R_j^+

$$\rho_1(x) = \bar{F}_{\alpha\delta, \beta}(L - x), \quad (\text{III.5})$$

- the condition probability of a CR after multiple inspection periods since R_j^+

$$\rho_2(x) = \int_x^\zeta \bar{F}_{\alpha\delta, \beta}(L - w) \sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w - x) dw, \quad (\text{III.6})$$

- the condition pdf of a PIR after one inspection period since R_j^+

$$p_1(y|x) = \int_{\zeta}^L g(y|x, r) f_{\alpha\delta, \beta}(r-x) dr, \quad (\text{III.7})$$

- the condition pdf of a PIR after multiple inspection periods since R_j^+

$$p_3(y|x) = \int_x^{\zeta} \left(\int_{\zeta}^L g(y|x, z) f_{\alpha\delta, \beta}(z-w) dz \right) \sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w-x) dw, \quad (\text{III.8})$$

- for $\zeta \leq x < L$,

- the condition probability of a CR after one inspection period since R_j^+

$$\rho_1(x) = \bar{F}_{\alpha\delta, \beta}(L-x), \quad (\text{III.9})$$

- the condition pdf of a PIR after one inspection period since R_j^+

$$p_2(y|x) = \int_x^L g(y|x, r) f_{\alpha\delta, \beta}(r-x) dr, \quad (\text{III.10})$$

in which $f_{\alpha(\cdot), \beta}(\cdot)$ is given from (II.1) and $\bar{F}_{\alpha\delta, \beta}(\cdot)$ is derived from (II.2). Figures III.1b and III.1a show the shapes of the

Figure III.1: Shapes of the transition kernel $P(x, dy)$ of the Markov chain $\{Y_j\}_{j \in \mathbb{N}}$ when $\delta = 4.6$ and $\zeta = 11$

transition kernel $P(x, dy)$ for the two above system configurations when $\delta = 4.6$, $\zeta = 11$. In each figure, the sub-figures on the left and on the right are with $x = 3$ and $x = 12$ respectively. In Figures III.1b and III.1a, the solid black curves are given from the Monte Carlo simulation and kernel density estimation [43], while the dashed red curves are obtained by

the numerical computation of (III.4). The identical results returned by both these approaches justify the exactitude of the formulation.

2) *Numerical Solution of The Stationary Law $\pi(dy)$* : Since the expression of $P(x, dy)$ consists of a pdf and a Dirac mass function, the solution $\pi(dy)$ of (III.3) is also in the form of a convex combination of a pdf and a Dirac mass function. By substituting this form in (III.3), we obtain, after some transformations given in Appendix B, the mathematical expression of $\pi(dy)$ as follows

$$\pi(dy) = a \cdot \delta_0(dy) + (1-a) \cdot b_1(y) \cdot 1_{\{0 < y < \zeta\}} dy + (1-a) \cdot b_2(y) \cdot 1_{\{\zeta \leq y < L\}} dy, \quad (\text{III.11})$$

where

$$a = \frac{1}{1 + \int_0^\zeta B_1(y) \cdot dy + \int_\zeta^L B_2(y) \cdot dy}, \quad b_1(y) = \frac{a}{1-a} \cdot B_1(y) \quad \text{and} \quad b_2(y) = \frac{a}{1-a} \cdot B_2(y). \quad (\text{III.12})$$

$B_1(y)$ and $B_2(y)$ are obtained by solving

$$B_1(y) = p_1(y | 0) + p_3(y | 0) + \int_0^y B_1(x) \cdot (p_1(y | x) + p_3(y | x)) \cdot dx, \quad 0 < y < \zeta, \quad (\text{III.13})$$

and

$$B_2(y) = p_1(y | 0) + p_3(y | 0) + \int_0^\zeta B_1(x) \cdot (p_1(y | x) + p_3(y | x)) \cdot dx + \int_\zeta^y B_2(x) \cdot p_2(y | x) \cdot dx, \quad \zeta \leq y < L, \quad (\text{III.14})$$

in which $p_1(\cdot | \cdot)$ and $p_2(\cdot | \cdot)$ are given from (III.8) and (III.7). Solving analytically (III.13) and (III.14) being difficult, we propose to use the Heun's method to derive their numerical solutions (see Appendix C). Let continue with the example introduced in Subsection III-A1, we sketch the shapes of $B(y)$ and the stationary law $\pi(y)$ in Figure III.2. As above, the results are given by both the numerical computation of (III.13), (III.14) and (III.11) (i.e., dashed red curves) and the Monte Carlo simulation and kernel density estimation [43] (i.e., solid black curves). The identical results justify the exactitude of the formulation.

B. Expectation Quantities With Respected to The Stationary Law π

Given the stationary law π of the Markov chain $\{Y_j\}_{j \in \mathbb{N}}$, we continue formulating here the three important expectations of (III.2): $E_\pi[N_i([0^+, R_1^+])]$, $E_\pi[N_c([0^+, R_1^+])]$ and $E_\pi[W([0^+, R_1^+])]$.

1) *Expected Number of Inspections Over The First Markov Renewal Cycle*: Let consider the first ‘‘artificial’’ Markov renewal cycle $[0^+, R_1^+]$ with $X_{0^+} = y$, then

- 1) the system is inspected only one time, if $0 \leq X_{0^+} < \zeta \leq X_{R_1^-}$ or $\zeta \leq X_{0^+} < L$,
- 2) the system is inspected $(k+1)$ times, $k = 1, 2, \dots$, if $0 \leq X_{0^+} \leq X_{R_1^- - \delta} < \zeta \leq X_{R_1^-}$.

Thus, the number of inspections over the first Markov renewal cycle $[0^+, R_1^+]$ can be expressed by

$$N_i([0^+, R_1^+]) = 1_{\{0 \leq X_{0^+} < \zeta \leq X_{\delta^-}, X_{0^+} = y\}} + 1_{\{\zeta \leq X_{0^+} < L, X_{0^+} = y\}} + \sum_{k=1}^{\infty} (k+1) \cdot 1_{\{0 \leq X_{0^+} \leq X_{k \cdot \delta^-} < \zeta \leq X_{(k+1) \cdot \delta^-}, X_{0^+} = y\}}. \quad (\text{III.15})$$

As shown in Appendix D, the expected value of $N_i([0^+, R_1^+])$ with respect to the stationary law π is given by

$$E_\pi[N_i([0^+, R_1^+])] = a \cdot \bar{F}_{\alpha, \delta, \beta}(\zeta) + (1-a) \cdot \int_0^\zeta \bar{F}_{\alpha, \delta, \beta}(\zeta - y) b_1(y) dy + (1-a) \cdot \int_\zeta^L b_2(y) dy + a \cdot \int_0^\zeta \bar{F}_{\alpha, \delta, \beta}(\zeta - w) \times \left(\sum_{k=1}^{\infty} (k+1) f_{\alpha k \delta, \beta}(w) \right) dw + (1-a) \cdot \int_0^\zeta \left(\int_y^\zeta \bar{F}_{\alpha, \delta, \beta}(\zeta - w) \left(\sum_{k=1}^{\infty} (k+1) f_{\alpha k \delta, \beta}(w - y) \right) dw \right) b_1(y) dy, \quad (\text{III.16})$$

where a , $b_1(y)$ and $b_2(y)$ are given in (III.12).

Figure III.2: Shapes of $B(y)$ and $\pi(y)$ when $\delta = 4.6$ and $\zeta = 11$

2) *Expected Number of Corrective Replacements Over The First Markov Renewal Cycle:* Let consider the first “artificial” Markov renewal cycle $[0^+, R_1^+]$ with $X_{0^+} = y$, then

- 1) the system is correctively replaced after one inspection period δ since 0^+ (i.e., at $R_1 = \delta$), if $0 \leq X_{0^+} < L \leq X_{R_1^-}$,
- 2) the system is correctively replaced after a multiple of inspection period $(k+1)\delta$, $k = 1, 2, \dots$, since 0^+ (i.e., at $R_j = (k+1)\delta$), if $0 \leq X_{0^+} \leq X_{R_j^- - \delta} < \zeta < L \leq X_{R_j^-}$.

