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Chapter 5

USING SYMBOLIC CALCULATORS TO STUDY MATHEMATICS.

The case of tasks and techniques.

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Abstract: This chapter will consider in more depth the possible contribution of technology -especially CAS- to the study of mathematical domains Using a theoretical approach to treat examples of classroom activities, we will show how a didactical reflection can help to understand this contribution. A variety of new techniques will be presented and related to paper-and-pencil techniques. Examining the pragmatic and epistemic value of both types of technique will help to make sense of classroom situations. It will also help to clarify the situation of teachers wanting to integrate new tools. Consideration of other approaches will show that educators emphasize the use of computer algebra to promote 'conceptual' mathematics. Nevertheless, they cannot ignore 'instrumented' techniques when considering the real potentialities of new tools and the conditions for their integration.

Key words: study, tasks, techniques, conceptualization.

1. INTRODUCTION

Using new tools, students can now easily perform numerical and symbolic calculation that would be very painstaking by hand. As we saw in chapter 3, transposing experimental approaches from mathematical sciences into teaching seems to be a realistic and stimulating prospect, but the question of the contribution that experimental approaches inspired by mathematical sciences might bring to students' conceptualizations remains open. We concluded that this question would require the addressing as a whole of the study of a mathematical domain and the way in which it is changed by new approaches that tools make possible.

Box 5-1.

The anthropological approach

(Artigue 2002)

The anthropological approach (Chevallard 1999) shares with socio-cultural approaches in the educational field (Sierpinska and Lerman 1996) a vision of mathematics as the product of a human activity. Mathematical productions and thinking modes are thus seen as dependent on the social and cultural contexts where they develop. As a consequence, mathematical objects are not absolute objects, but are entities which arise from the practices of given institutions. The word ‘institution’ has to be understood in this theory in a very broad sense: the family is an institution for instance. Any social or cultural practice takes place within an institution. Didactic institutions are those devoted to the intentional apprenticeship of specific contents of knowledge. As regards the objects of knowledge which it takes in hand, any didactic institution develops specific practices, and this results in specific norms and visions as regards the meaning of knowing or understanding such and such an object. Thus to analyze the life of a mathematical object in an institution, to understand the meaning in the institution of ‘knowing/understanding this object’, one needs to identify and analyze the practices which bring it into play.

These practices, or ‘praxeologies’, as they are called in Chevallard’s approach, are described by four components: a type of task in which the object is embedded; the techniques used to solve this type of task; the ‘technology’, that is to say the discourse which is used in order to both explain and justify these techniques; and the ‘theory’ which provides a structural basis for the technological discourse itself and can be seen as a technology of the technology. Since we have already assigned a meaning to the word ‘technology’ in this book, so as to avoid misunderstanding, in the following we combine Chevallard’s ‘technological’ and ‘theoretical’ components into a single ‘theoretical’ component. The word ‘theoretical’ has thus to be given a wider interpretation than is usual in the anthropological approach. Note also that the term ‘technique’ has to be given a wider meaning than is usual in educational discourse. A technique is a manner of solving a task and, as soon as one goes beyond the body of routine tasks for a given institution, each technique is a complex assembly of reasoning and routine work. We would like to stress that techniques are most often perceived and evaluated in terms of *pragmatic value*, that is to say, by focusing on their productive potential (efficiency, cost, field of validity). But they have also an *epistemic value*, as they contribute to the understanding of the objects they involve, and thus techniques are a source of questions about mathematical knowledge.

For obvious reasons of efficiency, the advance of knowledge in any institution requires the routinization of some techniques. This routinization is accompanied by a weakening of the associated theoretical discourse and by a ‘naturalization’ or ‘internalization’ of associated knowledge which tends to become transparent, to be considered as ‘natural’. A technique which has become routine in an institution tends thus to become ‘de-mathematized’ for the members of that institution. It is important to be aware of this naturalization process, because through this process techniques lose their mathematical ‘nobility’ and become simple acts. Thus, in mathematical work, what is finally considered as mathematical is reduced to being the tip of the iceberg of actual mathematical activity, and this dramatic reduction strongly influences our vision of mathematics and mathematics learning and the values attached to these.

The anthropological approach opens up a complex world whose ‘economy’ obeys subtle laws that play an essential role in the actual production of mathematics knowledge as well as in the learning of mathematics. A traditional constructivist approach does not help us to perceive this complexity, much less to study it. Nevertheless, this study is essential because, as pointed out by Lagrange (2000), it is through practices where technical work plays a decisive role that one constructs the mathematical objects and the connections between these that are part of conceptual understanding.

First we have to define what we mean by the study of a domain. This notion comes from Chevallard (1999, Box 5-1). To study a domain is to do mathematical work on this domain for educational purposes. In education as in professional research, working on a mathematical domain is trying to solve a set of problems through using and creating concepts. An important issue is how concepts are produced. As far as we can say in general, the work of a researcher is to structure a domain so as to make good questions appear. Good questions are not just problems, but specific questions that can be addressed in a mathematically appealing way. Concepts appear when structure becomes visible. Their formulation is the product of further structuring work.

In a teaching and learning context, we consider that working in a mathematical domain is done at three structural levels. The first level is that of ‘tasks’. Here, tasks are taken not just to be individual problems but rather as more general structures for problems. For instance, consider the domain of real functions. Problems can be expressed enactively from ‘real life’ situations. A task like ‘find the intervals of growth of a given function’ constitutes a common reference for some problems but not for others; likewise another relevant reference task is ‘find the zeros…’.

