Noether's-type theorems on time scales

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BAPTISTE ANEROT, JACKY CRESSON, KHALED HARIZ BELGACEM, AND FREDERIC PIERRET

Abstract. We prove a time scales version of the Noether's theorem relating group of symmetries and conservation laws in the framework of the shifted and nonshifted Δ calculus of variations. Our result extends the continuous version of the Noether's theorem as well as the discrete one and corrects a previous statement of Bartosiewicz and Torres in [3]. This result implies also that the second Euler–Lagrange equation on time scales as derived by Bartosiewicz, Martins and Torres is false. Using the Caputo duality principle, we provide the corresponding Noether's theorem on time scales in the framework of the shifted and nonshifted ∇ calculus of variations. All our results are illustrated with numerous examples supported by numerical simulations.

Contents

1. Introduction

The calculus on time scales was introduced by S. Hilger in his PhD thesis [17] in 1988 (see also [18]). The time scales calculus gives a convenient way to deal with discrete, continuous or mixed processes using a unique formalism. In 2001, this theory was used by M. Bohner [5] and R. Hilscher and V. Zeidan [19] to develop a *calculus of variations on time scales*. This first attempt was then followed by numerous generalizations. In this article we focus on two specific calculus of variations, namely the *shifted calculus of variations* as introduced in [5] and the *nonshifted* one as considered for example in [9]. In this context, many natural problems arise. One of them is to generalize to the time scales setting classical results of the calculus of variation in the continuous case. One of these problem is to obtain a time scales analogue of the Noether's Theorem relating group of symmetries and conservation laws.

The aim of this article is precisely to derive a time scales version of the Noether's theorem. We refer to the books of Olver [26] and Jost [23] for the classical case.

This problem was initially considered by Z. Bartosiewicz and D.F.M. Torres in [3] in the context of the shifted calculus of variations and then in [4].Two different strategies of proof are used:

- First, they proposed in [3] to derive the Noether's theorem for transformations depending on time from the easier result obtained for transformations without changing the time. In [12], we call *Jost's method* this way of proving the Noether's theorem as a classical reference is contained in the book [23].
- Another method is proposed in [4], where a *second Euler–Lagrange equation* is derived ([4], Theorem 5 p.12) and from which the Noether's theorem is deduced (see [4],Section 4, Theorem 6).

Unfortunately, **all these results are not correct** and need to be amended. Indeed, implementing numerically the conservation law and the second order Euler–Lagrange equation states in [3] on a specific example, we obtain incoherent results. Precisely, we use Example 3 of [3] defined as follows:

We consider the Lagrangian introduced in [3]

$$L(t,x,v) = \frac{x^2}{t} + tv^2 \tag{1}$$

for $t \in \mathbb{R} \setminus \{0\}$ and $(x,v) \in \mathbb{R}^2$. In [3], the authors consider the time scales

$$\mathbb{T} = \{2^n : n \in \mathbb{N} \cup \{0\}\}. \tag{2}$$

In that case, $\sigma(t) = 2t$ for all $t \in \mathbb{T}$ and $\Delta\sigma(t) = 2$.

The shifted Euler–Lagrange equation associated with L is given by

$$\Delta\big[t\Delta x(t)\big] = \frac{x^\sigma}{t}, \tag{3}$$

and the shifted time scales Noether's theorem given in [3] asserts that the following quantity

$$C(t,x^\sigma,v) = 2t\left(\frac{(x^\sigma)^2}{t} - tv^2\right), \tag{4}$$

is a constant of motion.

In [4], it is stated that the following equation

$$\Delta\left[\mathscr{H}(t,x^\sigma,\Delta x)\right] + \partial_t L(t,x^\sigma,\Delta x) = 0, \tag{5}$$

where

$$\mathscr{H}(t,x,v) = -L(t,x,v) + \partial_v L(t,x,v)v + \partial_t L(t,x,v)\mu(t), \tag{6}$$

is satisfied over the solutions of the shifted Euler-Lagrange equation for all $t \in \mathbb{T}^\kappa$ and is called the *second Euler-Lagrange equation.*

We then test numerically if the function C is constant over the solutions of the Euler–Lagrange equation and at the same time if the right hand side of equation (5) is equal to zero.

The simulations give the following results:

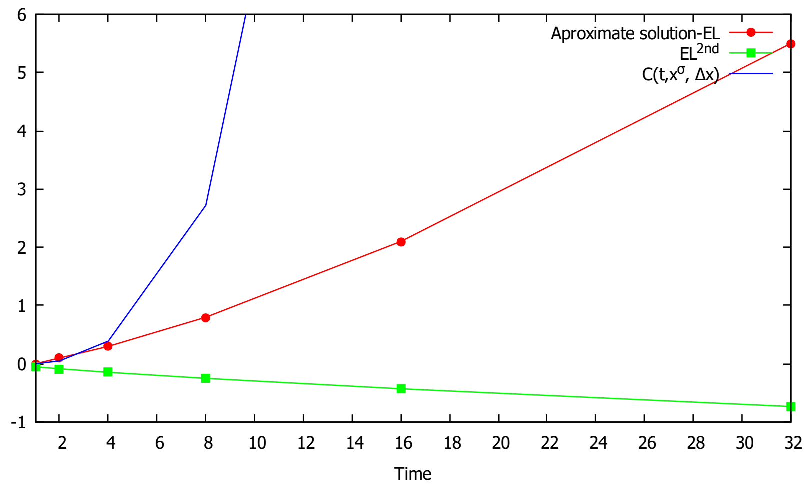


FIGURE 1. $x_0 = 0, \Delta x_0 = 0.1, n = 5$

These simulations clearly show that the function C is not a constant of motion and that the second Euler–Lagrange equation is not satisfied. It must be noted that this invalidates many other results which use the former results (see for example [29], Theorem 3 p. 5 where the second order Euler–Lagrange equation is used in the proof (see equation (33) in [29])).

In this article, we state and prove a time scales Noether's theorem in the shifted and non-shifted calculus of variations settings. We provide two different proofs:

- First, we follows the initial strategy used by Z. Bartosiewicz and D.F.M. Torres in [3] which refers to a time scales analogue of a classical proof exposed by J. Jost and X. Li-Jost in [23]. We point out several difficulties which are in fact inherent to the Jost's method (see [12]). This first proof is not the simplest one but it explains where and why the initial proof given in [3] is not correct.
- Second, a more classical one which can be called "*direct*", which consists in deriving the invariance relation with respect to the parameter of the transformation group and manipulating the obtained expression in order to provide a constant of motion. Although less elegant than the previous one, it is the most easiest one.

The plan of the paper is as follows.

In Section 2, we remind some definitions and notations about time scales and give some particular statements about the chain rule formula and the substitution formula for Δ and ∇ derivatives in the time scales setting, as well as the corresponding Leibniz formula.

In Section 3.1, we first define transformation groups in the context of time scales calculus. We introduce for a given time scales $\mathbb{T}$ the notion of $(\Delta, \mathbb{T})$ (resp. $(\nabla, \mathbb{T})$) admissible projectable transformations groups which imposes some conditions on the time scales as well as the transformation in time which can be considered.

In Section 7 we state the Noether's theorem on time scales in the context of the Δ shifted or nonshifted calculus of variations.

Section 4 gives the proof of our main result. The proof of several technical Lemmas are given in Section 9.

In Section 6.1, we discuss several examples and provide numerical simulations. We first study an example given by Bartosiewicz and Torres in [3]. We then discuss results obtained in the same context by X.H. Zhai and L.Y. Zhang in [29] about a time scales version of the Kepler problem in the plane. Here again, we prove that the results presented in [29] are not correct.

In Section 7, we use the *Caputo duality principle* in time scales as presented in [11] to obtain the Noether's theorem on time scales for the ∇ shifted and nonshifted calculus of variations. Our result differs also from the one obtained by N. Martins and D.F. Torres in [27]. We discuss an example proposed by X.H. Zhai and L.Y. Zhang in [29] and prove that the result of [27] are indeed incorrect.

The final Section contains the proof of several technical results used in the paper.

2. Preliminaries on time scales

In this Section, we remind some results about the chain rule formula, the change of variable formula for Δ-antiderivative which will be used during the proof of the main result. We refer to [1, 6, 7, 8] and references therein for more details on time scales calculus.

2.1. Time scales.

In this Section, we denote by $\mathbb{T}$ a *time scale*, i.e. an arbitrary non-empty closed subset of $\mathbb{R}$.

Two operators play a central role studying time scales: the backward and forward jump operators.