Thus, the number of CR $N_c([0^+, R_1^+])$ over the semi-renewal cycle $[0^+, R_1^+]$ can be expressed by

$$N_c([0^+, R_1^+]) = 1_{\{0 \leq X_{0^+} < L \leq X_{\delta^-}, X_{0^+} = y\}} + \sum_{k=1}^{\infty} 1_{\{0 \leq X_{0^+} \leq X_{k\delta^-} < \zeta < L \leq X_{(k+1)\delta^-}, X_{0^+} = y\}}. \quad (\text{III.17})$$

As shown in Appendix E, the expected value of $N_c([0^+, R_1^+])$ with respect to the stationary law π is

$$\begin{aligned} E_{\pi}[N_c([0^+, R_1^+])] &= a \cdot \bar{F}_{\alpha, \delta, \beta}(L) + (1-a) \cdot \int_0^{\zeta} \bar{F}_{\alpha, \delta, \beta}(L-y) b_1(y) dy \\ &+ (1-a) \cdot \int_{\zeta}^L \bar{F}_{\alpha, \delta, \beta}(L-y) b_2(y) dy + a \cdot \int_0^{\zeta} \bar{F}_{\alpha, \delta, \beta}(L-w) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w) \right) dw \\ &+ (1-a) \cdot \int_0^{\zeta} \left(\int_y^{\zeta} \bar{F}_{\alpha, \delta, \beta}(L-w) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w-y) \right) dw \right) b_1(y) dy, \quad (\text{III.18}) \end{aligned}$$

where a , $b_1(y)$ and $b_2(y)$ are given in (III.12).

3) *Expected Length of System Downtime Over The First Markov Renewal Cycle:* Let consider the first “artificial” Markov renewal cycle $[0^+, R_1^+]$ with $X_{0^+} = y$,

- 1) if $[0^+, R_1^+]$ corresponds to the first inspection period, then the system downtime $W([0^+, R_1^+])$ can be expressed by

$$W([0^+, R_1^+]) = \int_0^\delta 1_{\{0 < \tau_L < t, X_{0^+} = y\}} dt = \int_0^\delta 1_{\{0 \leq X_{0^+} < L < X_t, X_{0^+} = y\}} dt, \quad (\text{III.19})$$

- 2) if $[0^+, R_1^+]$ corresponds to $(k+1)$ first inspection period, $k = 1, 2, \dots$, then the system downtime $W([0^+, R_1^+])$ can be expressed by

$$W([0^+, R_1^+]) = \int_{k\delta}^{(k+1)\delta} 1_{\{k\delta < \tau_L < t, X_{0^+} = y\}} dt = \int_{k\delta}^{(k+1)\delta} 1_{\{0 \leq X_{0^+} \leq X_{k\delta-} < \zeta < L \leq X_t, X_{0^+} = y\}} dt, \quad (\text{III.20})$$

where τ_L is the system failure time given from (II.3). In other words,

$$W([0^+, R_1^+]) = \int_0^\delta 1_{\{0 \leq X_{0^+} < L < X_t, X_{0^+} = y\}} dt + \sum_{k=1}^{\infty} \int_{k\delta}^{(k+1)\delta} 1_{\{0 \leq X_{0^+} \leq X_{k\delta-} < \zeta < L \leq X_t, X_{0^+} = y\}} dt. \quad (\text{III.21})$$

Its expected value with respect to the stationary law π is then

$$\begin{aligned} E_\pi [W([0^+, R_1^+])] &= \int_0^\delta \left(a \cdot \bar{F}_{\alpha, t, \beta}(L) + (1-a) \cdot \int_0^\zeta \bar{F}_{\alpha, t, \beta}(L-y) b_1(y) dy \right. \\ &\quad \left. + (1-a) \cdot \int_\zeta^L \bar{F}_{\alpha, t, \beta}(L-y) b_2(y) dy + a \cdot \int_0^\zeta \bar{F}_{\alpha, t, \beta}(L-w) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w) \right) dw \right. \\ &\quad \left. + (1-a) \cdot \int_0^\zeta \left(\int_y^\zeta \bar{F}_{\alpha, t, \beta}(L-w) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w-y) \right) dw \right) b_1(y) dy \right) dt, \quad (\text{III.22}) \end{aligned}$$

where a , $b_1(y)$ and $b_2(y)$ are given in (III.12). The proof of (III.22) is given by Appendix F.

4) *Validation of Expectation Quantities and Cost Model:* To validate the mathematical formula of $E_\pi [N_i([0^+, R_1^+])]$, $E_\pi [N_c([0^+, R_1^+])]$ and $E_\pi [W([0^+, R_1^+])]$, as well as the cost rate $C_\infty(\delta, \zeta)$, we effectuate the numerical comparison between (III.16), (III.18), (III.22) and (III.2) and the results given by Monte Carlo simulation. A simple way to derive the value of $E_\pi [N_i([0^+, R_1^+])]$, $E_\pi [N_c([0^+, R_1^+])]$ and $E_\pi [W([0^+, R_1^+])]$ by the Monte Carlo simulation method is proposed in Appendix G. As an illustration, the results given by the two above approaches for the two system configurations introduced at the beginning of Section III are shown in Table I., The duration for Monte Carlo simulation has been chosen by $D = 10^7$

System parameters	Approach	$E_\pi [N_i([0^+, R_1^+])]$	$E_\pi [N_c([0^+, R_1^+])]$	$E_\pi [W([0^+, R_1^+])]$	$C_\infty(\delta, \zeta)$
$\alpha = 5, \beta = 5$	Num. Comp.	1.0004	0.2540	0.6141	8.6651
$\zeta = 4, \delta = 7$	M.C. Sim.	1.0004	0.2533	0.6105	8.6430
$\alpha = 0.2, \beta = 0.2$	Num. Comp.	1.8537	0.6006	2.1019	8.7571
$\zeta = 11, \delta = 8$	M.C. Sim.	1.8538	0.6001	2.0988	8.7488

Table I: Results for validation the mathematical formulation

time units (see also Appendix G). We find that the results returned by both the approaches are almost the same. This means that the mathematical developments are exact.

C. Existence of Optimum and Searching

As aforementioned, optimizing the (δ, ζ) policy is to seek the couple of decision variables $(\delta_{opt}, \zeta_{opt})$ that minimizes $C_\infty(\delta, \zeta)$ in (III.2). Due to the complexity of the mathematical expression of $C_\infty(\delta, \zeta)$, we cannot demonstrate the existence of optimum analytically, but rather numerically. In fact, numerous numerical examples under various configurations of system characteristics and intervention costs show that $C_\infty(\delta, \zeta)$ is a bivariate convex function of δ and ζ (see also numerical examples

in Section IV). Thus, the optimal (δ, ζ) policy exists. Figures III.3a and III.3b illustrate the graphs of $C_\infty(\delta, \zeta)$ for the two system configurations considered as above when δ and ζ vary.

Figure III.3: Shapes of $C_\infty(\delta, \zeta)$ for $L = 15, C_i = 5, C_r = 20, C_c = 100$ and $C_d = 25$.

To find the optimal decision variables $(\delta_{opt}, \zeta_{opt})$ as well as the associated maintenance cost rate $C_\infty(\delta_{opt}, \zeta_{opt})$, we apply the generalized pattern search algorithm, a derivative free algorithm for black-box optimization [44], to (III.2). The *patternsearch* solver of Matlab's Global Optimization Toolbox has been used. Figures III.4a and III.4b represent respectively the optimization procedure for the two considered system configurations. According to this algorithm, we obtain the optimal

Figure III.4: Optimization with generalized pattern search algorithm for $L = 15, C_i = 5, C_r = 20, C_c = 100$ and $C_d = 25$.

tuning $\zeta_{opt} = 10.125, \delta_{opt} = 6.125$ and $C_\infty(\delta_{opt}, \zeta_{opt}) = 7.9383$ for the first configuration (i.e., $\alpha = 5, \beta = 5$), and $\zeta_{opt} = 11.8906, \delta_{opt} = 4.125$ and $C_\infty(\delta_{opt}, \zeta_{opt}) = 8.0783$ for the second configuration (i.e., $\alpha = 0.2, \beta = 0.2$).

IV. SENSITIVITY STUDIES FOR THE (δ, ζ) POLICY

To better understand how the (δ, ζ) policy behaves under the variation of maintenance costs and system deterioration characteristics, numerical sensitivity studies are proposed.

A. Sensitivity Studies to Maintenance Costs

We study at first the impact of maintenance costs on the performance of the (δ, ζ) policy. To this end, we fix the values of system characteristics (i.e., α, β, L and $g(\cdot | \cdot, \cdot)$) and the CR cost C_c . Next, by varying inspection cost C_i , PIR cost C_r

and downtime cost rate C_d one after another in a wide rank, we observe the evolution of the optimal decision parameters $(\delta_{opt}, \zeta_{opt})$ and the associated cost rates $C_\infty(\delta_{opt}, \zeta_{opt})$. Repeating this procedure for various configurations, we can draw interesting conclusions on the impact of maintenance costs on the (δ, ζ) policy. Note that the sensitivity to CR cost is not mentioned because it is a redundant case of other studies. Although numerous numerical experiments have been done, only one of them is illustrated in this subsection. The following illustrations are based on $\alpha = 1$, $\beta = 1$, $L = 15$, a continuous uniform pdf for $g(\cdot | \cdot, \cdot)$ and $C_c = 100$. The values of C_i , C_r and C_d , which depend on special studies, are introduced later.