‘Techniques’ are the second structural level. ‘Technique’ has to be taken in the general sense of ‘a way of doing tasks’. Techniques help to distinguish and reorganize tasks. For instance different techniques exist for the task ‘find the intervals of growth of a given function’ depending on what is known about the function. If the function is differentiable the task can then be related to the task ‘find the zeros’ of another function. In other cases, a search based on a more direct algebraic treatment can be more effective…

The third level is that of ‘theories’. While the first two levels are related to action, this level is related to assertion. At this level, the consistency and effectiveness of techniques are discussed. Mathematical properties, concepts and a specific language appear.

In some respects, this three level structure defining the study of a domain has to be taken as a postulate. On one hand, doing mathematics in a domain necessarily involves structuring problems in terms of concepts. On the other hand, why choose to focus on tasks and techniques as intermediate structures? The reason is that, as we saw in chapter 3, the potentialities of technologies –and especially of CAS– are expressed in terms of the expanded possibilities of action in solving problems. Thus, if access to concepts is seen as depending on the possibilities of investigating problems, technology should automatically enhance this access. We have stressed that things are not so simple. Observing several windows does not necessarily stimulate multi-representation thinking. Easily obtained symbolic results do not automatically provoke real inductive activity. The postulate we make in this chapter is that taking the above structural levels of tasks and techniques into consideration can account for these difficulties and help to think better about the support that technology can bring. Such a perspective has its origin in our surprise that consideration of tasks and techniques is often omitted in technological innovations, whereas they have an important place in 'real life' teaching and learning. Like all postulates, it will be justified inductively by its productivity.

2. THE IMPACT OF TECHNOLOGY ON STUDY

Our interest in techniques led us towards the anthropological approach developed by Chevallard (1999). The main elements are presented in Box 5-1. From these, the impact of computers on teaching and learning can be thought of at the level of techniques: traditional paper-and-pencil techniques are challenged by 'push button' techniques while, as we shall see, the use of technology requires *new techniques* dependent on the tool. The *pragmatic* and *epistemic* value of traditional techniques (definition in Box 5-1) have to be reconsidered and new techniques have to be examined for a possible epistemic contribution.

To facilitate understanding, let us take the example of a small praxeology and examine the impact of a tool on techniques. Chevallard (ibid. p. 243) considers the domain of expressions like $\frac{a+b\sqrt{2}}{c+d\sqrt{2}}$, *a*, *b*, *c*, and *d* integers. The study of this domain can be seen as a praxeology whose central task is the reduction of such expressions into $\alpha+\beta\sqrt{2}$, α and β rationals. The technique to accomplish this task is to multiply numerator and denominator by a suitable expression to obtain an integer denominator. This (tedious) technique provides for a *pragmatic* canonical writing of expressions from the domain, helping for instance to recognize that quotients like $\frac{\sqrt{2}-2}{3\sqrt{2}-4}$ and $\frac{\sqrt{2}-1}{2\sqrt{2}-3}$ are equal. Since performing the technique includes several elementary actions, each action implying an algebraic analysis of the expression -especially before it has been routinized-, it can play an *epistemic* role towards developing knowledge of algebraic properties of quotients and radicals. At a theoretical level, it is a basis for the field structure of the algebraic extension Q [$\sqrt{2}$] and provides an algorithm to transform expressions into the canonical form.

In contrast a symbolic calculator accomplished the reduction in just one operation. Using a paper-and-pencil technique, a human being will necessarily limit the number of expressions (s)he will try, and focus on the underlying *algebraic property of radicals*. The use of a symbolic calculator for this task makes it possible to do more examples and orients the activity towards *pattern discovery* –for instance recognizing that every quotient can be expressed as the sum of a rational and a rational multiplied by $\sqrt{2}$, that the expression is rational whenever *ad=bc* …- and *generalization* –building a praxeology for Q[$\sqrt{k}$] or Q[$\sqrt[3]{2}$]. Clearly, the paper-and-pencil technique is linked to knowledge of elementary algebraic properties while the symbolic calculator technique opens up stable structures more directly, while hiding properties explaining the stability.

3. ‘PUSH BUTTON’ TECHNIQUES AND THEIR EFFECT ON CONCEPTUALISATION

CAS was created to ease the simplification of most common symbolic expressions. Corresponding tasks can be performed without great reflection. Thus, for students, ‘push button’ techniques tend to predominate over ordinary more painstaking techniques. For instance, after overcoming syntactical difficulties, an 11th grader will get limits even as simple as $\lim_{x\to\infty}\frac{1}{x+1}$ by using the symbolic calculator *limit* command with less pain than by reasoning. While by reasoning, s(he) would have to think of a graphic asymptotical representation of the function $x \to \frac{1}{x+1}$ or of a bounding by $x \to \frac{1}{x}$, with a calculator (s)he only has to enter *limit(1/(x+1),x,∞)*. Students adopt ‘push button’ techniques like this because of their simplicity and efficiency and there is a chance that they will link a concept too closely with the corresponding technique (Monaghan & al. 1994).