Definition 1. *The backward and forward jump operators* $\rho, \sigma : \mathbb{T} \longrightarrow \mathbb{T}$ *are respectively defined by:*

$$\forall t \in \mathbb{T},\ \rho(t) = \sup\{s \in \mathbb{T},\ s < t\} \text{ and } \sigma(t) = \inf\{s \in \mathbb{T},\ s > t\},$$

where we put $\sup \emptyset = \sup \mathbb{T}$ *and* $\inf \emptyset = \inf \mathbb{T}$*.*

Definition 2. *A point* $t \in \mathbb{T}$ *is said to be left-dense (resp. left-scattered, right-dense and right-scattered) if* $\rho(t) = t$ *(resp.* $\rho(t) < t$, $\sigma(t) = t$ *and* $\sigma(t) > t$*).*

Let LD (resp. LS, RD and RS) denote the set of all left-dense (resp. left-scattered, right-dense and right-scattered) points of $\mathbb{T}$.

Definition 3. *The graininess (resp. backward graininess) function* $\mu : \mathbb{T} \longrightarrow \mathbb{R}^+$ *(resp.* $\nu : \mathbb{T} \longrightarrow \mathbb{R}^+$ *) is defined by* $\mu(t) = \sigma(t) - t$ *(resp.* $\nu(t) = t - \rho(t)$*) for any* $t \in \mathbb{T}$*.*

We denote by $\mathbb{T}_\kappa = \mathbb{T} \setminus [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))$, $\mathbb{T}^\kappa = \mathbb{T} \setminus (\sup \mathbb{T}, \rho(\sup \mathbb{T})]$ and $\mathbb{T}^\kappa_\kappa = \mathbb{T}_\kappa \bigcap \mathbb{T}^\kappa$.

2.2. The Δ and ∇ derivatives.

Let us recall the usual definitions of Δ and ∇-differentiability.

Definition 4. *A function* $u : \mathbb{T} \longrightarrow \mathbb{R}^n$ *, where* $n \in \mathbb{N}$*, is said to be* Δ*-differentiable at* $t \in \mathbb{T}^\kappa$ *(resp.* ∇*-differentiable at* $t \in \mathbb{T}_\kappa$*) if the following limit exists in* $\mathbb{R}^n$*:*

$$\lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s} \quad \left(\textit{resp.} \lim_{\substack{s \to t \\ s \neq \rho(t)}} \frac{u(s) - u(\rho(t))}{s - \rho(t)} \right). \tag{7}$$

In such a case, this limit is denoted by $\Delta u(t)$ *(resp.* $\nabla u(t)$*).*

The characterization of constant of motion is related to the following fundamental result (see [6], Corollary 1.68 p.25):

Proposition 1. *Let* $u : \mathbb{T} \longrightarrow \mathbb{R}^n$ *. Then,* u *is* Δ*-differentiable on* $\mathbb{T}^\kappa$ *with* $\Delta u = 0$ *if and only if there exists* $c \in \mathbb{R}^n$ *such that* $u(t) = c$ *for every* $t \in \mathbb{T}$*.*

The analogous results for ∇-differentiability are also valid.

2.3. **Some functional spaces.**

Definition 5. *A function* u *is said to be rd-continuous (resp. ld-continuous) on* $\mathbb{T}$ *if it is continuous at every* $t \in \mathrm{RD}$ *(resp.* $t \in \mathrm{LD}$*) and if it admits a left-sided (resp. righ-sided) limit at every* $t \in \mathrm{LD}$ *(resp.* $t \in \mathrm{RD}$*).*

We respectively denote by $C^0_{\mathrm{rd}}(\mathbb{T})$ and $C^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$ the functional spaces of rd-continuous functions on $\mathbb{T}$ and of Δ-differentiable functions on $\mathbb{T}^\kappa$ with rd-continuous Δ-derivative. Similarly, we denote by $C^0_{\mathrm{ld}}(\mathbb{T})$ and $C^{1,\nabla}_{\mathrm{ld}}(\mathbb{T})$, respectively, the functional spaces of ld-continuous functions on $\mathbb{T}$ and of ∇-differentiable functions on $\mathbb{T}_\kappa$ with ld-continuous ∇-derivative.

Let us denote by $\int \Delta\tau$ the Cauchy Δ-integral defined in [6, p.26] with the following result (see [6, Theorem 1.74 p.27]):

Theorem 1. *For every* $u \in C^0_{\mathrm{rd}}(\mathbb{T}^\kappa)$*, there exist a unique* Δ*-antiderivative* U *of* u *in sense of* $\Delta U = u$ *on* $\mathbb{T}^\kappa$ *vanishing at* $t = a$*. In this case the* Δ*-integral is defined by*

$$U(t) = \int_a^t u(\tau)\Delta\tau$$

for every $t \in \mathbb{T}$*.*

2.4. **Algebraic properties of Δ and ∇ derivatives.**

2.4.1. *Leibniz property.* The Δ derivative satisfies a Leibniz formula given by (see [6],Corollary 1.20 p.8):

Theorem 2 (Leibniz formula for the Δ-derivative)**.** *Let* $v, w : \mathbb{T} \longrightarrow \mathbb{R}^n$ *. If* v *and* w *are* Δ*-differentiable at* $t \in \mathbb{T}_\kappa$*, then the scalar product* $v \cdot w$ *is* Δ*-differentiable at* t *and the following Leibniz formula holds:*

$$\begin{aligned} \Delta(v \cdot w)(t) &= v^\sigma(t) \cdot \Delta w(t) + \Delta v(t) \cdot w(t), \\ &= v(t) \cdot \Delta w(t) + \Delta v(t) \cdot w^\sigma(t). \end{aligned} \tag{8}$$

We have a time scales Leibniz formula for the ∇-derivative (see [9, Proposition 7]).

Theorem 3 (Leibniz formula for ∇-derivative)**.** *Let* $v, w : \mathbb{T} \longrightarrow \mathbb{R}^n$ *and* $t \in \mathbb{T}^\kappa_\kappa$*. If the following properties are satisfied:*

- σ *is* ∇*-differentiable at* t*,*
- v *is* Δ*-differentiable at* t*,*
- w *is* ∇*-differentiable at* t*,*

then, $v^\sigma \cdot w$ *is* ∇*-differentiable at* t *and the following Leibniz formula holds:*

$$\nabla(v^\sigma \cdot w)(t) = v(t) \cdot \nabla w(t) + \nabla\sigma(t) \cdot \Delta v(t) \cdot w(t). \tag{9}$$

2.4.2. *Chain rule formula and the substitution formula.* We have a time scales chain rule formula (see [6, Theorem 1.93]).

Theorem 4 (Time scales Chain Rule)**.** *Assume that* $v : \mathbb{T} \longrightarrow \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := v(\mathbb{T})$ *is a time scales. Let* $w : \widetilde{\mathbb{T}} \longrightarrow \mathbb{R}$ *. If* $\Delta v(t)$ *and* $\Delta_{\widetilde{\mathbb{T}}} w(v(t))$ *exist for* $t \in \mathbb{T}^\kappa$*, then*

$$\Delta (w \circ v) = \left(\Delta_{\widetilde{\mathbb{T}}} w \circ v\right) \Delta v \tag{10}$$

With the time scales chain rule, we obtain a formula for the derivative of the inverse function (see [6, Theorem 1.97]).

Theorem 5 (Derivative of the inverse)**.** *Assume that* $v : \mathbb{T} \longrightarrow \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := v(\mathbb{T})$ *is a time scales. Then*

$$\frac{1}{\Delta v} = \Delta_{\widetilde{\mathbb{T}}} \left(v^{-1}\right) \circ v \tag{11}$$

at points where Δv *is different from zero.*

Another formula from the chain rule is the substitution rule for integrals (see [6, Theorem 1.98]).

Theorem 6 (Substitution)**.** *Assume that* $v : \mathbb{T} \longrightarrow \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := v(\mathbb{T})$ *is a time scales. If* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *is a rd-continuous function and* v *is differentiable with rd-continuous derivative, then for* $a, b \in \mathbb{T}$*,*

$$\int_a^b f(t) \Delta v(t) \Delta t = \int_{v(a)}^{v(b)} \left(f \circ v^{-1}\right)(s) \Delta_{\widetilde{\mathbb{T}}} s. \tag{12}$$

3. Main results

In this Section, $\mathbb{T}$ denotes a bounded time scales with $a = \min \mathbb{T}$ and $b = \max \mathbb{T}$. We assume that $\mathrm{card}(\mathbb{T}) \geq 3$ ensuring that $\mathbb{T}^\kappa_\kappa \neq \emptyset$.

A function L defined by

$$L : \begin{array}{ccc} [a,b] \times \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (t,x,v) & \longmapsto & L(t,x,v) \end{array}, \tag{13}$$

is said to be a Lagrangian function if L is of class C^2 with respect to all its arguments.