1) *Sensitivity to Inspection Costs:* For this study, $C_r = 30$, $C_d = 25$ and C_i varies from 3 to 9 with step 2. Figure IV.1 sketches the associated iso-level curves of $C_\infty(\delta, \zeta)$ when δ and ζ vary. In each sub-figure, the red dot location represents

Figure IV.1: Sensitivity to inspection costs when $C_r = 30$, $C_c = 100$ and $C_d = 25$

the optimal decision parameters $(\delta_{opt}, \zeta_{opt})$, and the associated optimal cost rate $C_\infty(\delta_{opt}, \zeta_{opt})$ is shown in the title.

We find that the value of δ_{opt} increases with respect to the increasing of C_i . This means the (δ, ζ) policy requires to inspect the system less frequently to avoid expensive cost. Meanwhile, less frequent inspections also cause late detection of system failure, thence longer system downtime. This is why the (δ, ζ) policy adjusts ζ_{opt} smaller to enable more often PIR. The cost rate $C_\infty(\delta_{opt}, \zeta_{opt})$ is thus a compromise between the gain and the loss of inspection and PIR. Notwithstanding, $C_\infty(\delta_{opt}, \zeta_{opt})$ always increases for higher inspection cost.

2) *Sensitivity to Preventive Imperfect Repair Costs:* Figure IV.2 is obtained when $C_i = 5$, $C_d = 25$ and C_r varies from 5 to 47 with step 14. Its meanings are the same as Figure IV.1.

Figure IV.2: Sensitivity to preventive imperfect repair costs when $C_i = 5$, $C_c = 100$ and $C_d = 25$

When C_r increases, ζ_{opt} also increases to avoid more frequent PIR, while δ_{opt} is more or less constant. This means that the inspection period in the (δ, ζ) policy is less sensible to the variation of PIR costs, and that only ζ_{opt} contributes to keep

a suitable maintenance performance with respect to the variation of C_r . Consequently, $C_\infty(\delta_{opt}, \zeta_{opt})$ increases when PIR become more costly.

3) *Sensitivity to Downtime Cost Rates:* Figure IV.3 represents the results for the case that $C_i = 5$, $C_r = 30$ and C_d varies from 10 to 56 with step 12. Its meanings are the same as the two above case studies.

Figure IV.3: Sensitivity to downtime cost rate when $C_i = 5$, $C_r = 30$ and $C_d = 25$

We remark that the optimal (δ, ζ) policy sets δ_{opt} at a very small value when C_d becomes important. The system state is thus followed very closely to trigger a timely CR and to shorten the system downtime. Accordingly, it does not need to set ζ_{opt} at a small value, but rather at a high value to extend as most as possible the system lifetime. As such, both δ_{opt} and ζ_{opt} contribute to the optimal cost rate $C_\infty(\delta_{opt}, \zeta_{opt})$, which becomes higher when the system downtime is costly.

B. Sensitivity Studies to System Deterioration Characteristics

Besides interventions costs, the deterioration speed and associated variance are important factors deciding the performance of maintenance policy. It is then necessary to study their impacts on the (δ, ζ) policy. Fixing C_i , C_r , C_c , C_d and L , we vary the deterioration speed $\mu = \frac{\alpha}{\beta}$ and the deterioration variance $\sigma^2 = \frac{\alpha}{\beta^2}$ one after another, and we observe how the optimal decision parameters $(\delta_{opt}, \zeta_{opt})$ and the associated cost rates $C_\infty(\delta_{opt}, \zeta_{opt})$ evolve. The following illustrations are obtained for $C_i = 5$, $C_r = 19$, $C_c = 100$, $C_d = 25$, $L = 15$, and a continuous uniform pdf for $g(\cdot | \cdot, \cdot)$. The values of μ and σ^2 are introduced later.

1) *Sensitivity to Deterioration Speed:* Setting $\sigma^2 = 1$ and varying μ from 1 to 4 with step 1, we obtain the results as in Figure IV.4.

Figure IV.4: Sensitivity to system deterioration speed when $C_i = 5$, $C_r = 19$, $C_c = 100$ and $C_d = 25$

We can find that the optimal cost rate $C_\infty(\delta_{opt}, \zeta_{opt})$ is a increasing function of the deterioration speed μ . When μ is relatively small, $C_\infty(\delta_{opt}, \zeta_{opt})$ depends closely on both the values of δ_{opt} and ζ_{opt} , but it is almost independent of the ζ_{opt} value when μ becomes higher. In all cases, δ_{opt} decreases with respect to the increasing of μ .

2) *Sensitivity to Deterioration Variance*: Using $\mu = 1$ and setting $\sigma^2 = 0.2, 1, 5$ and 10 respectively, we obtain the results as in Figure IV.5.

Figure IV.5: Sensitivity to system deterioration variance when $C_i = 5$, $C_r = 19$, $C_c = 100$ and $C_d = 25$

Under the optimal (δ, ζ) policy, δ_{opt} and ζ_{opt} tend to decrease and increase with respect to higher values of deterioration variance σ^2 . We find that these tendencies are similar to what observed in Figure IV.3. Accordingly, δ_{opt} and ζ_{opt} have the same meanings. Normally, when the deterioration behavior of the system is chaotic, we should follow closely the system state to be able to make a proper decision. Numerous numerical experiments also show that the optimal cost rate $C_\infty(\delta_{opt}, \zeta_{opt})$ is a concave function of σ^2 . Moreover, the smaller values are concentrated at high values of σ^2 . In other words, the (δ, ζ) policy is especially suitable to systems with very chaotic deterioration process.

V. COMPARATIVE STUDIES FOR THE (δ, ζ) POLICY

To assess the performances of the (δ, ζ) policy, as well as the importance of incorporating the past dependency in maintenance modeling, comparative studies with two more classical benchmark policies are done. These policies share exactly the same inspection and maintenance decision structures as the (δ, ζ) policy (see Section II-B). The only difference is the efficiency and incurred cost of employed preventive maintenance actions.

- 1) (δ, λ) policy. At an inspection time $T_{j,k} = R_{j-1} + k \cdot \delta$, if $\lambda \leq X_{T_{j,k}}^- < L$, a preventive perfect replacement (PPR) with constant unit cost $C_p \in (C_r, C_c)$ is immediately carried out. Just after an instantaneous PPR, the system is put back to an AGAN condition (i.e., $X_{T_{j,k}}^+ = 0$). Such a policy is called (δ, λ) policy because the inspection period δ and the PPR threshold λ are two decision variables to be optimized.
- 2) (δ, η) policy: At an inspection time $T_{j,k} = R_{j-1} + k \cdot \delta$, if $\eta \leq X_{T_{j,k}}^- < L$, a memoryless PIR with the same constant unit cost C_r as in the (δ, ζ) policy is immediately carried out. Just after an instantaneous memoryless PIR, the system deterioration is set to a level between 0 and $X_{T_{j,k}}^-$ independently of previous maintenance actions following the pdf $X_{T_{j,k}}^+ \sim p(y | 0, z)$, where z is a realization of $X_{T_{j,k}}^-$. We call this policy (δ, η) because the inspection period δ and the PIR threshold η are two decision variables to be optimized.

An illustration of the deterioration behavior of the maintained system under the two benchmark policies is shown in Figures V.1a and V.1b respectively. We note that the (δ, λ) policy and the (δ, η) policy have been proposed in [33] and [6] respectively. In this paper, the former stands for another choice of preventive actions (i.e., PPR instead of PIR), while the latter represents a wrong choice of PIR models (i.e., memoryless PIR instead of past-dependent PIR).

Figure V.1: Schematic evolution of the maintained system state under benchmark maintenance policies

To illustrate the performances of the three studied maintenance policies, we consider a system defined by $\alpha = 1$, $\beta = 1$ and $L = 15$. The functions $g(\cdot | \cdot, \cdot)$ and $p(\cdot | \cdot, \cdot)$ are chosen as continuous uniform pdf. Next, we fix $C_i = 5$, $C_p = 75$, $C_c = 100$, $C_d = 20$, vary C_r from 10 to 70 with step 5, and we sketch the evolution of the long-run maintenance cost rate $C_{\infty, opt}$ of the three considered policies in Figure V.2. The figure shows clearly that the benefit of PIR against PIR is a

Figure V.2: Evolution of the long-run maintenance cost rate when $C_i = 5$, $C_p = 75$, $C_c = 100$ and $C_d = 20$

compromise between the efficiency and incurred cost of maintenance actions. Especially, the PIR is more profitable than the PPR when C_r is relatively smaller than C_p . Comparing $C_{\infty, opt}$ of the (δ, ζ) policy and of the (δ, η) policy, we find that a mistake in PIR modeling will overestimate the benefit of PIR, hence a wrong choice between PIR and PPR.