This is an example that, with new tools, painstaking paper-and-pencil techniques retain little pragmatic value because they are challenged by ‘push button’ techniques. Routinization is no longer a necessity and thus their epistemic value could become more visible. However, paper-and-pencil techniques tend to become obsolete because of the ease of using CAS commands. This obsolescence is a problem because traditional techniques can no longer play their role in conceptualization and ‘push button’ techniques cannot take over this role directly.

Mathematics education has thus to reconsider the study of a domain, taking the obsolescence of traditional techniques into account, and to conceive new techniques as components of new praxeologies for this domain. Thinking of new techniques linked to the use of computer tools and of their possible epistemic value is not easy because mathematical culture is implicitly linked with paper-and-pencil techniques and is not accustomed to the idea that other tools can support conceptualization. However, this is indeed possible, as the next section will show.

4. A CAS TECHNIQUE AND ITS EPISTEMIC VALUE

The situation described in Box 5-2 illustrates how a technique linked to the use of CAS can be more than just ‘pushing a button’ and how it contributes to students’ problem solving activity *and* mathematical conceptualization. Two teachers trialled three versions of the same situation.

The first version was deceptive as students worked with paper-and-pencil and could not go far in the value of n and thus in their conjectures. In the second version the teachers tried to use the CAS DERIVE to liberate students “from the technical aspects of computing by hand” while encouraging them “to keep sight of the main goal”. While students actually engaged in experimental activity and learned about algebraic facts (degree of factors…), the situation did not produce real conjectures in spite of the teachers’ expectation that students could find motivating conjectures by observing CAS factorizations with enough detachment so that they would not miss general factorizations.

Students could not actually distinguish between DERIVE’s and ‘general’ factorizations because they could not grasp the following idea: for a given n, several factorizations may exist, but only some of them are ‘general’ or true for a large set of values of *n*.

This situation is remarkable because students will have learned much about algebra if they can say, “Yes, for given polynomials DERIVE gave us factored forms that are not the general factorization, but the general factorization is still valid because we can find it by collecting and expanding parts in the factored forms”. An important point to emphasize is that students who are able to make this statement know what it means to collect and expand parts of an expression. Mathematicians may think that this *technique* is obvious because they recognize complete and incomplete factorizations and understand that CAS provides the means to pass from one form to another.

To collect and develop several factors in a factorization is not so easy a manipulation for students: one must understand software-specific copy and paste functionalities and link them with an understanding of the structure of a factorized expression. In the situation reported above, this ‘DERIVE technique’ was missing, and this arose both from an insufficient knowledge of DERIVE *and* from a lack of understanding of the concept of factorization. Classroom elaboration of this technique is a condition for enabling experimental activity on the part of students *and* for giving this activity a mathematical dimension.

This observation illustrates what we said in chapter 3. Experimenting ‘like professional mathematicians’ with the help of new tools is not so easily transposed into education. When not enough emphasis is put on techniques specific to the tool, a potentially rich situation may fail to bring students to conceptualizations. We observed that teachers using CAS were often reluctant to give time to these techniques. Since working regularly in a computer room presented difficulties, most of the teachers taught only a few sessions with DERIVE. In this context, they saw little pragmatic use for DERIVE techniques and tried -often unsuccessfully- to focus on conceptual issues. In contrast, experienced teachers and researchers successfully integrated DERIVE techniques into the classroom.

In a third version of the situation, introducing students to techniques for manipulating factors in DERIVE and making it possible for them to practice at home was more productive as the final report (Mounier & Aldon ibid. p. 59) illustrates: for instance, students found and proved a non-trivial factorization where *n* is a power of 2. This proof by induction uses the expansion of a part of a factorization, a finding obviously linked to the technique. According to the authors, students did learn DERIVE as a new tool *and* changed their image of the concept of factorization. These teachers recognized the need to build techniques for using DERIVE and the epistemic role of these techniques in understanding algebra.

Box 5-2.

The factorization of $x^n - 1$
(Mounier & Aldon 1996)

11th grade students (scientific stream) were asked to conjecture and prove "general" -true for every *n*- factorizations of these polynomials by observing examples for some values of *n*. Three successive versions of this situation were developed

Version 1: solving this problem by hand in a single session
Students found easily that x-1 is a factor, then they used polynomial division to obtain a factorization for two or three values of *n* and were able to generalize into $(x-1)\left(x^{n-1}+x^{n-2}+\ldots+x+1\right)$. Polynomial division was a tedious manipulation and after that, students did not look for other factorizations.

Version 2: with CAS in a single session
Students had to observe a set of outputs from DERIVE's *Factor Rational* command. A difficulty is that this command gives a most factored form while the general factorizations expected by the teachers are not complete for every *n*. For instance, the two-factor decomposition above is obtained only for prime values of *n*.
This is how students typically behaved. They used the *Factor Rational* command for n=2 and 3 and conjectured the above two-factor general factorization. Factoring for n=4 they thought that even *n* are not regular and trying n = 5, 6, 7 provided confirmation. At this point, they conjectured that there were two separate general factorizations, the above for odds and a three-factor one for evens. Wanting a confirmation for n=9, they got a three-factor decomposition $(x-1)(x^2+x+1)(x^6+x^3+1)$. The conjecture was then rejected and students tried a variety of new conjectures but without success because they always found anomalous values of *n*. They did not go farther because a theory of cyclotomic polynomials that would explain DERIVE's factorizations was beyond their reach.