3.1. **Admissible transformations group.** We refer to the classical book of P.J.Olver [26] for more details in particular Chapter 4. In the following, we consider a special class of symmetry groups of differential equations called *projectable* or *fiber-preserving* (see [26],p.93) and given by

$$g_s : \begin{array}{ccc} [a,b] \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \times \mathbb{R}^n \\ (t,x) & \longmapsto & (g_s^0(t), g_s^1(x)) \end{array} \tag{14}$$

where $\{g_s\}_{s\in\mathbb{R}}$ is a one parameter group of diffeomorphisms satisfying $g_0 = \mathbb{1}$, where $\mathbb{1}$ is the identity function. The associated *infinitesimal (or local) group action* (see [26],p.51) or transformations is obtained by making a Taylor expansion of g_s around $s = 0$:

$$g_s(t,x) = g_0(t,x) + s \left.\frac{\partial g_s(t,x)}{\partial s}\right|_{s=0} + o(s). \tag{15}$$

The *transform* (see [26],p.90) of a given function $x(t)$ identified with its graph $\Gamma_x = \{(t, x(t)),\ t \in [a, b]\}$ by g_s is easily obtained introducing a new variable τ defined by $\tau = g_s^0(t)$. The transform of x denoted by $\tilde{x}$ is then given by

$$\tau \longrightarrow (\tau, g_s^1 \circ x \circ (g_s^0)^{-1}(\tau)).$$

Remark 1. *In general, the transform of a given function is not so easy to determine explicitly (see* [26]*, Example 2.21, p.90-91) and one must use the implicit function theorem in order to recover the transform of x. This is precisely the reason why we restrict our attention to projectable or fiber-preserving symmetry groups.*

Working with time scales imposes some restrictions on the transformation groups that one can consider. In the following, we need the notion of $(\Delta, \mathbb{T})$**-admissible projectable group of transformations**:

Definition 6 ($(\Delta, \mathbb{T})$-admissible projectable group of transformations)**.** *A projectable group of transformations $\{g_s\}_{s\in\mathbb{R}}$ is called a $(\Delta, \mathbb{T})$-admissible projectable group of transformations if for all $s \in \mathbb{R}$, the function g_s^0 verifies:*

- *g_s^0 is strictly increasing,*
- *$\Delta g_s^0 \neq 0$ and Δg_s^0 is rd-continuous and such that,*
- *the set defined by $\widetilde{\mathbb{T}}_s = g_s^0(\mathbb{T})$ is a time scales.*
- *$\Delta_{\widetilde{\mathbb{T}}_s} \left(g_s^0\right)^{-1}$ exists.*

3.2. **Noether's theorem on time scales in the nonshifted calculus of variations.** Let L be a Lagrangian function. We can associate to L a functional denoted by $\mathcal{L}_{L,[a,b],\mathbb{T}} : C^{1,\Delta}_{\mathrm{rd}}(\mathbb{T}) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{L}_{L,[a,b],\mathbb{T}}(x) = \int_a^b L(t, x(t), \Delta x(t))\Delta t, \tag{16}$$

called the *nonshifted Lagrangian functional* over the time scales $\mathbb{T}$.

If σ is ∇-differentiable on $\mathbb{T}_\kappa$, then the critical points of $\mathcal{L}_{L,[a,b],\mathbb{T}}$ are solutions of the $\nabla \circ \Delta$-*differential Euler–Lagrange equation* (see [9, Theorem 1,p.548]):

$$\nabla \left[\frac{\partial L}{\partial v}(t, x(t), \Delta x(t))\right] = \nabla\sigma(t)\frac{\partial L}{\partial x}(t, x(t), \Delta x(t)), \tag{EL$^{\nabla\circ\Delta}$}$$

for every $t \in \mathbb{T}^\kappa_\kappa$.

3.2.1. *Invariance of functionals and variational symmetries.* We have the following time scales generalization of the definition of a *variational symmetry group* of a *nonshifted Lagrangian functional on time scales* (see [26, Definition 4.10, p.253]):

Definition 7 (Variational symmetries)**.** *The $(\Delta, \mathbb{T})$-admissible group of transformations $\{g_s\}_{s\in\mathbb{R}}$ is a variational symmetry group of the nonshifted functional* (16) *if whenever $I = [t_a, t_b]$ is a subinterval of $[a, b]$ with $t_a, t_b \in \mathbb{T}$ and $x \in C^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$ such that its transform under g_s denoted by $\tilde{x}$ is defined over $\tilde{I}_s = [\tilde{a}_s, \tilde{b}_s]$ which is a subset of $g_s^0([a, b]) = [\tau_a, \tau_b]$, then*

$$\mathcal{L}_{L,[t_a,t_b],\mathbb{T}}(x) = \mathcal{L}_{L_s,[\tilde{a}_s,\tilde{b}_s],\widetilde{\mathbb{T}}_s}(\tilde{x}). \tag{17}$$

It is interesting to give an explicit formulation of this definition. Indeed, according to definition (16) we can write (17) as

(18)
$$\int_{t_a}^{t_b} L\left(t, x(t), \Delta x(t)\right) \Delta t = \int_{\tilde{a}_s}^{\tilde{b}_s} L_s\left(\tau, g_s^1 \circ x \circ (g_s^0)^{-1}(\tau), \Delta_{\widetilde{\mathbb{T}}_s}\left(g_s^1 \circ x \circ (g_s^0)^{-1}\right)(\tau)\right) \Delta_{\widetilde{\mathbb{T}}_s} \tau$$

where $\tilde{a}_s = g_s^0(t_a)$ and $\tilde{b}_s = g_s^0(t_b)$.

3.2.2. *Noether's Theorem on time scales-nonshifted case.* Our main result is the following nonshifted version of the Noether's theorem on time scales:

Theorem 7 (Noether's theorem - Nonshifted case)**.** *Let $\mathbb{T}$ be a time scales such that σ is ∇-differentiable on $\mathbb{T}_\kappa$ and $G = \{g_s(t,x) = (g_s^0(t), g_s^1(x))\}_{s\in\mathbb{R}}$ be a $(\Delta, \mathbb{T})$-admissible projectable group of transformations which is a variational symmetry of the nonshifted Lagrangian functional on time scales $\mathbb{T}$ given by*

$$\mathcal{L}_{L,[a,b],\mathbb{T}}(x) = \int_a^b L\left(t, x(t), \Delta x(t)\right)\, \Delta t$$

and

$$X = \zeta(t)\frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x}, \tag{19}$$

be the infinitesimal generator of G. Then, the function

$$I(t,x,v) = -\zeta^\sigma(t)\cdot \mathcal{H}(\star) + \xi^\sigma(x)\cdot \partial_v L(\star) + \int_a^t \zeta\Big[\nabla\sigma \partial_t L(\star) + \nabla\big(\mathcal{H}(\star)\big)\Big]\,\nabla t, \tag{20}$$

where $\mathcal{H} : \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d \longrightarrow \mathbb{R}$ is defined by

$$\mathcal{H}(t,x,v) = -L(t,x,v) + \partial_v L(t,x,v)\cdot v, \tag{21}$$

and $(\star) = (t, x(t), \Delta x(t))$, is a constant of motion over the solution of the time scales Euler–Lagrange equation $(\mathrm{EL}^{\nabla\circ\Delta})$*, i.e., that*

$$\nabla\left[I\left(\cdot\,, x(\cdot)\right)\right](t) = 0, \tag{22}$$

for all solutions x of the time scales Euler–Lagrange equations and any $t \in \mathbb{T}_\kappa^\kappa$.

The proof is given in Section 4.

In the continuous case $\mathbb{T} = [a,b]$, one obtains the classical form of the integral of motion

$$I(t,x) = -\zeta(t)\cdot\mathcal{H}(t,x,\dot{x}) + \xi(x)\cdot\partial_v L(t,x,\dot{x}). \tag{23}$$

Indeed, if $\mathbb{T} = [a,b]$ then σ is ∇-differentiable on $\mathbb{T}_\kappa$ with $\nabla[\sigma] = 1$ and moreover, on the solutions of the Euler–Lagrange equation one has the identity

$$-\partial_t L(t,x,\dot{x}) = \frac{d}{dt}\big[\mathcal{H}(t,x,\dot{x})\big] \tag{24}$$

which is called the *second Euler–Lagrange equation* [28].

In the discrete case, $\mathbb{T} = \mathbb{Z}$ and transformations without changing time, one recovers the classical integral (see [10, Theorem 12, p.885] and also [21]):

$$I(x) = \xi^\sigma(x)\cdot\partial_v L(t,x,\dot{x}). \tag{25}$$

3.3. **Noether's Theorem on time scales in the shifted calculus of variations.** Let L be a Lagrangian function. We consider the functional $\mathcal{L}^{\sigma}_{L,[a,b],\mathbb{T}}(x)$ defined for all $x \in C^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$ by

$$\mathcal{L}^{\sigma}_{L,[a,b],\mathbb{T}}(x) = \int_a^b L\left(t, x^{\sigma}(t), \Delta x(t)\right) \Delta t. \tag{26}$$

The critical points of $\mathcal{L}^{\sigma}_{L,[a,b],\mathbb{T}}$ are solutions of the **shifted time scales Euler–Lagrange equation** given by (see [5, Theorem 4.2, p.344])

$$\Delta\left[\frac{\partial L}{\partial v}(t, x^{\sigma}(t), \Delta x(t))\right] = \frac{\partial L}{\partial x}(t, x^{\sigma}(t), \Delta x(t)), \tag{$\mathrm{EL}^{\Delta\circ\Delta}$}$$

for every $t \in \mathbb{T}^{\kappa}$.