VI. CONCLUSIONS AND PERSPECTIVES

This paper develops a deterioration-based maintenance model with past-dependent PIR for single-unit systems subject to stochastic continuous deterioration process. A complete procedure, including deterioration and failure modeling, maintenance policy elaboration, mathematical cost model formulation and validation, and sensitivity and comparative studies, has been made. Especially, the paper applies truncated probability distributions to describe the efficiency of past-dependent PIR on the system deterioration. Such a model is much more simple and can break the memory assumption existing in well-known arithmetic reduction type and Kijima's type models. Compared to many memoryless CBIM models in the literature, our model is more general and more realistic.

The work presented in this paper is merely theoretical; and even if the theoretical results are encouraging, we should validate the proposed CBIM model with real-world experiments. Building a deterioration and maintenance data-set, and estimating

parameters of the CBIM model are recognized as key steps of this future work. Besides, proposing a new maintenance policy combining the advantages of imperfect repairs and perfect replacements for preventive actions could be an interesting perspective. For such a policy, how to switch from an imperfect repair to a perfect replacement and vice versa will be a main problem. Other perspective is to continue to work on other past-dependent effects of imperfect repairs.

APPENDIX

A. Proof of (III.4)

Considering the Markov renewal cycle $[R_j^+, R_{j+1}^+]$ and assuming that $X_{R_j^+} = x$ and $X_{R_{j+1}^+} = y$, the transition kernel of the Markov chain $\{Y_j\}_{j \in \mathbb{N}}$ is defined by

$$P(x, dy) = \sum_{k=0}^{\infty} P(X_{R_{j+1}^+} \in dy \mid X_{R_j^+} = x). \quad (\text{A.1})$$

If a replacement is performed at R_{j+1} , then $y = 0$. If an imperfect repair is done, then $y \neq 0$ and $x \leq y < L$. Hereafter, we consider different scenarios to compute $P(x, dy)$ following the length to replacement/repair time since R_j .

1) *Replacement/Repair After One Inspection Period*: Let consider the situation that the next replacement/repair on the system is performed after one inspection period δ since R_j (i.e., $R_{j+1}^+ = R_j^+ + \delta$). We further then distinguish two cases $0 \leq x < \zeta$ and $\zeta \leq x < L$.

1) If $0 \leq x < \zeta$, then the transition kernel $P(x, dy)$ is expressed as

$$P(x, dy) = \underbrace{P(X_{R_{j+1}^+} \in dy, L \leq X_{R_{j+1}^-} \mid X_{R_j^+} = x)}_{P_1(x, dy)} + \underbrace{P(X_{R_{j+1}^+} \in dy, \zeta \leq X_{R_{j+1}^-} < L \mid X_{R_j^+} = x)}_{P_2(x, dy)}. \quad (\text{A.2})$$

In (A.2), $P_1(x, dy)$ stands for the transition pdf associated with a CR (i.e., $y = 0$)

$$\begin{aligned} P_1(x, dy) &= P(X_{R_{j+1}^+} \in dy \mid L \leq X_{R_{j+1}^-}, X_{R_j^+} = x) P(L \leq X_{R_{j+1}^-} \mid X_{R_j^+} = x) \\ &= \delta_0(dy) P(L - x \leq X_{R_{j+1}^-} - X_{R_j^+}) = \delta_0(dy) \bar{F}_{\alpha\delta, \beta}(L - x), \end{aligned} \quad (\text{A.3})$$

and $P_2(x, dy)$ stands for the transition pdf associated with a PIR (i.e., $y \neq 0$ and $x \leq y < L$)

$$\begin{aligned} P_2(x, dy) &= P(X_{R_{j+1}^+} \in dy, \zeta - x \leq X_{R_{j+1}^-} - X_{R_j^+} < L - x \mid X_{R_j^+} = x) \\ &= \int_{\zeta-x}^{L-x} P(X_{R_{j+1}^+} \in dy \mid X_{R_j^+} = x, X_{R_{j+1}^-} - X_{R_j^+} = v) f_{\alpha\delta, \beta}(v) dv \\ &= \int_{\zeta-x}^{L-x} P(X_{R_{j+1}^+} \in dy \mid X_{R_j^+} = x, X_{R_{j+1}^-} = x + v) f_{\alpha\delta, \beta}(v) dv \\ &= dy \int_{\zeta-x}^{L-x} g(y \mid x, x + v) f_{\alpha\delta, \beta}(v) dv = dy \int_{\zeta}^L g(y \mid x, r) f_{\alpha\delta, \beta}(r - x) dr, \end{aligned} \quad (\text{A.4})$$

in which the change of variable $r = x + v$ is used. Thus,

$$P(x, dy) = \delta_0(dy) \cdot \underbrace{\bar{F}_{\alpha\delta, \beta}(L - x)}_{\rho_1(x)} + 1_{\{y \neq 0, x \leq y < L\}} \cdot dy \underbrace{\int_{\zeta}^L g(y \mid x, r) f_{\alpha\delta, \beta}(r - x) dr}_{p_1(y|x)}, \quad (\text{A.5})$$

where $0 \leq x < \zeta$.

2) If $\zeta \leq x < L$, then the transition kernel $P(x, dy)$ is expressed as

$$P(x, dy) = \underbrace{P(X_{R_{j+1}^+} \in dy, L \leq X_{R_{j+1}^-} \mid X_{R_j^+} = x)}_{P_3(x, dy)} + \underbrace{P(X_{R_{j+1}^+} \in dy, X_{R_{j+1}^-} < L \mid X_{R_j^+} = x)}_{P_4(x, dy)}. \quad (\text{A.6})$$

In (A.6), $P_3(x, dy)$ stands for the transition pdf associated with a CR (i.e., $y = 0$)

$$P_3(x, dy) = P_1(x, dy) = \delta_0(dy) \cdot \rho_1(x), \quad (\text{A.7})$$

and $P_4(x, dy)$ stands for the transition pdf associated with a PIR (i.e., $y \neq 0$ and $x \leq y < L$)

$$\begin{aligned} P_4(x, dy) &= P\left(X_{R_{j+1}^+} \in dy, X_{R_{j+1}^-} - X_{R_j^+} < L - x \mid X_{R_j^+} = x\right) \\ &= \int_0^{L-x} P\left(X_{R_{j+1}^+} \in dy \mid X_{R_j^+} = x, X_{R_{j+1}^-} - X_{R_j^+} = v\right) f_{\alpha\delta, \beta}(v) dv \\ &= \int_0^{L-x} P\left(X_{R_{j+1}^+} \in dy \mid X_{R_j^+} = x, X_{R_{j+1}^-} = x + v\right) f_{\alpha\delta, \beta}(v) dv \\ &= dy \int_0^{L-x} g(y \mid x, x + v) f_{\alpha\delta, \beta}(v) dv = dy \int_x^L g(y \mid x, r) f_{\alpha\delta, \beta}(r - x) dr, \quad (\text{A.8}) \end{aligned}$$

in which the change of variable $r = x + v$ is used. Thus,

$$P(x, dy) = \delta_0(dy) \cdot \rho_1(x) + 1_{\{y \neq 0, x \leq y < L\}} \cdot dy \underbrace{\int_x^L g(y \mid x, r) f_{\alpha\delta, \beta}(r - x) dr}_{p_2(y|x)}, \quad (\text{A.9})$$

where $\zeta \leq x < L$.

2) *Replacement/Repair After Multiple Inspection Periods:* We are in the situation that the next system replacement/repair is performed after a multiple of inspection periods since R_j (i.e., $R_{j+1}^+ = R_j^+ + (k+1)\delta$, where $k \in \mathbb{N}^*$). For this situation, $0 \leq x < \zeta$, and then the transition kernel $P(x, dy)$ is thus computed as

$$\begin{aligned} P(x, dy) &= \sum_{k=1}^{\infty} \left[\underbrace{P\left(X_{R_{j+1}^+} \in dy, X_{R_{j+1}-\delta} < \zeta < L \leq X_{R_{j+1}^-} \mid X_{R_j^+} = x\right)}_{P_{5,k+1}(x, dy)} \right. \\ &\quad \left. + \underbrace{P\left(X_{R_{j+1}^+} \in dy, X_{R_{j+1}-\delta} < \zeta \leq X_{R_{j+1}^-} < L \mid X_{R_j^+} = x\right)}_{P_{6,k+1}(x, dy)} \right], \quad (\text{A.10}) \end{aligned}$$