Version 3: a long-term problem
Students had access to Derive on laptops in classroom and for personal work. The teachers assigned the factorization of $x^n - 1$ as "a long term problem." A first session introduced the problem and students were initiated to techniques for manipulating factors in DERIVE. Then students practiced at home and found conjectures and proofs that they reported in classroom discussions. For instance they recognized that the above factorization is true for all *n* and they proved factorizations like
$x^{2^n}-1=(x-1)(x+1)(x^2+1)(x^{2^2}+1)\ldots(x^{2^{n-1}}+1)$.

A provisional conclusion is that there is a great variety of new techniques, including ‘push button’ techniques and techniques needed to manage expressions. Obviously, in a paper-and-pencil context, one cannot bypass such techniques because of their pragmatic utility and one can easily overlook the epistemic value of such techniques. That is why recognizing new techniques and their epistemic value is not obvious for mathematics educators. In our analysis, instead of trying to reduce their importance or to bypass them, teaching has to considerer their pragmatic and epistemic value and their evolution during the mathematical work in a domain, in order to understand how the use of technology can support conceptualization.

In the next sections we will have a closer look at the variety of new techniques, emphasizing possible specificities, and we will consider how this approach to techniques helps to look at the teacher’s role in the classroom use of tools.

5. LINKING CAS AIDED PATTERN DISCOVERY AND PAPER-AND-PENCIL TECHNIQUES

Since CAS first appeared in classrooms, there has been a recurrent debate about what should happen to paper-and-pencil techniques. Authors who see these techniques simply as skills tend to think that, if there is some necessity for students to learn them, this learning should be ‘resequenced’ as late as possible in order to avoid interference with conceptualization (see section 9). Other authors refer to paper-and-pencil proficiencies as sometimes valuable and meaningful. They recognize that technology changes the scene and try to identify ‘lists of basic skills that mathematics educators would agree are necessary for students to know how to perform by hand, even in a technological environment.

Goldenberg (2003 p.15) wonders whether “algebra is dead” now that “CAS do, with no effort, what we previously thought we wanted the students to do” (ibid. p.13). He reminds us that the role of algebra is not just to solve practical problems. Algebra can play a role in “opening up various black boxes, including the ones we called patterns” and thus “some algebra skills that are no longer needed for *finding* answers still remain essential for *understanding* answers…” (ibid. p. 17) He proposes the example of the expansion of $(x - 1)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$ ‘collapsing’ to produce $(x^8 - 1)$. CAS gives the answer, but does not give insight into the process involved (ibid. p. 29).

Box 5-3.

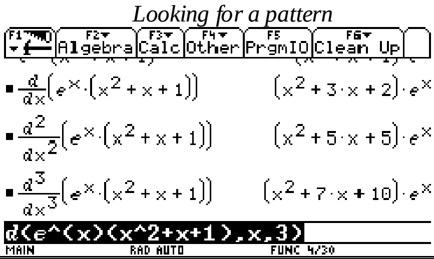
A challenge: Find the n^{th} order derivative of $(x^2 + x + 1)e^x$.

(Trouche & al. 1998)

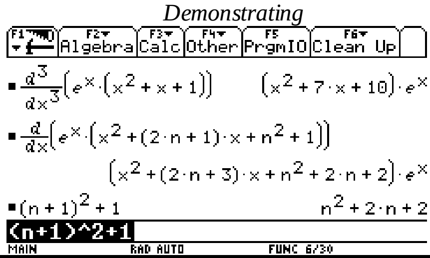
This situation comes from a booklet made up from reports by 12th grade students on their solutions to a number of ‘challenges’. Here, the challenge was: for every positive integer n find the n^{th} order derivative of $(x^2 + x + 1)e^x$. Two students presented their work.

Their first solution appears on the two TI-92 screens below.

Looking for a pattern



Demonstrating



Then the students wrote:

"Using the TI-92, we discovered a pattern and proved it. We had then to look again at this exercise. Actually, we searched for the derivative of a product of two functions u and v, with $u(x) = e^x$ and $v(x) = x^2 + x + 1$. Every derivative of u is u, the first derivative of v is $v'(x) = 2x + 1$, the second is $v''(x) = 2$ and the other derivatives of v are zero".

From this, they calculated the first, second and third derivative of uv, then they referred to the ‘Leibniz formula’ and found $(uv)^{(n)} = uv + nuv' + \frac{n(n-1)}{2}uv''$. From this, they obtained again the expression for the n^{th} order derivative of $(x^2 + x + 1)e^x$.

We can think of ‘discovering patterns’ and ‘getting insight into patterns’ as two activities, one with the use of a tool and the other with paper-and-pencil. The associated techniques make possible different epistemic contribution to the learning of algebra as the example in Box 5-3 will show.

This example deals with differentiation. The algebraic paper-and-pencil techniques for differentiation are developed in secondary mathematics education mainly for their pragmatic utility and they tend to be seen as meaningless skills. Their epistemic contribution to the understanding of algebraic aspects of calculus is nevertheless important. For instance one cannot understand why CAS simplifies the antiderivative of $x^n e^{x^2}$ only for odd numbers n, without some knowledge of the differentiation of products and chain expressions.