Remark 2 (A remark on the shifted calculus of variations)**.** *Although the shifted calculus of variations was introduced first in the literature, the definition of the functional (26) seems to be non-natural with respect to a discretisation procedure of the continuous Lagrangian functional and in fact leads to very bad numerical integrator of the continuous equation. This is due to the fact that in this case, the second order derivative* d^2/dt^2 *is approximated by* $\Delta \circ \Delta$ *which is an operator of order one with respect to the time step used as a discretization step, instead of order* 2 *for the* $\nabla \circ \Delta$ *operator which appears in the non-shifted case.*

However, leaving this aspect, one can justify the use of the shifted calculus of variations as follows: Going back to I. Newton's seminal work Philosophiae Naturalis Principia Mathematica published first in 1866 (a reprint can be found in [20] *with other texts of interest), we can take a look at the first place were he derived the now famous law of motions for a body under the gravitational force. We refer to the discussion given by R. Feynman in* [15] *for more details.*

He explains that the motion of a body around a massive body with an initial speed v_0 *evolves during a short amount of time* $t_1 - t_0 = h$ *following the inertia principle introduced by Galileo. The particle then follows a straight line between the initial position* x_0 *and* $\tilde{x_1}$ *whose length is given by* $v_0 h$*. However, at time* t_1*, the effect of the force* F *during the time* h *is taken into account and assumed to be of magnitude* $F(x_0)h^2$*. This reasoning is illustrated by I. Newton in his book by the following picture (see* [20]*,p.431 and also* [15]*,p.84):*

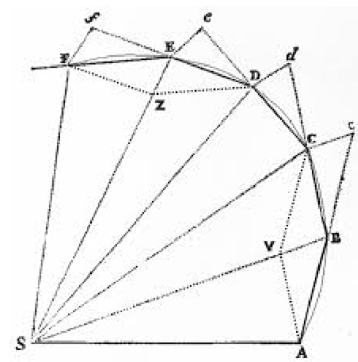


FIGURE 2. Newton's illustration for the motion of a planet around a star

As the force is assumed to be directed toward the massive body, I. Newton deduces that the position of the particle at time t_1 *satisfies* $x_1 - \tilde{x_1} = F(x_0)h^2$, *which leads to* $x_1 - x_0 = v_0 h + F(x_0)h^2$ *and finally, denoting* $\Delta x(t_1) = (x_1 - x_0)/h$ *and* $\Delta x(t_0) = v_0$, *to the equation*

$$\Delta x(t_1) - \Delta x(t_0) = F(x_0)h, \tag{27}$$

and to the classical writing of Newton's fundamental law of motion

$$\Delta(\Delta x)(t_0) = F(x_0). \tag{28}$$

As a consequence, I. Newton's first derivation of the law of motion leads to an equation where only the Δ *derivative appears. This equation can only be recovered using the shifted calculus of variations.*

The notion of invariance is adapted to the shifted case as follows:

Definition 8 (Shifted invariance). *A time scales Lagrangian functional* $\mathcal{L}^{\sigma}_{L,[a,b],\mathbb{T}}$ *is said to be invariant under a* $(\Delta, \mathbb{T})$*-admissible projectable group of transformations* $G = \{g_s(t,x) = (g_s^0(t), g_s^1(x))\}_{s\in\mathbb{R}}$ *if and only if for any subinterval* $[t_a, t_b] \subset [a,b]$ *with* $t_a, t_b \in \mathbb{T}$, *for any* $s \in \mathbb{R}$ *and* $x \in C^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$

$$\int_{t_a}^{t_b} L\left(t, x^{\sigma}(t), \Delta x(t)\right) \Delta t = \int_{\tilde{a}_s}^{\tilde{b}_s} L_s\Big(\tau, \left[g_s^1 \circ x \circ (g_s^0)^{-1}\right]^{\tilde{\sigma}_s}(\tau), \Delta_{\widetilde{\mathbb{T}}_s}\left[g_s^1 \circ x \circ (g_s^0)^{-1}\right](\tau)\Big)\, \Delta_{\widetilde{\mathbb{T}}_s}\tau \tag{29}$$

where $\tilde{a}_s = g_s^0(t_a)$ *and* $\tilde{b}_s = g_s^0(t_b)$, $\widetilde{\mathbb{T}}_s = g_s^0(\mathbb{T})$ *and* $\tilde{\sigma}_s$ *is the forward jump operator over* $\widetilde{\mathbb{T}}_s$.

Theorem 8 (Noether's theorem - σ-shifted case). *Let* $G = \{g_s(t,x) = (g_s^0(t), g_s^1(x))\}_{s\in\mathbb{R}}$ *be a* $(\Delta,\mathbb{T})$*-variational symmetry of* $\mathcal{L}^{\sigma}_{L,[a,b],\mathbb{T}}$ *with the corresponding infinitesimal generator given by*

$$X = \zeta(t)\frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x}. \tag{30}$$

Then, the quantity

$$I(t, x^{\sigma}, v) = -\mathscr{H}(\star_{\sigma})\zeta(t) + \partial_v L(\star_{\sigma})\xi(x) + \int_a^t \zeta^{\sigma}(t)\Big(\Delta\big[\mathscr{H}(\star_{\sigma})\big] + \partial_t L(\star_{\sigma})\Big)\Delta t \tag{31}$$

where

$$\mathscr{H}(t,u,v) = \mathcal{H}(t,u,v) + \partial_t L(t,u,v)\mu(t) \tag{32}$$

and $\mathcal{H}$ *is given by* (21)*, with* $(\star_\sigma) = (t, x^\sigma(t), \Delta x(t))$*, is a constant of motion over the solution of the time scales Euler–Lagrange equation* $(\mathrm{EL}^{\Delta\circ\Delta})$*, i.e., that*

$$\Delta\left[I\left(\cdot\,, x^\sigma(\cdot), \Delta x(\cdot)\right)\right](t) = 0, \tag{33}$$

for all solutions x *of* $(\mathrm{EL}^{\Delta\circ\Delta})$ *and any* $t \in \mathbb{T}^\kappa$*.*

In the continuous case $\mathbb{T} = [a,b]$, we have $\sigma(t) = t$ and $\mu(t) = 0$, so that one obtains the classical form of the integral of motion (23).

3.4. **Comparison with the Noether's theorem on time scales obtained by Z. Bartosiewicz and D.F.M Torres.** In [3], Z. Bartosiewicz and D.F.M. Torres prove a Noether's theorem on time scale which leads to the statement that the quantity

$$C\left(t, x^\sigma, v\right) = -\mathscr{H}(\star_\sigma)\cdot\zeta(t,x) + \partial_v L(\star_\sigma)\cdot\xi(t,x) \tag{34}$$

is a constant of motion over the solution of $(\mathrm{EL}^{\Delta\circ\Delta})$.

As we can see, we have an extra term in our result given by

$$\int_a^t \zeta^\sigma(t)\Big(\Delta\big[\mathscr{H}(\star_\sigma)\big] + \partial_t L(\star_\sigma)\Big)\Delta t.$$

The difference comes from the fact that Z. Bartosiewicz and D.F.M. Torres [3] assume that the following equation

$$\Delta\big[\mathscr{H}(t, x^\sigma, \Delta x)\big] = -\frac{\partial L}{\partial t}(t, x^\sigma(t), \Delta x(t)), \tag{$\mathrm{EL}_\sigma^{2^{nd}}$}$$

called the second order Euler-Lagrange equation is satisfied over the solutions of the shifted Euler-Lagrange equation. As already showed in the introduction simulations on an explicit example, this is not true. In the following, we give a counter-example to the second order Euler-Lagrange equation where all computations can be made explicitly.

3.4.1. *Explicit counter-example to the second order Euler-Lagrange equation on time scales.* Let us consider the Lagrangian

$$L(x^\sigma, \Delta x) = (\Delta x)^2 + 4x^\sigma. \tag{35}$$

The shifted Euler-Lagrange equation is given by

$$\Delta\left[\Delta x\right] = 2. \tag{36}$$

As $\partial_t L = 0$, the quantity $\mathscr{H}$ reduces to

$$\mathscr{H}(x^\sigma, \Delta x) = (\Delta x)^2 - 4x^\sigma. \tag{37}$$

We have the following Lemma:

Lemma 1. *The function* $\Delta\mathscr{H}$ *is equal to*

$$\Delta\mathscr{H} = 4\left[-\mu - 2\mu\Delta\mu - \Delta x\Delta\mu\right], \tag{38}$$

over the solutions of the shifted Euler-Lagrange equation.