In (A.10), $P_{5,k+1}(x, dy)$ represents the transition pdf associated with a CR (i.e., $y = 0$)

$$\begin{aligned} P_{5,k+1}(x, dy) &= P\left(X_{R_{j+1}^+} \in dy, X_{R_{j+1}-\delta} - X_{R_j^+} < \zeta - x < L - x \leq X_{R_{j+1}^-} - X_{R_j^+} \mid X_{R_j^+} = x\right) \\ &= \int_0^{\zeta-x} P\left(X_{R_{j+1}^+} \in dy, L - x \leq X_{R_{j+1}^-} - X_{R_j^+} \mid X_{R_j^+} = x, X_{R_{j+1}^-} - X_{R_j^+} = u\right) f_{\alpha k \delta, \beta}(u) du \\ &= \int_0^{\zeta-x} P\left(X_{R_{j+1}^+} \in dy, L - x - u \leq X_{R_{j+1}^-} - X_{R_{j+1}-\delta} \mid X_{R_j^+} = x, X_{R_{j+1}^-} - X_{R_{j+1}-\delta} = x + u\right) f_{\alpha k \delta, \beta}(u) du \\ &= \int_0^{\zeta-x} \delta_0(dy) \bar{F}_{\alpha\delta, \beta}(L - x - u) f_{\alpha k \delta, \beta}(u) du, \quad (\text{A.11}) \end{aligned}$$

and $P_{6,k+1}(x, dy)$ represents the transition pdf associated with a PIR (i.e., $y \neq 0$ and $x \leq y < L$)

$$\begin{aligned}
P_{6,k+1}(x, dy) &= P\left(X_{R_{j+1}^+} \in dy, X_{R_{j+1}-\delta} - X_{R_j^+} < \zeta - x \leq X_{R_{j+1}^-} - X_{R_j^+} < L - x \mid X_{R_j^+} = x\right) \\
&= \int_0^{\zeta-x} P\left(X_{R_{j+1}^+} \in dy, \zeta - x \leq X_{R_{j+1}^-} - X_{R_j^+} < L - x \mid X_{R_j^+} = x, X_{R_{j+1}-\delta} - X_{R_j^+} = u\right) f_{\alpha k \delta, \beta}(u) du \\
&= \int_0^{\zeta-x} P\left(X_{R_{j+1}^+} \in dy, \zeta - x - u \leq X_{R_{j+1}^-} - X_{R_{j+1}-\delta} < L - x - u \mid X_{R_j^+} = x, X_{R_{j+1}-\delta} = x + u\right) f_{\alpha k \delta, \beta}(u) du \\
&= \int_0^{\zeta-x} \int_{\zeta-x-u}^{L-x-u} P\left(X_{R_{j+1}^+} \in dy, \mid X_{R_j^+} = x, X_{R_{j+1}-\delta} = x + u, X_{R_{j+1}^-} - X_{R_{j+1}-\delta} = v\right) f_{\alpha \delta, \beta}(v) f_{\alpha k \delta, \beta}(u) dv du \\
&= \int_0^{\zeta-x} \int_{\zeta-x-u}^{L-x-u} P\left(X_{R_{j+1}^+} \in dy, \mid X_{R_j^+} = x, X_{R_{j+1}^-} = x + u + v\right) f_{\alpha \delta, \beta}(v) f_{\alpha k \delta, \beta}(u) dv du \\
&= \int_0^{\zeta-x} \int_{\zeta-x-u}^{L-x-u} g(y \mid x, x + u + v) dy f_{\alpha \delta, \beta}(v) f_{\alpha k \delta, \beta}(u) dv du. \quad (\text{A.12})
\end{aligned}$$

Thus,

$$\begin{aligned}
P(x, dy) &= \sum_{k=1}^{\infty} \left[\delta_0(dy) \int_0^{\zeta-x} \bar{F}_{\alpha \delta, \beta}(L - x - u) f_{\alpha k \delta, \beta}(u) du \right. \\
&\quad \left. + 1_{\{y \neq 0, x \leq y < L\}} \cdot dy \int_0^{\zeta-x} \int_{\zeta-x-u}^{L-x-u} g(y \mid x, x + u + v) f_{\alpha \delta, \beta}(v) f_{\alpha k \delta, \beta}(u) dv du \right] \\
&= \sum_{k=1}^{\infty} \int_0^{\zeta-x} (\delta_0(dy) \bar{F}_{\alpha \delta, \beta}(L - x - u) + 1_{\{y \neq 0, x \leq y < L\}} \cdot dy \times \\
&\quad \int_{\zeta-x-u}^{L-x-u} g(y \mid x, x + u + v) f_{\alpha \delta, \beta}(v) dv) f_{\alpha k \delta, \beta}(u) du \\
&= \int_0^{\zeta-x} (\delta_0(dy) \bar{F}_{\alpha \delta, \beta}(L - x - u) + 1_{\{y \neq 0, x \leq y < L\}} \cdot dy \times \\
&\quad \int_{\zeta-x-u}^{L-x-u} g(y \mid x, x + u + v) f_{\alpha \delta, \beta}(v) dv) \sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(u) du, \quad (\text{A.13})
\end{aligned}$$

Using the change of variables $z = x + u + v$ and $w = x + u$, we obtain

$$\begin{aligned}
P(x, dy) &= \int_x^{\zeta} \left(\delta_0(dy) \bar{F}_{\alpha \delta, \beta}(L - w) + 1_{\{y \neq 0, x \leq y < L\}} \cdot dy \int_{\zeta}^L g(y \mid x, z) f_{\alpha \delta, \beta}(z - w) dz \right) \\
&\quad \times \sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w - x) dw = \underbrace{\delta_0(dy) \int_x^{\zeta} \bar{F}_{\alpha \delta, \beta}(L - w) \sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w - x) dw}_{\rho_2(x)} \\
&\quad + 1_{\{y \neq 0, x \leq y < L\}} \cdot dy \underbrace{\int_x^{\zeta} \left(\int_{\zeta}^L g(y \mid x, z) f_{\alpha \delta, \beta}(z - w) dz \right) \sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w - x) dw}_{p_3(y|x)}, \quad (\text{A.14})
\end{aligned}$$

where $0 \leq x < \zeta$.

By summarizing the above cases, we obtain (III.4)

B. Proof of (III.11)

The stationary measure π of $\{Y_j\}_{j \in \mathbb{N}}$ is the solution of the invariance equation

$$\pi(dy) = \int_{[0, L)} P(x, dy) \pi(dx), \quad (\text{B.1})$$

where $P(x, dy)$, given by (III.4), is rewritten as

$$P(x, dy) = (\rho_1(x) \cdot 1_{\{0 \leq x < L\}} + \rho_2(x) \cdot 1_{\{0 \leq x < \zeta\}}) \cdot \delta_0(dy) + (p_1(y|x) + p_3(y|x)) \cdot 1_{\{y \neq 0, 0 \leq x \leq y < \zeta\}} \cdot dy \\ + ((p_1(y|x) + p_3(y|x)) \cdot 1_{\{y \neq 0, 0 \leq x < \zeta \leq y < L\}} + p_2(y|x) \cdot 1_{\{y \neq 0, \zeta \leq x \leq y < L\}}) \cdot dy. \quad (\text{B.2})$$

We search a solution in the form

$$\pi(dy) = a \cdot \delta_0(dy) + (1-a) \cdot b_1(y) \cdot 1_{\{0 < y < \zeta\}} dy + (1-a) \cdot b_2(y) \cdot 1_{\{\zeta \leq y < L\}} dy, \quad (\text{B.3})$$

where $0 < a < 1$ and $b(y) = b_1(y) \cdot 1_{\{0 < y < \zeta\}} + b_2(y) \cdot 1_{\{\zeta \leq y < L\}}$ is a pdf on $(0, L)$ (i.e., $\int_0^L b(y) dy = \int_0^\zeta b_1(y) dy + \int_\zeta^L b_2(y) dy = 1$). Substituting (B.2) and (B.3) in (B.1), we obtain

$$a \cdot \delta_0(dy) + (1-a) \cdot b_1(y) \cdot 1_{\{0 < y < \zeta\}} \cdot dy + (1-a) \cdot b_2(y) \cdot 1_{\{\zeta \leq y < L\}} \cdot dy \\ = \int_{[0, L)} ((\rho_1(x) \cdot 1_{\{0 \leq x < L\}} + \rho_2(x) \cdot 1_{\{0 \leq x < \zeta\}}) \cdot \delta_0(dx) + (p_1(y|x) + p_3(y|x)) \cdot 1_{\{y \neq 0, 0 \leq x \leq y < \zeta\}} \cdot dy \\ + ((p_1(y|x) + p_3(y|x)) \cdot 1_{\{y \neq 0, 0 \leq x < \zeta \leq y < L\}} + p_2(y|x) \cdot 1_{\{y \neq 0, \zeta \leq x \leq y < L\}}) \cdot dy) \\ \times (a \cdot \delta_0(dx) + (1-a) \cdot b_1(x) \cdot 1_{\{0 < x < \zeta\}} \cdot dx + (1-a) \cdot b_2(x) \cdot 1_{\{\zeta \leq x < L\}} \cdot dx). \quad (\text{B.4})$$