We noted above that the pragmatic value of paper-and-pencil techniques is challenged by push button techniques, and that putting their epistemic value to the fore is not obvious. The situation of Box 5-3 can help to make sense of techniques for differentiation of products. The stability of sets of expressions, like the set of products of the exponential with quadratic polynomials is a consequence of the algebraic properties giving these techniques an epistemic value. The CAS technique of pattern discovery helps to conjecture and prove this stability but it hides the underpinning algebraic properties.

Students challenged to ‘explain’ the stability had to use the algebraic techniques in a different way as compared to the usual paper-and-pencil differentiation. They produced a second non-CAS solution based on properties of the differentiation of the exponential and quadratic expressions and on the Leibniz formula generalizing the product differentiation to the n^{th} order derivative.

The interest of this situation is the following: a CAS technique of pattern discovery helps to find and prove a property but students recognize that this solution ‘tells only part of the story’. Actually, pattern discovery gives the property a meaning at a local level and algebraic techniques are a link with general calculus objects (polynomial, exponential, derivatives…) One can then expect from this interrelation of techniques a more general and reflective understanding of algebraic techniques than in the usual paper-and-pencil exercises on differentiation.

More generally, we cannot envisage students doing mathematics only by using CAS. Rather, we envisage a ‘CAS assisted’ practice intertwining technology and paper-and-pencil. Thus we should think of the use of CAS as calling for an interrelation between new techniques and paper-and-pencil techniques. Goldenberg’s reflection and the above example help to illustrate that this interrelation can be mathematically productive as a specific epistemic contribution can be expected from each type of technique.

6. ACCESSING GENERALIZATION THROUGH SYMBOLIC TECHNIQUES

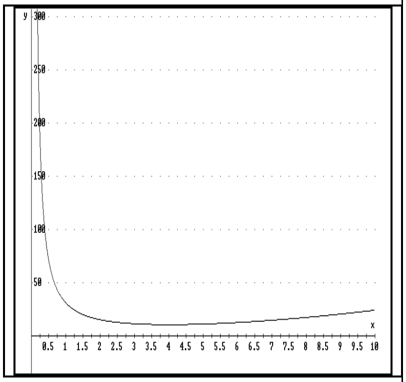
We will now present a situation which is interesting for similar reasons, but in reverse. In the n^{th} order derivative task, CAS techniques gave a local meaning to the solution and traditional paper-and-pencil work provided for a wider sense. In the tank problem situation (Boxes 5-4 and 5-5), access to generalization is provided by CAS. It is primarily a problem of optimization.

Box 5-4.

The concrete tank problem

(Artigue & Lagrange 1999)

A man wants to build a tank. The walls and base of the tank are to be made of concrete 20 cm thick, the base is to be a square, and the tank must contain 32 cubic meters. Let x be the horizontal dimension of the side of the inner square, and let h be the inner vertical of the tank, both measured in meters. What should be the values of x and h to use as little concrete as possible?



Solution

The function giving the quantity of concrete simplifies into

$$\frac{25x^4 + 20x^3 + 4x^2 + 3200x + 640}{125x^2}.$$

Generally, students do a graphical or numerical study of this function and observe that a minimum seems to appear near the value $x = 4$.

In a more mathematical approach to the problem, students can compute, with or without the TI-92, the derivative of this function.

They can find that it has two zeros one for $x = 4$, and another for $x = -0.4$, and generally, they do a numerical study of the sign of this derivative, finding that it is negative between 0 and 4 and positive above 4.

These problems are popular because students can work using precalculus concepts. The graphic and numerical facilities of calculators are excellent supports to encourage students to consider multiple approaches to these problems. In France, students learn the derivatives in eleventh grade and can then solve optimization problems symbolically. They have learned various techniques to tackle these problems and are able to approach calculus concepts when working out these techniques and reflecting on them (Lagrange 1999). By designing lessons for students using a TI-92, we thought that the availability of CAS could help amplify the tasks of optimizing and the associated techniques passing from a particular to a generalized configuration.

Studying a numerical case (Box 5-4) is interesting and the graphical, numerical, and even symbolic capabilities of the TI-92 can help but it has two significant limitations. It primarily ‘encourages’ the numerical or graphical approach to the problem. Students are not encouraged to use more powerful approaches, such as studying the sign of the derivative. The answer is not remarkable because it does not pave the way for new questions, such as “Is it a general result that there is a minimum and only one? What can you say of this minimum?” and so on. Limitations such as those described above do not exist in the generalized problem (Box 5-5).

Box 5-5.

The generalized concrete tank problem

The walls and base of the tank are to be made of concrete of the same thickness e, the base is to be a square, and the tank's inner volume must be V. Let x be the horizontal dimension of the side of the inner square, and let h be the inner vertical of the tank, both measured in meters.

The aim of the problem is to know whether a value of *x* and *h* exists that uses as little concrete as possible. In addition, we want to know how this value depends on *e* and *V*.

Solution

Students can adapt the function giving the quantity of concrete from the numerical case and get its derivative from the TI_92. Then the task is more difficult because they cannot conduct a graphical study of the function nor a numerical study of the derivative as they usually do, but study symbolically its sign by factoring the derivative. The TI-92, give this factorization $\frac{2e(x+2e)\left(x-(2V)^{1/3}\right)\left(x^2+(2V)^{1/3}x+(2V)^{2/3}\right)}{x^3}$ showing that the derivative has the same sign as $\left(x-(2V)^{1/3}\right)$ for positive *x*.