Proof. We have (see [6],1.36 p.337) that for any function $u \in C_{\rm rd}^{1,\Delta}$ such that $\Delta(u^\sigma)$ exists, the relation

$$\Delta(u^\sigma) = (1+\Delta\mu)\,(\Delta u)^\sigma\,. \tag{39}$$

Moreover, using the Leibniz formula we have

$$\begin{aligned}\Delta\left((\Delta x)^2\right) &= \Delta(\Delta x)\Delta x + (\Delta x)^\sigma \Delta(\Delta x),\\ &= \Delta(\Delta x)\left(\Delta x + (\Delta x)^\sigma\right).\end{aligned} \tag{40}$$

As a consequence, we obtain

$$\begin{aligned}\Delta\mathscr{H} &= \Delta\left((\Delta x)^2\right) - 4\Delta(x^\sigma),\\ &= \Delta(\Delta x)\left(\Delta x + (\Delta x)^\sigma\right) - 4(1+\Delta\mu)\,(\Delta x)^\sigma\,.\end{aligned} \tag{41}$$

Using the shifted Euler-Lagrange equation, one obtain

$$\Delta\mathscr{H} = 2\left(\Delta x + (\Delta x)^\sigma\right) - 4(1+\Delta\mu)\,(\Delta x)^\sigma\,. \tag{42}$$

Moreover, we have the classical relation for $u \in C_{\rm rd}^{1,\Delta}$ (see [6],(iv),p.6):

$$u^\sigma = u + \mu\Delta u, \tag{43}$$

which gives

$$(\Delta x)^\sigma = \Delta x + \mu\Delta(\Delta x) = \Delta x + 2\mu, \tag{44}$$

thanks to the shifted Euler-Lagrange equation.

As a consequence, replacing in the expression of $\Delta\mathscr{H}$, one obtain

$$\begin{aligned}\Delta\mathscr{H} &= 2\left(2\Delta x + 2\mu\right) - 4(1+\Delta\mu)(\Delta x + 2\mu),\\ &= 4\left[-\mu - 2\mu\Delta\mu - \Delta x\Delta\mu\right],\end{aligned} \tag{45}$$

which concludes the proof. □

As a consequence, any time scales such that μ is a non zero constant lead to a counter example to the second order Euler-Lagrange equation. In particular, we have

Lemma 2. *Let $\mathbb{T} = \mathbb{Z}$, then $\Delta\mathscr{H} = -4$.*

Proof. For $\mathbb{T} = \mathbb{Z}$, we have $\mu = 1$ for all $t \in \mathbb{T}$. As a consequence, we have $\Delta\mu = 0$. Replacing in the formula (38), we obtain $\Delta\mathscr{H} = -4$. □

3.4.2. *Connexion with energy preserving variational integrators.* We can go further relying on the fact that for uniform time scales, the shifted Euler-Lagrange equation can be interpreted as a *variational integrator* (see [25] and [21]):

Assuming that $\mathbb{T}$ is the uniform time scale over $[a,b]$, i.e. that $\mathbb{T} = \{t_i = a + ih,\ i = 0,\dots,N\}$ with $h = (b-a)/N$. Then $\mu(t) = h$ for all $t \in \mathbb{T}^\kappa$. If the Lagrangian L is independent of the time variable, then $\partial_t L = 0$ and the quantity $(\mathrm{EL}_\sigma^{2^{nd}})$ reduced to

$$\Delta\left[\mathscr{H}(\cdot, x^\sigma(\cdot), \Delta x(\cdot))\right](t) = 0, \quad \forall\ t \in \mathbb{T}^\kappa. \tag{46}$$

The quantity $\mathscr{H}$ corresponds to the Hamiltonian associated to the Lagrangian systems and its value to the *energy* of the system. However, it is well known since the work of Z. Ge and J.E. Marsden [16] that "fixed time step variational integrators derived from the discrete variational principle cannot preserve the energy of the system exactly". This implies precisely that the time scales second Euler-Lagrange equation is not valid in full generality.

We refer to the book of E. Hairer, C. Lubich and G. Wanner *Geometric numerical integration* [21] for more details, in particular Chapter VI.6 about variational integrators and Chapter IX.8 for a a discussion of long-term energy conservation of symplectic numerical schemes.

Remark 3. *In* [24]*, A.B. Malinowska and N. Martins discuss in full generality the derivation of a second Noether Theorem on time scales. In (*[24]*, Remark 23,p.8) they recover the second Euler-Lagrange equation derived in* [4] *as a special case. As a consequence, the previous discussion invalidate also the results proved in* [24]*.*

4. Proof of the main results using the Jost's method

The terminology of *Jost's method* was introduced in [12] to designate a particular way of proving the classical Noether's theorem which can be found in [23]. The idea is very simple and elegant. One extend the set of variables, incorporating the time variable, in order to see the invariance of the functional under a symmetry group with transformation in time as an invariance of a new functional but for a symmetry group without transformation in the new "time" variable. The idea being then to apply the well known Noether's theorem in this case to obtain the desired constant of motion. In [12], we have identified several steps in the method:

- First, rewrite the invariance condition in order to have an equality between two integrals over the same domain.
- The first step leads to the introduction of an *extended Lagrangian* and a new set of paths.
- Rewrite the initial invariance condition with transformation in time as an invariance condition for the extended Lagrangian for a transformation without transforming "time".
- Look for the correspondence between the solution of the initial Euler-Lagrange equation and the Euler-Lagrange equation associated to the extended Lagrangian.
- Apply the invariance characterization and derive a constant of motion.

The first three steps impose some specific constraints in the time scales framework due to the fact that the chain rule formula and the substitution formula are not always valid. However, the main problem comes from the Euler-Lagrange equation satisfied by the extended Lagrangian. Although this equation is always satisfied by solution of the initial Euler-Lagrange equation in the continuous case, this implication is no longer valid in general for an arbitrary time scales. This is precisely where some arguments given in [3] are incomplete. The end of the computations are only technical.

4.1. **The nonshifted case.**

4.1.1. *Rewriting the invariance condition and the extended Lagrangian.* We first rewrite the invariance relation (18) in order to have the same domain of integration.

Lemma 3. *Let G be a $(\Delta,\mathbb{T})$-variational symmetry of the nonshifted time scales Lagrangian functional $\mathcal{L}_{L,[a,b],\mathbb{T}}$, then, we have*

$$\int_a^b L\left(t,x(t),\Delta x(t)\right)\Delta t=\int_a^b L_s\left(g_s^0(t),(g_s^1\circ x)(t),\Delta\left(g_s^1\circ x\right)(t)\frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t)\Delta t. \tag{47}$$

The proof is given in Section 9.

As for the classical case, we construct an extended Lagrangian functional which enables us to rewrite the invariance condition for a transformation group changing time as the invariance of a new functional under a transformation group without changing time.

Let us denote by $\mathbb{L}:\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^*\times\mathbb{R}^d\longrightarrow\mathbb{R}$ the Lagrangian function defined by

$$\mathbb{L}(t,x,w,v)=L\left(t,x,\frac{v}{w}\right)w. \tag{48}$$

which is the same as the classical case and called the **extended Lagrangian**.

We denote by $\mathcal{L}_{\mathbb{L}}(t,x)$ the nonshifted Lagrangian functional associated to $\mathbb{L}$ defined for all $t\in C_{\rm rd}^{1,\Delta}(\mathbb{T})$ strictly increasing and $x\in C_{\rm rd}^{1,\Delta}(\mathbb{T})$ such that $\Delta_{\widetilde{\mathbb{T}}}(x\circ t)$ exists where $\widetilde{\mathbb{T}}=t(\mathbb{T})$ by

$$\mathcal{L}_{\mathbb{L}}(t,x)=\int_a^b \mathbb{L}\left(t(\tau),(x\circ t)(\tau)),\Delta[t](\tau),\Delta[x\circ t](\tau))\right)\Delta\tau, \tag{49}$$

is called the **nonshifted extended Lagrangian functional**.