This leads to the three following equalities:

1) when $y = 0$,

$$a = \int_{[0, L)} (\rho_1(x) \cdot 1_{\{0 \leq x < L\}} + \rho_2(x) \cdot 1_{\{0 \leq x < \zeta\}}) \\ \times (a \cdot \delta_0(dx) + (1-a) \cdot b_1(x) \cdot 1_{\{0 < x < \zeta\}} \cdot dx + (1-a) \cdot b_2(x) \cdot 1_{\{\zeta \leq x < L\}} \cdot dx) = a \cdot \rho_1(0) + a \cdot \rho_2(0) \\ + \int_0^\zeta (1-a) \cdot b_1(x) \cdot (\rho_1(x) + \rho_2(x)) \cdot dx + \int_\zeta^L (1-a) \cdot b_2(x) \cdot \rho_1(x) \cdot dx, \quad (\text{B.5})$$

2) when $0 < y < \zeta$,

$$(1-a) \cdot b_1(y) = \int_{[0, L)} ((p_1(y|x) + p_3(y|x)) \cdot 1_{\{y \neq 0, 0 \leq x \leq y < \zeta\}}) \\ \times (a \cdot \delta_0(dx) + (1-a) \cdot b_1(x) \cdot 1_{\{0 < x < \zeta\}} \cdot dx + (1-a) \cdot b_2(x) \cdot 1_{\{\zeta \leq x < L\}} \cdot dx) \\ = a \cdot (p_1(y|0) + p_3(y|0)) + \int_0^y (1-a) \cdot b_1(x) \cdot (p_1(y|x) + p_3(y|x)) \cdot dx, \quad (\text{B.6})$$

3) when $\zeta \leq y < L$,

$$(1-a) \cdot b_2(y) = \int_{[0, L)} ((p_1(y|x) + p_3(y|x)) \cdot 1_{\{y \neq 0, 0 \leq x < \zeta \leq y < L\}} + p_2(y|x) \cdot 1_{\{y \neq 0, \zeta \leq x \leq y < L\}}) \\ \times (a \cdot \delta_0(dx) + (1-a) \cdot b_1(x) \cdot 1_{\{0 < x < \zeta\}} \cdot dx + (1-a) \cdot b_2(x) \cdot 1_{\{\zeta \leq x < L\}} \cdot dx) = a \cdot (p_1(y|0) + p_3(y|0)) \\ + \int_0^\zeta (1-a) \cdot b_1(x) \cdot (p_1(y|x) + p_3(y|x)) \cdot dx + \int_\zeta^y (1-a) \cdot b_2(x) \cdot p_2(y|x) \cdot dx. \quad (\text{B.7})$$

In these equalities, $\rho_1(\cdot)$, $p_1(\cdot|\cdot)$, $\rho_2(\cdot)$ and $p_2(\cdot|\cdot)$ are given from (III.6), (III.8), (III.5) and (III.7) respectively. Let $B_i(\cdot) = \frac{1-a}{a} b_i(\cdot)$, $i = 1, 2$, the above equalities become:

1) when $y = 0$,

$$1 = \rho_1(0) + \rho_2(0) + \int_0^\zeta B_1(x) \cdot (\rho_1(x) + \rho_2(x)) \cdot dx + \int_\zeta^L B_2(x) \cdot \rho_1(x) \cdot dx, \quad (\text{B.8})$$

2) when $0 < y < \zeta$,

$$B_1(y) = p_1(y|0) + p_3(y|0) + \int_{0+}^y B_1(x) \cdot (p_1(y|x) + p_3(y|x)) \cdot dx, \quad (\text{B.9})$$

3) and when $\zeta \leq y < L$,

$$B_2(y) = p_1(y|0) + p_3(y|0) + \int_0^\zeta B_1(x) \cdot (p_1(y|x) + p_3(y|x)) \cdot dx + \int_\zeta^y B_2(x) \cdot p_2(y|x) \cdot dx. \quad (\text{B.10})$$

Solving (B.9) and (B.10) gives the expressions $B_1(y)$ and $B_2(y)$. Moreover, since $\int_0^L \pi(y) dy = 1$, from (B.3), we have

$$a + \int_0^\zeta (1-a) \cdot b_1(y) \cdot dy + \int_\zeta^L (1-a) \cdot b_2(y) \cdot dy = 1. \quad (\text{B.11})$$

Since $a \cdot B_i(\cdot) = (1-a) \cdot b_i(\cdot)$, $i = 1, 2$, then

$$a \cdot \left(1 + \int_0^\zeta B_1(y) \cdot dy + \int_\zeta^L B_2(y) \cdot dy \right) = 1. \quad (\text{B.12})$$

Thence,

$$a = \frac{1}{1 + \int_0^\zeta B_1(y) \cdot dy + \int_\zeta^L B_2(y) \cdot dy}, \quad b_1(y) = \frac{a}{1-a} \cdot B_1(y) \quad \text{and} \quad b_2(y) = \frac{a}{1-a} \cdot B_2(y), \quad (\text{B.13})$$

where $B_1(y)$ and $B_2(y)$ are derived from (B.9) and (B.10).

C. Heun's Method for Solving (III.13) and (III.14)

This section aims at using the Heun's method presented in [45, pages 334-335] to approximate the solution $B_1(y)$ of (III.13), and $B_2(y)$ of (III.14). In fact, (III.13) and (III.14) share the same form of a non-homogeneous linear Volterra integral equation of the second kind, which can be generally expressed by

$$B_n(y) = q_n(y) + \int_{a_n}^y K_n(y, x) \cdot B_n(x) \cdot dx, \quad a_n < y < b_n, \quad (\text{C.1})$$

where

1) when $n = 1$, then

$$a_n = 0, \quad b_n = \zeta, \quad K_n(y, x) = p_1(y|x) + p_3(y|x), \quad q_n(y) = p_1(y|0) + p_3(y|0),$$

2) when $n = 2$, then

$$a_n = \zeta, \quad b_n = L, \quad K_n(y, x) = p_2(y|x), \quad q_n(y) = p_1(y|0) + p_3(y|0) + \int_0^\zeta (p_1(y|x) + p_3(y|x)) \cdot B_1(x) \cdot dx.$$

The idea of the Heun's method is to approximate the integral part of (C.1) by the well-known trapezoid rule

$$\int_{a_n}^y K_n(y, x) B_n(x) \cdot dx \simeq h \cdot \left(\frac{1}{2} K_n(y, x_0) B_n(x_0) + K_n(y, x_1) B_n(x_1) + \dots + K_n(y, x_{N-1}) B_n(x_{N-1}) + \frac{1}{2} K_n(y, x_N) B_n(x_N) \right), \quad (\text{C.2})$$

where $h = \frac{y-a_n}{N}$ is the mesh spacing given by dividing the interval of integration (a_n, y) into $N \geq 1$ equal sub-intervals, $x_0 = a_n$, $x_j = a_n + j \cdot h$ with $j = 1, \dots, N$, $x_j \leq y$. Dividing the interval (a_n, b_n) by equal sub-intervals with length h (i.e., $y_0 = a_n$, $y_N = x_N = y = b_n$, $y_i = a_n + i \cdot h$, $i = 1, \dots, N$), and denoting $B_{n,i} = B_n(y_i)$, $q_{n,i} = q_n(y_i)$ and

$K_{n,i,j} = K_n(y_i, x_j)$, we can recursively compute $B_n(y)$ of (C.1) from the following equalities

$$\begin{aligned}
B_{n,0} &= q_{n,0} \\
B_{n,1} &= q_{n,1} + h \cdot \left(\frac{1}{2} K_{n,1,0} B_{n,0} + \frac{1}{2} K_{n,1,1} B_{n,1} \right) \\
B_{n,2} &= q_{n,2} + h \cdot \left(\frac{1}{2} K_{n,2,0} B_{n,0} + K_{n,2,1} B_{n,1} + \frac{1}{2} K_{n,2,2} B_{n,2} \right) \\
&\vdots \\
B_{n,N} &= q_{n,N} + h \cdot \left(\frac{1}{2} K_{n,N,0} B_{n,0} + K_{n,N,1} B_{n,1} + \cdots + K_{n,N,N-1} B_{n,N-1} + \frac{1}{2} K_{n,N,N} B_{n,N} \right)
\end{aligned}$$