A minimum quantity of concrete is used at $x=(2V)^{1/3}$.The problem was trialled during the TI-92 experiment and chapter 9 will report on its place in the experiment and on students' work. We discuss here a possible student solution. Solving the numerical case should help students to understand the problem and try a numerical or graphical approach. The main difficulty will be to find an algebraic expression of the concrete volume with respect to *x*. With the generalized problem, students will meet the limitations of their graphic calculator techniques. They will be driven towards a symbolic technique that they learned before, but do not use in numerical cases when they see more sense in graphic and numerical approaches. They will be able to perform this technique only with the help of the TI-92 not just because the expression is 'big' but really because 11$^{\text{th}}$ grade students' knowledge of algebraic manipulation and differentiation is too weak to handle parameterized expressions.

We expected that students could answer the above questions in the following terms –"The minimum depends on *V*, but it does not depend on *e*"– and see that this issue of dependence and independence is more important than the value of *x* itself. The general problem is thus not simply a continuation of the numerical problem; it reveals the limitations of an existing technique and promotes a new, more general and symbolic technique. The objects that the general technique handles bring more sense to the problem. It is an example of CAS providing new techniques in interrelating with old techniques, opening a new understanding of optimization. These new techniques are possible because the use of CAS allows students to interpret calculations with symbolic constants, or parameters, as a continuation of the same calculation with explicit values. As another example, chapter 7 of this book offers a report on students' solutions of two-variable systems and a discussion of the role of parameters and of techniques -or schemas- in the CAS context.

The techniques presented in the two last sections are richer than simple 'push button' techniques. Their value follows naturally from the potentialities of computer tools. Easy computation helps pattern discovery. Recovery of a memorized numerical calculation allows students to rethink a technique so as to introduce generalization and make use of CAS for proof. Situations to work on these techniques can then be easily introduced into teaching. However, we are aware that not all obstacles will be suppressed. For instance it is relatively easy to introduce parameters to generalize a numerical situation, but students may have difficulties resulting from the plurality of roles that a letter can play, as we will see in the example of chapter 7.

7. TECHNIQUES FOR MANAGING EXPRESSIONS

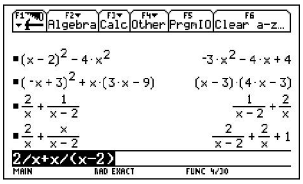
In this section, we return to the techniques for managing expressions; their importance and difficulty were shown above in the situation involving factorization. For instance, when students use a symbolic calculator on an everyday basis, these techniques are a necessity for effectiveness and reflectivity. As an example, students using the TI-92 algebraic window have to learn to consciously use the items of the Algebra menu (*Factor*, *Expand*, *ComDenom*), to decide whether expressions are equivalent, and to anticipate the output of a given transformation on a given expression. Since a CAS does not generally check conditions for the validity of a transformation involving for instance radicals or quotients[1], a student must learn what (s)he has to control and what (s)he can trust in CAS operation.

Box 5-6.

Working on techniques for finding equivalent expressions (11th grade)
(Artigue & Lagrange 1999)

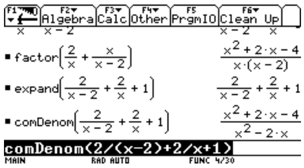
The following screens illustrate three tasks that we proposed for work on the equivalence of expressions. This work is a consolidation of high school algebra *and* an introduction to the TI-92 algebraic techniques.

In the first task, (screen A) students had to enter expressions and observe the TI-92 simplification. They then had to identify the mathematical treatment that the simplification carried out. We chose the expressions (see left side of screen A) to obtain a variety of simplifications (see right side of screen A). expanding, factoring, reordering, partial fractional expanding, and cancellation by a possibly null expression

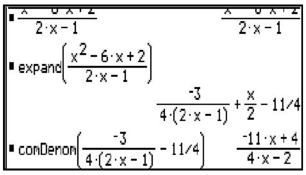


Screen A

In the second task, (screen B) students had to explore the effect of algebra menu items on the same expressions. In this task, they learned to identify the algebraic transformations and their TI-92 syntax. Students also learned how to copy an expression into the entry line. Therefore, they saved time and effort by not entering expressions several times.

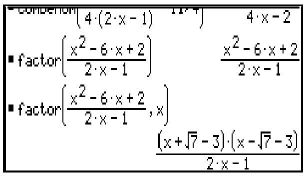


Screen B



Screen C

In the third task (screens C and D) students had to look for equivalence in four expressions *G, H, I J.* To encourage students to use various transformations, we offered expressions in different forms: reduced, more or less expanded, and factorized.



Screen D

With this aim, we developed a set of three tasks (Box 5-6) to help students acquire flexible use of the TI-92 commands for algebraic transformations useful in working on the equivalence of expressions[2]. The goal of the first task was to make students aware of the output of the TI-92 simplification and the many possible equivalent forms of an expression. In the second task, our goal was to link the understanding of general forms of expressions with the various items of the Algebra menu. In addition, we wanted students to remember the associated TI-92 syntax. They could also notice that the number of transformed expressions depends on the original expression itself. For instance, every transformation of $\frac{x-2}{x\times x-2x}$ gives $\frac{1}{x}$, whereas *Expand*, *ComDenom* and *Factor* have different effects on

$\frac{2}{x}+\frac{x}{x-2}$. Interesting discussion may follow this observation. In this task, students may also learn how to copy an expression into the entry line, saving time and effort by not entering expressions several times.