We define the *time scales bundle path class* denoted by $\mathcal{F}$ and defined by

$$\mathcal{F}=\{(t,x)\in C_{\rm rd}^{1,\Delta}(\mathbb{T})\times C_{\rm rd}^{1,\Delta}(\mathbb{T})\ ;\ \tau\longmapsto(t(\tau),(x\circ t)(\tau)))=(\tau,x(\tau))\}. \tag{50}$$

We have the following proposition:

Proposition 2. *The restriction of the Lagrangian function $\mathcal{L}_{\mathbb{L}}$ to a path $\gamma=(t,x)\in\mathcal{F}$ satisfies*

$$\mathcal{L}_{\mathbb{L}}(t,x)=\mathcal{L}_{L,[a,b],\mathbb{T}}(x). \tag{51}$$

Proof. Let $\gamma=(t,x)\in\mathcal{F}$. By definition, we have

$$\mathbb{L}\left(t(\tau),x(\tau),\Delta[t](\tau),\Delta[x\circ t](\tau)\right)=L\left(t(\tau),(x\circ t)(\tau)),\Delta[x\circ t](\tau))\frac{1}{\Delta[t](\tau)}\right)\Delta[t](\tau). \tag{52}$$

As γ is a bundle path, we have $t(\tau)=\tau$ and $\Delta[t](\tau)=1$. As t is strictly increasing, $t\in C_{\rm rd}^{1,\Delta}(\mathbb{T})$ and $x\circ t=x$ belongs to $C_{\rm rd}^{1,\Delta}(\mathbb{T})$, the functional (49) is well defined and we obtain

$$\begin{aligned}\mathcal{L}_{\mathbb{L}}(t,x)&=\int_a^b \mathbb{L}\left(t(\tau),(x\circ t)(\tau),\Delta[t](\tau),\Delta[x\circ t](\tau))\right)\Delta\tau,\\&=\int_a^b L\left(\tau,x(\tau),\Delta x(\tau)\right)\Delta\tau=\mathcal{L}_{L,[a,b],\mathbb{T}}(x),\end{aligned} \tag{53}$$

which concludes the proof. ☐

4.1.2. *Invariance of the extended Lagrangian.* We now reformulate the initial existence of a variational symmetry for $\mathcal{L}_{L,[a,b],\mathbb{T}}$ under the group G as an invariance of the extended Lagrangian:

Lemma 4. *Let $\mathcal{L}_{L,[a,b],\mathbb{T}}$ be a time scales Lagrangian functional invariant under the $(\Delta,\mathbb{T})$-admissible projectable group of transformations $\{g_s\}_{s\in\mathbb{R}}$. Then, the time scales Lagrangian functional $\mathcal{L}_{\mathbb{L}}$ is invariant over $\mathcal{F}$ under the $(\Delta,\mathbb{T})$-admissible projectable group of transformations $\{g_s\}_{s\in\mathbb{R}}$.*

The proof is given in Section 9.

In order to apply the Noether's theorem for transformations without changing time, one needs to check that the solutions of the time scales Euler-Lagrange equation produce solutions of the extended Lagrangian systems.

Lemma 5. *A path $\gamma=(t,x)\in\mathcal{F}$ is a critical point of $\mathcal{L}_{\mathbb{L}}$ if, and only if, x is a critical point of $\mathcal{L}_{L,[a,b],\mathbb{T}}$ and for all $t\in\mathbb{T}_{\kappa}^{\kappa}$ we have*

$$\nabla\sigma(t)\frac{\partial L}{\partial t}(t,x(t),\Delta x(t))+\nabla\left[\Delta x(t)\frac{\partial L}{\partial v}(t,x(t),\Delta x(t))-L(t,x(t),\Delta x(t))\right]=0. \tag{$\circledast$}$$

The proof is given in Section 9.

Contrary to the continuous case, Lemma 5 implies that extended solutions of the initial Lagrangian are not automatically solutions of the extended Euler–Lagrange equation. This implies that one can not use the Noether's theorem but only the infinitesimal invariance criterion as formulated in ([3],Theorem 2 p.1223).

4.1.3. *Proof of the nonshifted time scales Noether's Theorem.* We deduce from Lemma 4 and the *necessary condition of invariance* given in ([3],Theorem 2 p.1223) that

$$\partial_t L(\star)\cdot\zeta+\partial_x L(\star)\cdot\xi+\partial_v L(\star)\cdot\Delta\xi+\big[L(\star)-\partial_v L(\star)\cdot\Delta x\big]\cdot\Delta\zeta=0. \tag{54}$$

Multiplying equation (54) by $\nabla\sigma$ and using the Time scales Euler–Lagrange equation ($\mathrm{EL}^{\nabla\circ\Delta}$), we obtain

$$\partial_t L(\star)\cdot\nabla\sigma\cdot\zeta+\nabla\sigma\cdot\partial_v L(\star)\cdot\Delta[\xi]+\nabla\big[\partial_v L(\star)\big]\cdot\xi+\big[L(\star)-\partial_v L(\star)\cdot\Delta x\big]\cdot\nabla\sigma\cdot\Delta\zeta=0. \tag{55}$$

Using the Leibniz formula (9), we have

$$\partial_t L(\star)\cdot\nabla\sigma\cdot\zeta+\nabla\big[\partial_v L(\star)\cdot\xi^{\sigma}\big]+\big[L(\star)-\partial_v L(\star)\cdot\Delta x\big]\cdot\nabla\sigma\cdot\Delta\zeta=0. \tag{56}$$

Trying to be as close as possible to the continuous case, we can use again the formula (9) on the last term, we obtain

$$\partial_t L(\star)\cdot\nabla\sigma\cdot\zeta+\nabla\big[\partial_v L(\star)\cdot\xi^{\sigma}\big]+\nabla\big[\zeta^{\sigma}\cdot(L(\star)-\partial_v L(\star)\cdot\Delta x)\big]-\zeta\cdot\nabla\big[L(\star)-\partial_v L(\star)\cdot\Delta x\big]=0. \tag{57}$$

Taking the ∇-antiderivative of this expression, we deduce the conservation law (20). This concludes the proof.

4.2. **The σ-shifted case.** The shifted case follows essentially the same line as the non shifted case. However, due to the the shift, after the initial change of variables, one needs another rewriting of the invariance condition in order to identify the corresponding extended Lagrangian.

4.2.1. *Rewriting the invariance condition and the extended Lagrangian.* Following Section, we have:

Lemma 6. *Let the functional $\mathcal{L}^{\sigma}_{L,[a,b],\mathbb{T}}$ satisfying condition* (29)*, then we have*

$$\int_{t_a}^{t_b} L\left(t, x^{\sigma}(t), \Delta x(t)\right)\Delta t = \int_{t_a}^{t_b} L_s\left(g_s^0(t), \left[g_s^1\circ x\right]^{\sigma}(t), \Delta\left[g_s^1\circ x\right](t)\cdot\frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t)\Delta t. \tag{58}$$

However, in order to consider the time as a new variable, one must rewrite the left-hand side of the invariance condition taking into account that ([3, Theorem 4, p.1224])

$$g_s^0(t) = (g_s^0)^{\sigma}(t) - \mu(t)\Delta g_s^0(t). \tag{59}$$

One then obtain:

Lemma 7. *The invariance condition* (58) *can be written as*

$$\begin{aligned}\int_{t_a}^{t_b} L\left(t, x^{\sigma}(t), \Delta x(t)\right)\Delta t = \\ \int_{t_a}^{t_b} L_s\left((g_s^0)^{\sigma}(t) - \mu(t)\Delta g_s^0(t), \left[g_s^1\circ x\right]^{\sigma}(t), \Delta\left[g_s^1\circ x\right](t)\cdot\frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t)\Delta t\end{aligned} \tag{60}$$

We are now ready to introduce the extended Lagrangian.

4.2.2. *Invariance of the extended Lagrangian.* Introducing the **shifted extended Lagrangian** denoted by $\mathbb{L} : \mathbb{R}\times[a,b]\times\mathbb{R}^d\times\mathbb{R}^*\times\mathbb{R}^d \longrightarrow \mathbb{R}$

$$\mathbb{L}_{\sigma}(\tau; t, x, w, v) = L\left(t - \mu(\tau)w, x, \frac{v}{w}\right)w. \tag{61}$$

Introducing the functional denoted by $\mathcal{L}_{\mathbb{L}_{\sigma}}$ and defined by

$$\mathcal{L}_{\mathbb{L}_{\sigma}}(t,x) = \int_{t_a}^{t_b} \mathbb{L}_{\sigma}\left(\tau; t^{\sigma}(\tau), (x^{\sigma}\circ t)(\tau)), \Delta t(\tau), \Delta x(\tau)\right)\Delta\tau. \tag{62}$$

Taking into account the bundle path $\mathcal{F}$ defined in (50), we obtain that $\Delta[t] = 1$, so that the restriction of $\mathbb{L}_{\sigma}$ to $\mathcal{F}$ satisfies

$$\mathbb{L}_{\sigma}\left(\tau; t^{\sigma}(\tau) = \tau^{\sigma}, x^{\sigma}(\tau), \Delta\tau, \Delta x(\tau)\right) = L\left(\tau, x^{\sigma}(\tau), \Delta x(\tau)\right). \tag{63}$$