Rewriting the above equalities in matrix form, we have

$$\begin{bmatrix} B_{n,0} \\ B_{n,1} \\ B_{n,2} \\ B_{n,3} \\ \vdots \\ B_{n,N} \end{bmatrix} = \begin{bmatrix} q_{n,0} \\ q_{n,1} \\ q_{n,2} \\ q_{n,3} \\ \vdots \\ q_{n,N} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{h}{2} K_{n,1,0} & \frac{h}{2} K_{n,1,1} & 0 & 0 & 0 & 0 & 0 \\ \frac{h}{2} K_{n,2,0} & h K_{n,2,1} & \frac{h}{2} K_{n,2,2} & 0 & 0 & 0 & 0 \\ \frac{h}{2} K_{n,3,0} & h K_{n,3,1} & h K_{n,3,2} & \frac{h}{2} K_{n,3,3} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{h}{2} K_{n,N,0} & h K_{n,N,1} & h K_{n,N,2} & h K_{n,N,3} & \cdots & h K_{n,N,N-1} & \frac{h}{2} K_{n,N,N} \end{bmatrix} \cdot \begin{bmatrix} B_{n,0} \\ B_{n,1} \\ B_{n,2} \\ B_{n,3} \\ \vdots \\ B_{n,N} \end{bmatrix}. \quad (\text{C.3})$$

Thence,

$$\mathbf{M} \cdot \mathbf{B}_n = \mathbf{q}_n, \quad (\text{C.4})$$

where $\mathbf{B}_n = \begin{bmatrix} B_{n,0} & \cdots & B_{n,N} \end{bmatrix}^T$, $\mathbf{q}_n = \begin{bmatrix} q_{n,0} & \cdots & q_{n,N} \end{bmatrix}^T$, and $\mathbf{M}$ is a lower triangular matrix such that

$$M_{i,i} = 1 - \frac{h}{2} \cdot K_{n,i,i}, \quad i = 1, \dots, N,$$

$$M_{i,j} = -h \cdot K_{n,i,j}, \quad 1 \leq j < i \leq N,$$

$$M_{i,0} = -\frac{h}{2} \cdot K_{n,i,0}, \quad i = 1, \dots, N,$$

$$M_{0,0} = 1.$$

Solving (C.4), we obtain $\mathbf{B}_n$, an approximation of $B_n(y)$ for $a_n < y < b_n$, $n = 1, 2$.

D. Proof of (III.16)

We have

$$E_\pi \left[1_{\{0 \leq X_{0+} < \zeta \leq X_{\delta-}, X_{0+} = y\}} \right] = \int_{[0, \zeta)} P(\zeta \leq X_{\delta-} \mid X_{0+} = y) \pi(dy) = \int_{[0, \zeta)} \bar{F}_{\alpha, \delta, \beta}(\zeta - y) \pi(dy), \quad (\text{D.1})$$

$$E_\pi \left[1_{\{\zeta \leq X_{0+} < L, X_{0+} = y\}} \right] = \int_{[\zeta, L)} \pi(dy), \quad (\text{D.2})$$

and

$$\begin{aligned}
E_\pi \left[\sum_{k=1}^{\infty} (k+1) \cdot 1_{\{0 \leq X_{0+} \leq X_{k \cdot \delta^-} < \zeta \leq X_{(k+1) \cdot \delta^-}, X_{0+}=y\}} \right] &= \sum_{k=1}^{\infty} (k+1) \int_{[0, \zeta)} P(X_{k \cdot \delta^-} < \zeta \leq X_{(k+1) \cdot \delta^-} \mid X_{0+}=y) \pi(dy) \\
&= \sum_{k=1}^{\infty} (k+1) \int_{[0, \zeta)} P(X_{k \cdot \delta^-} - X_{0+} < \zeta - y \leq X_{(k+1) \cdot \delta^-} - X_{0+}) \pi(dy) \\
&= \int_{[0, \zeta)} \left(\int_0^{\zeta-y} P(\zeta - y \leq X_{(k+1) \cdot \delta^-} - X_{0+} \mid X_{k \cdot \delta^-} - X_{0+} = u) \left(\sum_{k=1}^{\infty} (k+1) f_{\alpha k \delta, \beta}(u) \right) du \right) \pi(dy) \\
&= \int_{[0, \zeta)} \left(\int_0^{\zeta-y} P(\zeta - y - u \leq X_{(k+1) \cdot \delta^-} - X_{k \cdot \delta^-}) \left(\sum_{k=1}^{\infty} (k+1) f_{\alpha k \delta, \beta}(u) \right) du \right) \pi(dy) \\
&= \int_{[0, \zeta)} \left(\int_0^{\zeta-y} \bar{F}_{\alpha \cdot \delta, \beta}(\zeta - y - u) \left(\sum_{k=1}^{\infty} (k+1) f_{\alpha k \delta, \beta}(u) \right) du \right) \pi(dy) \\
&= \int_{[0, \zeta)} \left(\int_y^{\zeta} \bar{F}_{\alpha \cdot \delta, \beta}(\zeta - w) \left(\sum_{k=1}^{\infty} (k+1) f_{\alpha k \delta, \beta}(w - y) \right) dw \right) \pi(dy). \quad (D.3)
\end{aligned}$$

Thus, from (III.15), $E_\pi [N_i([0^+, R_1^+))]$ is computed by

$$\begin{aligned}
E_\pi [N_i([0^+, R_1^+)))] &= \int_{[0, \zeta)} \bar{F}_{\alpha \cdot \delta, \beta}(\zeta - y) \pi(dy) + \int_{[\zeta, L)} \pi(dy) + \\
&\quad \int_{[0, \zeta)} \left(\int_y^{\zeta} \bar{F}_{\alpha \cdot \delta, \beta}(\zeta - w) \left(\sum_{k=1}^{\infty} (k+1) f_{\alpha k \delta, \beta}(w - y) \right) dw \right) \pi(dy). \quad (D.4)
\end{aligned}$$

Substituting $\pi(dy)$ in (D.4) by its expression (III.11), we obtain

$$\begin{aligned}
E_\pi [N_i([0^+, R_1^+)))] &= a \cdot \bar{F}_{\alpha \cdot \delta, \beta}(\zeta) + (1-a) \cdot \int_0^{\zeta} \bar{F}_{\alpha \cdot \delta, \beta}(\zeta - y) b_1(y) dy + (1-a) \cdot \int_{\zeta}^L b_2(y) dy + a \cdot \int_0^{\zeta} \bar{F}_{\alpha \cdot \delta, \beta}(\zeta - w) \\
&\quad \times \left(\sum_{k=1}^{\infty} (k+1) f_{\alpha k \delta, \beta}(w) \right) dw + (1-a) \cdot \int_0^{\zeta} \left(\int_y^{\zeta} \bar{F}_{\alpha \cdot \delta, \beta}(\zeta - w) \left(\sum_{k=1}^{\infty} (k+1) f_{\alpha k \delta, \beta}(w - y) \right) dw \right) b_1(y) dy. \quad (D.5)
\end{aligned}$$

E. Proof of (III.18)

We have

$$E_\pi \left[1_{\{0 \leq X_{0+} < L \leq X_{\delta^-}, X_{0+}=y\}} \right] = \int_{[0, L)} P(L \leq X_{\delta^-} \mid X_{0+}=y) \pi(dy) = \int_{[0, L)} \bar{F}_{\alpha \cdot \delta, \beta}(L - y) \pi(dy), \quad (E.1)$$

and

$$\begin{aligned}
E_\pi \left[\sum_{k=1}^{\infty} 1_{\{0 \leq X_{0+} \leq X_{k \cdot \delta^-} < \zeta < L \leq X_{(k+1) \cdot \delta^-}, X_{0+}=y\}} \right] &= \sum_{k=1}^{\infty} \int_{[0, \zeta)} P(X_{k \cdot \delta^-} < \zeta < L \leq X_{(k+1) \cdot \delta^-} \mid X_{0+}=y) \pi(dy) \\
&= \sum_{k=1}^{\infty} \int_{[0, \zeta)} P(X_{k \cdot \delta^-} - X_{0+} < \zeta - y < L - y \leq X_{(k+1) \cdot \delta^-} - X_{0+}) \pi(dy) \\
&= \int_{[0, \zeta)} \left(\int_0^{\zeta-y} P(L - y \leq X_{(k+1) \cdot \delta^-} - X_{0+} \mid X_{k \cdot \delta^-} - X_{0+} = u) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(u) \right) du \right) \pi(dy) \\
&= \int_{[0, \zeta)} \left(\int_0^{\zeta-y} P(L - y - u \leq X_{(k+1) \cdot \delta^-} - X_{k \cdot \delta^-}) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(u) \right) du \right) \pi(dy) \\
&= \int_{[0, \zeta)} \left(\int_0^{\zeta-y} \bar{F}_{\alpha \cdot \delta, \beta}(L - y - u) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(u) \right) du \right) \pi(dy) \\
&= \int_{[0, \zeta)} \left(\int_y^{\zeta} \bar{F}_{\alpha \cdot \delta, \beta}(L - w) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w - y) \right) dw \right) \pi(dy). \quad (E.2)
\end{aligned}$$