After completing this task, students could learn to use these transformations to decide whether two given expressions are equivalent. With CAS, the technique is as follows: enter the two expressions separated by the equals sign, and simplify. CAS generally returns *true* for equivalent expressions[3]. The epistemic value of this technique is poor because it provides no insight into the reasons for the equivalence. Although the technique is simple, it requires two expressions to be entered and thus can be tedious for complex expressions. We designed the third task to make the use of transformations which are more convenient than entering a test of equality.

The third task gives a rational expression *G* along with three other apparently equivalent rational expressions *H*, *I*, and *J*. For instance *G* was $\frac{x^2-6x+2}{2x-1}$, *H* was $\frac{-11x+4}{4x-2}+\frac{x}{2}$, *I* was $\frac{3}{4(2x-1)}-\frac{x}{2}+\frac{11}{4}$, and *J* was $\frac{\left(x+\sqrt{7}-3\right)\left(x-\sqrt{7}-3\right)}{2x-1}$. With these expressions, a good strategy is to expand *G*. It yields an expression opposite in sign to *I*. The user can then copy this expression into the entry line, split the first two terms, and apply the *ComDenom* command. *G* and *H* are thus proved equivalent. *J* is a factored form of *H*, but the mere Factor command does not transform *H* into *J* (Screen B and C). Because *J* is a “radical” factorization of *H*, a special form of the command must be applied. The solution described in Box 5-6 is one in several possible strategies. This topic can prompt many rich mathematical discussions in the classroom.

Pragmatically, these techniques are a necessity for proper use of CAS and their development in the classroom provides opportunities for mathematical discussion. Their epistemic role is clear, as they shed light on the structure and equivalence of algebraic expressions. However, this does not ensure that they will easily gain a place in ‘standard’ teaching, because of institutional obstacles: the institutional values of the school are defined relative to paper-and-pencil techniques and the dependence of these techniques on a tool is not recognized. We mentioned above that teachers are reluctant to devote time and discussion to techniques that they think too far from ‘official’ mathematics. They simply reflect the position of the institution: even techniques for managing the graphic window, that would be very useful for students and mathematically meaningful, have no official status in French secondary teaching[4].

8. THE OBSOLESCENCE OF PAPER-AND-PENCIL TECHNIQUES AND TEACHERS' WORK

Many teachers have not yet really considered classroom use of technology[5]. It is an indicator of the difficulties of this endeavor confirmed by the observation of teachers in chapter 4. The intriguing fact is that even when the introduction of a technology has been well prepared by an epistemological analysis and situations have been proposed, implementation by teachers still looks like a struggle to give birth to a more personal creation. As indeed it is. In our view, new techniques and the way in which they change the teaching and learning of a mathematical domain are not given with the tool and cannot even be just thought of in terms of the design of tool-aided lessons. When a teacher wants to introduce technology, (s)he has to integrate these techniques into his (her) own understanding of the domain, into his (her) own personality and to create relevant situations, certainly not an easy task.

Schneider (1999) offers an example where two teachers wanted to introduce students to TI-92 use in the study of logarithmic functions. They had to entirely rethink their teaching because the techniques they used to work with became obsolete. Without the TI-92, a central task was to solve exponential equations. Students progressively built techniques relevant for a variety of equations and learned about the properties of logarithmic functions by reflecting on these techniques. The teachers became rapidly aware that the TI-92 solved the equations in one easy action and that all had to be rebuilt. The outcome was an entirely new approach to the domain, where symbolic techniques were complemented by graphic and numeric exploration. It is striking that this elaboration appears to be work for the teachers themselves –or maybe on themselves- rather than a creation to share with colleagues. A praxeology (definition in Box 5-1) is not just an organization of mathematical contents. At classroom level it offers teachers 'command levers' with which to make students enter the study of a domain. Thus a teacher cannot just receive and apply a new praxeology. (S)he has to create something new.

9. THE POTENTIALITIES OF TECHNOLOGY

The conclusion of this chapter is that the potentialities of new tools can only be appreciated by considering the impact of technology on existing techniques and the possible new techniques that students can develop as a bridge between tasks and theories. This is certainly a different viewpoint from that of an influential mathematics-education tradition, which tends to stress an opposition between skills and understanding. In this section we will look at this tradition, see how it lives on in conceptualizations of the use of new tools, and how these conceptualizations converge on the idea of a direct access to concepts, an idea that, from our perspective, cannot really account for the potentialities of technology.

The opposition between manipulation and understanding is ancient especially in the study of algebra. Rachlin (1989) states, “Teachers in the USA even (in 1890) were opposed to what they saw as an overemphasis on manipulative skills and were calling for a meaningful treatment of algebra that would bring about more understanding.” In the past fifteen years, the idea that universal access to new technology would “enable us to modify our skill-dominated conception of school algebra and rebalance it in favor of objectives related to understanding and problem solving” (Kieran and Wagner 1989, p. 8) gained greater acceptance.

To many authors, CAS was an appropriate technology for this “new balance” because it is not limited by the approximate treatment of numbers or by the necessity of programming. Mayes (1997) states that authors of research papers on CAS often study how CAS may help “set a new balance between skills and understanding” or “resequence skills and understanding.”