As a consequence, one can rewrite the invariance condition (60) as follows

Lemma 8. *The invariance condition* (60) *over $\mathcal{F}$ can be written as*

$$\mathcal{L}_{\mathbb{L}_{\sigma}}(t,x) = \int_{t_a}^{t_b} \mathbb{L}_{\sigma}\left(\tau; \left[g_s^0\right]^{\sigma}(t(\tau)), \left[g_s^1\circ x\right]^{\sigma}(t(\tau)), \Delta_{\widetilde{\mathbb{T}}_s} g_s^0(t(\tau)), \Delta_{\widetilde{\mathbb{T}}_s}\left[g_s^1\circ x\right](t(\tau))\right)\Delta_{\widetilde{\mathbb{T}}_s}\tau. \tag{64}$$

One can obtain the necessary invariance condition of the functional $\mathcal{L}_{\mathbb{L}_{\sigma}}$ over $\mathcal{F}$ by differentiating both sides of (64) around $s = 0$, that is

$$\partial_t\mathbb{L}_{\sigma}(\bullet)\cdot\zeta^{\sigma}(\tau) + \partial_x\mathbb{L}_{\sigma}(\bullet)\cdot\xi^{\sigma}(x) + \partial_w\mathbb{L}_{\sigma}(\bullet)\cdot\Delta\zeta(\tau) + \partial_v\mathbb{L}_{\sigma}(\bullet)\cdot\Delta\xi(x) = 0 \tag{65}$$

where $(\bullet) := \left(\tau; \tau^{\sigma}, x^{\sigma}(\tau), \Delta_{\widetilde{\mathbb{T}}}\tau, \Delta_{\widetilde{\mathbb{T}}}x(\tau)\right)$.

4.2.3. *Proof of the shifted time scales Noether's Theorem.* Using the relation (61), we have

$$\begin{cases} \partial_t \mathbb{L}_\sigma(\tau; t^\sigma, x, w, v) & = \partial_t L\left(t^\sigma - \mu(\tau) w, x, \dfrac{v}{w}\right) \cdot w \\ \partial_x \mathbb{L}_\sigma(\tau; t^\sigma, x, w, v) & = \partial_x L\left(t^\sigma - \mu(\tau) w, x, \dfrac{v}{w}\right) \cdot w \\ \partial_w \mathbb{L}_\sigma(\tau; t^\sigma, x, w, v) & = L\left(t^\sigma - \mu(\tau) w, x, \dfrac{v}{w}\right) - \partial_v L\left(t^\sigma - \mu(\tau) w, x, \dfrac{v}{w}\right) \cdot \dfrac{v}{w} \\ & \qquad - \partial_t L\left(t^\sigma - \mu(\tau) w, x, \dfrac{v}{w}\right) \cdot \mu(\tau) \cdot w \\ \partial_v \mathbb{L}_\sigma(\tau; t^\sigma, x, w, v) & = \partial_v L\left(t^\sigma - \mu(\tau) w, x, \dfrac{v}{w}\right) \end{cases} \tag{66}$$

On the other hand, we reduce the equations (66) over $\mathcal{F}$ as follows

$$\begin{cases} \partial_t \mathbb{L}_\sigma\left(\tau; \tau^\sigma, x(\tau), 1, \Delta x(\tau)\right) & = \partial_t L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) \\ \partial_x \mathbb{L}_\sigma\left(\tau; \tau^\sigma, x(\tau), 1, \Delta x(\tau)\right) & = \partial_x L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) \\ \partial_w \mathbb{L}_\sigma\left(\tau; \tau^\sigma, x(\tau), 1, \Delta x(\tau)\right) & = L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) - \partial_v L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) \cdot \Delta x(\tau) \\ & \qquad - \partial_t L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) \cdot \mu(\tau) \\ \partial_v \mathbb{L}_\sigma\left(\tau; \tau^\sigma, x(\tau), 1, \Delta x(\tau)\right) & = \partial_v L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) \end{cases} \tag{67}$$

Substituting (67) into (65) gives

$$\begin{aligned} &\partial_t L\left(\star_\sigma\right) \cdot \zeta^\sigma(\tau) + \partial_x L\left(\star_\sigma\right) \cdot \xi^\sigma(x) \\ &\qquad + \Big[L\left(\star_\sigma\right) - \partial_v L\left(\star_\sigma\right) \Delta x(\tau) - \partial_t L\left(\star_\sigma\right) \cdot \mu(\tau)\Big] \cdot \Delta\zeta(\tau) + \partial_v L\left(\star_\sigma\right) \cdot \Delta\xi(x) = 0. \end{aligned} \tag{68}$$

Using the Euler–Lagrange equation ($\mathrm{EL}^{\Delta\circ\Delta}$) and the *time scales Leibniz rule*, we obtain

$$\begin{aligned} &\partial_t L\left(\star_\sigma\right) \cdot \zeta^\sigma(\tau) + \big[L\left(\star_\sigma\right) - \partial_v L\left(\star_\sigma\right) \Delta x(\tau) - \partial_t L\left(\star_\sigma\right) \cdot \mu(\tau)\big] \cdot \Delta\zeta(\tau) \\ &\qquad + \Delta\big[\partial_v L\left(\star_\sigma\right) \cdot \xi(x)\big] = 0. \end{aligned} \tag{69}$$

Observe that the term between brackets in (69) is the function $-\mathscr{H}$ defined in (32). Using the *time scales Leibniz rule*, we obtain

$$\big[-\mathscr{H}(\star_\sigma)\big] \cdot \Delta\zeta(\tau) = \Delta\big[-\mathscr{H}(\star_\sigma) \cdot \zeta(\tau)\big] + \Delta\big[\mathscr{H}(\star_\sigma)\big] \cdot \zeta^\sigma(\tau). \tag{70}$$

Substituting the formula (70) into (69) gives

$$\Big(\partial_t L\left(\star_\sigma\right) + \Delta\big[\mathscr{H}(\star_\sigma)\big]\Big) \cdot \zeta^\sigma(\tau) + \Delta\big[-\mathscr{H}(\star_\sigma) \cdot \zeta(\tau) + \partial_v L\left(\star_\sigma\right) \cdot \xi(x)\big] = 0. \tag{71}$$

We complete the proof by taking the Δ-antiderivative of this latter equation.

5. Direct proof of the main results

We follow in this Section the usual proof of the Noether's theorem consisting in deriving the invariance condition with respect to the parameter of the symmetry group and deducing a constant of motion.

5.1. **The nonshifted case.** Since the invariance condition (47) holds for any subinterval of $[a,b]$ and $x \in C^{1,\Delta}_{\text{rd}}(\mathbb{T})$, then we have:

$$L\left(t,x(t),\Delta x(t)\right) = L_s\left(g_s^0(t),(g_s^1\circ x)(t),\Delta\left(g_s^1\circ x\right)(t)\frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t).$$

Differentiating both sides of the latter equation with respect to s, it gives for $s=0$ that

$$\zeta\partial_t L + \xi\partial_x L + \left(\Delta\xi - \Delta\zeta\cdot\Delta x\right)\partial_v L + \Delta\zeta\cdot L = 0. \tag{72}$$

Since this equation and the equation (54) are the same, one can follow the proof in subsection 4.1.3.

Remark 4 (Prolongation of vector fields in a time-scales setting)**.** *The operator appearing in* (72) *can be rewritten using the vector field denoted by* $X^{(1)}$ *and defined by*

$$X^{(1)} = \zeta\partial_t + \xi\partial_x + \left(\Delta\xi - \Delta\zeta\cdot\Delta x\right)\partial_v \tag{73}$$

By analogy with the definition of the prolongation of vector fields given by P.J. Olver (see [26]*, Definition 2.28 p.101), we call this vector field the first prolongation of the vector field* $X = \zeta\partial_t + \xi\partial_x$*. Consequently, one can replace the condition* (47) *by the following invariance criterion*

$$X^{(1)}L + \Delta\zeta\cdot L = 0. \tag{74}$$

In the case when $\mathbb{T}=\mathbb{R}$*, one recover the usual formula for the first prolongation (see* [26, Theorem 2.36, p.110]*) of the vector field* X*, i.e.*

$$X^{(1)} = \zeta\partial_t + \xi\partial_x + (\dot{\xi} - \dot{\zeta}\dot{x})\partial_v. \tag{75}$$

In order to develop a full analogue of the theory of symmetries as presented in the book of P.J. Olver [26]*, one needs first to defined correctly the discrete analogue of vector fields which is still missing at that time.*

5.2. **The σ-shifted case.** Since the invariance condition (58) holds for any subinterval of $[a,b]$ and $x \in C^{1,\Delta}_{\text{rd}}(\mathbb{T})$, then we have:

$$L\left(t,x^\sigma(t),\Delta x(t)\right) = L_s\left((g_s^0)^\sigma(t) - \mu(t)\Delta g_s^0(t),\left[g_s^1\circ x\right]^\sigma(t),\Delta\left[g_s^1\circ x\right](t)\cdot\frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t)$$

In the same way as done in the nonshifted case, by differentiating both sides of the above equation with respect to s, it gives for $s=0$ that

$$\begin{aligned} 0 &= (\zeta^\sigma - \mu(t)\Delta\zeta)\partial_t L + \xi^\sigma\partial_x L + (\Delta\xi - \Delta x\Delta\zeta)\partial_v L + \Delta\zeta\cdot L \\ &= \zeta\partial_t L + \xi^\sigma\partial_x L + (\Delta\xi - \Delta x\Delta\zeta)\partial_v L + \Delta\zeta\cdot L. \end{aligned}$$

Since the latter equation and the equation (68) are the same, one can follow the same proof as in subsection 4.2.3.