From (III.17), the expected number of CR $N_c([0^+, R_1^+])$ over the semi-renewal cycle $[0^+, R_1^+]$ can be given by

$$E_\pi [N_c([0^+, R_1^+])] = \int_{[0, L)} \bar{F}_{\alpha \cdot \delta, \beta}(L - y) \pi(dy) + \int_{[0, \zeta)} \left(\int_y^\zeta \bar{F}_{\alpha \cdot \delta, \beta}(L - w) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w - y) \right) dw \right) \pi(dy). \quad (\text{E.3})$$

Substituting $\pi(dy)$ in (E.3) by its expression (III.11), we obtain

$$\begin{aligned} E_\pi [N_c([0^+, R_1^+])] &= a \cdot \bar{F}_{\alpha \cdot \delta, \beta}(L) + (1 - a) \cdot \int_0^\zeta \bar{F}_{\alpha \cdot \delta, \beta}(L - y) b_1(y) dy \\ &\quad + (1 - a) \cdot \int_\zeta^L \bar{F}_{\alpha \cdot \delta, \beta}(L - y) b_2(y) dy + a \cdot \int_0^\zeta \bar{F}_{\alpha \cdot \delta, \beta}(L - w) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w) \right) dw \\ &\quad + (1 - a) \cdot \int_0^\zeta \left(\int_y^\zeta \bar{F}_{\alpha \cdot \delta, \beta}(L - w) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w - y) \right) dw \right) b_1(y) dy. \quad (\text{E.4}) \end{aligned}$$

F. Proof of (III.22)

We have

$$\begin{aligned} E_\pi \left[\int_0^\delta 1_{\{0 \leq X_{0^+} < L < X_t, X_{0^+} = y\}} dt \right] &= \int_0^\delta \left(\int_{[0, L)} P(L \leq X_t \mid X_{0^+} = y) \pi(dy) \right) dt = \int_0^\delta \left(\int_{[0, L)} \bar{F}_{\alpha \cdot t, \beta}(L - y) \pi(dy) \right) dt, \quad (\text{F.1}) \end{aligned}$$

and

$$\begin{aligned} E_\pi \left[\sum_{k=1}^{\infty} \int_{k\delta}^{(k+1)\delta} 1_{\{0 \leq X_{0^+} \leq X_{k \cdot \delta^-} < \zeta < L \leq X_t, X_{0^+} = y\}} \right] &= \sum_{k=1}^{\infty} \int_{k\delta}^{(k+1)\delta} \left(\int_{[0, \zeta)} P(X_{k \cdot \delta^-} < \zeta < L \leq X_t \mid X_{0^+} = y) \pi(dy) \right) dt \\ &= \sum_{k=1}^{\infty} \int_{k\delta}^{(k+1)\delta} \left(\int_{[0, \zeta)} P(X_{k \cdot \delta^-} - X_{0^+} < \zeta - y < L - y \leq X_t - X_{0^+}) \pi(dy) \right) dt \\ &= \sum_{k=1}^{\infty} \int_{k\delta}^{(k+1)\delta} \left(\int_{[0, \zeta)} \left(\int_0^{\zeta - y} P(L - y \leq X_t - X_{0^+} \mid X_{k \cdot \delta^-} - X_{0^+} = u) f_{\alpha k \delta, \beta}(u) du \right) \pi(dy) \right) dt \\ &= \sum_{k=1}^{\infty} \int_{k\delta}^{(k+1)\delta} \left(\int_{[0, \zeta)} \left(\int_0^{\zeta - y} P(L - y - u \leq X_t - X_{k \cdot \delta^-}) f_{\alpha k \delta, \beta}(u) du \right) \pi(dy) \right) dt \\ &= \sum_{k=1}^{\infty} \int_{k\delta}^{(k+1)\delta} \left(\int_{[0, \zeta)} \left(\int_0^{\zeta - y} \bar{F}_{\alpha \cdot (t - k\delta), \beta}(L - y - u) f_{\alpha k \delta, \beta}(u) du \right) \pi(dy) \right) dt \\ &= \int_0^\delta \left(\int_{[0, \zeta)} \left(\int_y^\zeta \bar{F}_{\alpha \cdot l, \beta}(L - w) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w - y) \right) dw \right) \pi(dy) \right) dl. \quad (\text{F.2}) \end{aligned}$$

From (III.21), the expected length of the system downtime $W([0^+, R_1^+])$ over the semi-renewal cycle $[0^+, R_1^+]$ can be given by

$$\begin{aligned} E_\pi [W([0^+, R_1^+])] &= \int_0^\delta \left(\int_{[0, L)} \bar{F}_{\alpha \cdot t, \beta}(L - y) \pi(dy) + \int_{[0, \zeta)} \left(\int_y^\zeta \bar{F}_{\alpha \cdot t, \beta}(L - w) \left(\sum_{k=1}^{\infty} f_{\alpha k \delta, \beta}(w - y) \right) dw \right) \pi(dy) \right) dt. \quad (\text{F.3}) \end{aligned}$$

Substituting $\pi(dy)$ in (F.3) by its expression (III.11), we obtain

$$\begin{aligned} E_\pi [W([0^+, R_1^+])] &= \int_0^\delta \left(a \cdot \bar{F}_{\alpha,t,\beta}(L) + (1-a) \cdot \int_0^\zeta \bar{F}_{\alpha,t,\beta}(L-y) b_1(y) dy \right. \\ &\quad \left. + (1-a) \cdot \int_\zeta^L \bar{F}_{\alpha,t,\beta}(L-y) b_2(y) dy + a \cdot \int_0^\zeta \bar{F}_{\alpha,t,\beta}(L-w) \left(\sum_{k=1}^\infty f_{\alpha k \delta, \beta}(w) \right) dw \right. \\ &\quad \left. + (1-a) \cdot \int_0^\zeta \left(\int_y^\zeta \bar{F}_{\alpha,t,\beta}(L-w) \left(\sum_{k=1}^\infty f_{\alpha k \delta, \beta}(w-y) \right) dw \right) b_1(y) dy \right) dt. \quad (F.4) \end{aligned}$$

G. Expectation Quantities Evaluation By Monte Carlo Simulation Method

We propose here a simple way to deduce the value of $E_\pi [N_i([0^+, R_1^+])]$, $E_\pi [N_c([0^+, R_1^+])]$ and $E_\pi [W([0^+, R_1^+])]$ from the Monte Carlo simulation method.

1) *Expected Number of Inspections Over The First Markov Renewal Cycle:* Using the semi-regenerative property of the maintained system state process, we have

$$\lim_{t \rightarrow \infty} \frac{N_r(t) + N_c(t)}{t} = \frac{E_\pi [N_r([0^+, R_1^+])] + E_\pi [N_c([0^+, R_1^+])]}{E_\pi [\Delta R_1]} = \frac{1}{\delta \cdot E_\pi [N_i([0^+, R_1^+])]}.$$

Thus

$$E_\pi [N_i([0^+, R_1^+])] = \frac{1}{\delta} \cdot \lim_{t \rightarrow \infty} \frac{t}{N_r(t) + N_c(t)} \simeq \frac{1}{\delta} \cdot \frac{D}{N_r(D) + N_c(D)}, \quad (G.1)$$

where D is a large enough time duration.

2) *Expected Number of Corrective Replacement Over The First Markov Renewal Cycle:* Using once again the semi-regenerative property of the maintained system state process, we obtain

$$\lim_{t \rightarrow \infty} \frac{N_c(t)}{t} = \frac{E_\pi [N_c([0^+, R_1^+])]}{E_\pi [\Delta R_1]}.$$

Thus

$$E_\pi [N_c([0^+, R_1^+])] = E_\pi [\Delta R_1] \cdot \lim_{t \rightarrow \infty} \frac{N_c(t)}{t} = \lim_{t \rightarrow \infty} \frac{t}{N_r(t) + N_c(t)} \cdot \frac{N_c(t)}{t} \simeq \frac{N_c(D)}{N_r(D) + N_c(D)}. \quad (G.2)$$

3) *Expected Length of System Downtime Over The First Markov Renewal Cycle:* Similarly, we have

$$\lim_{t \rightarrow \infty} \frac{W(t)}{t} = \frac{E_\pi [W([0^+, R_1^+])]}{E_\pi [\Delta R_1]}.$$

Thus

$$E_\pi [W([0^+, R_1^+])] = E_\pi [\Delta R_1] \cdot \lim_{t \rightarrow \infty} \frac{W(t)}{t} = \lim_{t \rightarrow \infty} \frac{t}{N_r(t) + N_c(t)} \cdot \frac{W(t)}{t} \simeq \frac{W(D)}{N_r(D) + N_c(D)}. \quad (G.3)$$

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