As early as 1988 Heid published a paper about the educational use of CAS in the Journal for Research in Mathematics Education (JRME). Her guiding hypothesis was that:

> If mathematics instruction were to concentrate on meaning and concepts first, that initial learning would be processed deeply and remembered well. A stable cognitive structure could be formed on which later skill development could build. (Heid 1988, p. 4).

This paper had a great influence and authors very often refer to it as a confirmation for hypotheses about benefits of technology and especially CAS. One of these authors, Pérez (1998, p. 362) published a text following a talk at the International Conference on Mathematics Education (ICME 8) and he interpreted Heid’s study as a proof of the many advantages of CAS including “students’ definite progress toward higher levels of formal thinking and easier integration of conceptual representations.”

It is interesting to look in detail at Heid’s argumentation. Her research is about an experiment involving the use of early computer programs –a symbolic calculator software with a command line user interface and a graphing application without connection to the calculator- in a project involving a new approach to introducing calculus. The author was also the teacher and she chose to delay training in computational skill , to develop graphic approaches to concepts and encourage reflection on the meaning of computer results, and to set students wider classes of problem to solve. She compared her students' proficiency with that of a control group following a 'traditional' curriculum. Delayed skill training did not harm her students and they achieved some more varied representations of concepts.

Using computers, more varied approaches are certainly possible and Heid's experiment provides a remarkable example of such use in a calculus course. Although maybe not this definite progress "towards higher levels of formal thinking", students' tendency towards more varied representations is worth noting. Ruthven, (2002 p. 284) took a closer look at the conditions that made technology contribute to this tendency:

> In the experimental classes, the constitution of a quite different system of techniques appears to have played an important part. The shift to "reasoning in non algebraic modes of representation [which] characterized concept development in the experimental classes" (p. 10) not only created new types of task, but encouraged systematic attention to corresponding techniques (…) Not only did the 'conceptual' phase of the experimental course expose students to (…) wider techniques; it also appears to have helped students to develop proficiency in what had become standard tasks, even if they were not officially recognized as such, and had not been framed so algorithmically, taught so directly, or rehearsed so explicitly as those deferred to the final 'skill' phase.
>
> It deferred routinization of the customarily taught skills of symbolic manipulation until the final phase of the applied calculus course, while the attention given to a broader range of problems and representations in the innovative main phase supported development of a richer conceptual system. Equally, however, (…) this conceptual development grew out of new techniques constituted in response to this broader range of tasks, and from greater opportunities for the theoretical elaboration of these techniques. At the same time, standard elements emerged from these new tasks, characteristic of the types of problem posed and the forms of representation employed, creating a new corpus of skills distinct from those officially recognized.

Artigue (2002 p. 248) has noted that Chevallard's approach gives technique "a wider meaning than is usual in educational discourse" comprising not just recognized routines for standard tasks but more "complex assembl[ies] of reasoning and routine work", whereas mainstream mathematics education research delimits the technical domain more narrowly in terms of "routine manipulations", "computational procedures" and "algorithmic skills.

Ruthven's analysis above takes technique in this wider meaning and sheds helpful light: the traditional opposition of concepts and skills should be tempered by recognizing a technical dimension in mathematical activity which is not reducible to skills. A cause of misunderstanding is that, at certain moments, a technique can take the form of a skill. This is particularly the case when a certain 'routinization' is necessary. But techniques must not be considered only in their routinized form. In this chapter we tried to show that when CAS is used, the technical dimension is different, but it retains its importance in giving students understanding. The work of constituting techniques in response to tasks, and of theoretical formulation of the questions posed by these techniques remains fundamental to learning.

This chapter provided a first approach to the techniques appearing when new tools are used and a sense of their variety. We have restrained our reflection to the fact that new artifacts were designed as tools to facilitate some techniques and so necessarily have a strong impact on the technical level of mathematical activity, making new techniques possible and old techniques in some sense obsolete. However we have also mentioned that changes in the teaching and learning of a mathematical domain resulting from this impact are not directly determined by an artifact. These changes cannot be appreciated without considering the evolving relationship between users and tools, an idea that the next chapter will develop, stressing the transformation of an artifact into an *instrument* for mathematical work. There the term *instrumented techniques* will be used to denote the way in which new techniques are linked to the tool that makes them possible but also to the mathematical domain that they address and to the user's representations of both.

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[1] For instance, the TI-92 gives a solution -1 for the equation of a real unknown x $\sqrt{x}\sqrt{x-3} = -2$.

[2] Chapter 9, section 2 will discuss the implementation of this situation within a curriculum.

[3] Actually this is true only for expressions belonging to a set where a canonical form for equivalent expressions exists and is implemented in the calculator.

[4] In 1998 the French ministry for Education designed an experimental (non official) baccalaureate. The paper included an interesting question about characteristics of a window to conjecture the intersection of two curves. No change followed in the official exam.

[5] Little data is actually available on the use of technology by teachers and biases can often be suspected. For instance BECTa (2002) maintains that the proportion of upper secondary school pupils in the UK never -or hardly ever- using ICT in their mathematics lessons is as much as 82%. This statistic however ignores the extent to which graphic calculators are used, since the survey in question appears not to have classed these as ICT.

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