Remark 5. *One can replace the condition* (58) *by an alternative condition that is*

$$\zeta\partial_t L + \xi^\sigma\partial_x L + (\Delta\xi - \Delta x\Delta\zeta)\partial_v L + \Delta\zeta\cdot L = 0. \tag{76}$$

6. Examples and simulations

6.1. **The σ-shifted and nonshifted version of the Bartosiewicz and Torres example.** We consider the Lagrangian introduced in [3] and given by

$$L(t,x,v)=\frac{x^2}{t}+tv^2 \tag{77}$$

for $x,v\in\mathbb{R}$.

We discuss both the shifted and nonshifted Lagrangian functional associated to L and the corresponding conservation laws as obtained using the Noether's theorem on time scales proved in the previous Section.

One can prove that the nonshifted Lagrangian functional possesses a variational symmetry given by:

Lemma 9. *The Lagrangian functional associated to* (77) *is invariant under the family of transformation $G=\{g_s(t,x)=(te^s,x)\}_{s\in\mathbb{R}}$ where its infinitesimals are given by*

$$\zeta(t)=t \quad \text{and} \quad \xi(x)=0. \tag{78}$$

Proof. Indeed, we have $L\left(te^s,x,\dfrac{\Delta x}{e^s}\right)e^s=\left(\dfrac{x^2}{te^s}+te^s\dfrac{(\Delta x)^2}{e^{2s}}\right)e^s=L(t,x,\Delta x)$ so that condition (47) is satisfied. □

The same result is valid in the shifted case.

In the following, we consider two time scales given by

$$\mathbb{T}_1=\{a+nh,\ n\in\mathbb{N}\},\ h=(b-a)/N, N\in\mathbb{N}^* \quad \text{and} \quad \mathbb{T}_2=\{2^n,\ n\in\mathbb{N}\cup\{0\}\}, \tag{79}$$

which will be used to make simulations.

6.1.1. *The nonshifted case.* In our case, the (non-shifted) Euler–Lagrange equation associated with L is given by

$$\nabla\big[t\Delta x(t)\big]=\nabla\sigma(t)\frac{x}{t}, \tag{80}$$

with $\nabla\sigma(t)=1$ if $t\in\mathbb{T}_1$ and $\nabla\sigma(t)=2$ if $t\in\mathbb{T}_2$ and our time scales Noether's theorem generates the following first integral

$$I(t,x,v)=\sigma(t)\left(\frac{x^2}{t}-tv^2\right)+\int_a^t\left[-\nabla\sigma(t)\left(\frac{x^2}{t}-tv^2\right)-t\nabla\left(\frac{x^2}{t}-tv^2\right)\right]\nabla t. \tag{81}$$

6.1.2. *The shifted case.* We consider the following shifted Lagrangian

$$L\left(t,x^\sigma,v\right)=\frac{(x^\sigma)^2}{t}+tv^2 \tag{82}$$

and the family of transformation $G=\{\phi_s(t,x)=(te^s,x)\}_{s\in\mathbb{R}}$ which is a variational symmetry of L. Indeed, using the invariance criterion (76) we have that

$$t\left[-\left(\frac{x^\sigma}{t}\right)^2+v^2\right]-2tv^2+\frac{(x^\sigma)^2}{t}+tv^2=0.$$

The (shifted) Euler–Lagrange equation ($\mathrm{EL}^{\Delta\circ\Delta}$) associated to L is given by

$$\Delta\big[t\Delta x(t)\big] = \frac{x^\sigma}{t}. \tag{83}$$

According to Noether's theorem, we conclude the following first integral

$$I(t, x^\sigma, v) = \sigma(t)\left(\frac{(x^\sigma)^2}{t} - tv^2\right) + \int_a^t \sigma(t)\left(-\frac{(x^\sigma)^2}{t^2} + v^2 + \Delta\Big[\sigma(t)\left(-\frac{(x^\sigma)^2}{t^2} + v^2\right)\Big]\right)\Delta t.$$

Remark 6. *In* [3], *the authors consider* $\mathbb{T} = \{2^n : n \in \mathbb{N} \cup \{0\}\}$. *In that case,* $\sigma(t) = 2t$ *for all* $t \in \mathbb{T}$, *which gives the expression of* $C(t, x^\sigma, v)$ *in* [3, Example 3], *that is*

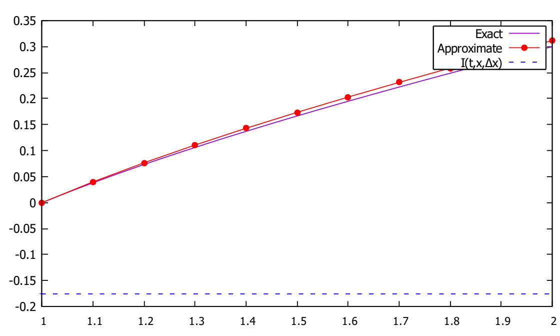
$$C(t, x^\sigma, v) = \sigma(t)\left(\frac{(x^\sigma)^2}{t} - tv^2\right). \tag{84}$$

6.1.3. *Simulations.* With the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$ as given before, we present simulations of both the Euler–Lagrange equations (80) and (83) which are called "approximate" on the picture as well as computations of the quantities $I(t, x, \Delta x)$ and $I(t, x^\sigma, \Delta x)$ on $\mathbb{T}_1$ and $\mathbb{T}_2$. In order to check the validity of our numerical scheme, we give also the exact solution of the Euler-Lagrange equation in the continuous case for the corresponding initial conditions.

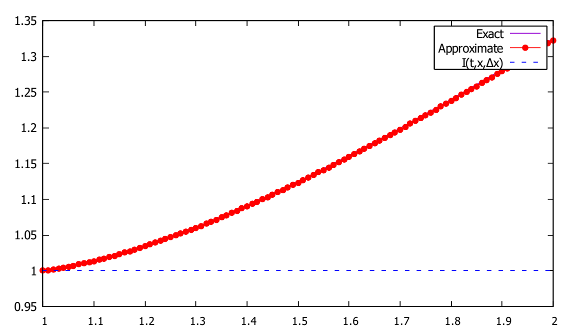
As we can see in Figures 3 and 5,over the time scales $\mathbb{T}_1$ when h is sufficiently small, the solution of the nonshifted or shifted Euler-Lagrange equation provide very good approximations of the exact solution.

We can not expect such a result for the time scales $\mathbb{T}_2$ as in this case, the time increment is very big at the beginning of the simulation.

As expected, all the computations given in Figures 3 and 5 over $\mathbb{T}_1$ and 4, and 6 over $\mathbb{T}_2$ show that the quantities obtained in the Noether's theorem on time scales are constant over the solutions of the time scales Euler–Lagrange equation (80) and (83)) respectively.

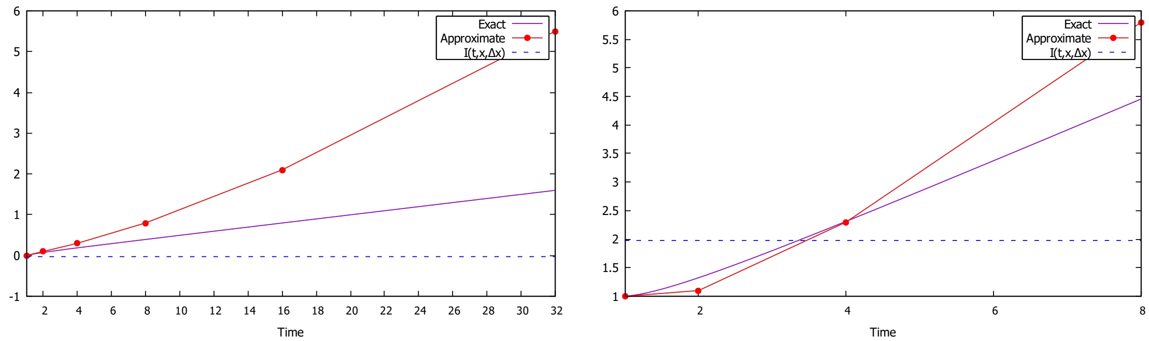


(A) $x_0 = 0, \Delta x_0 = 0.4, h = 0.1$



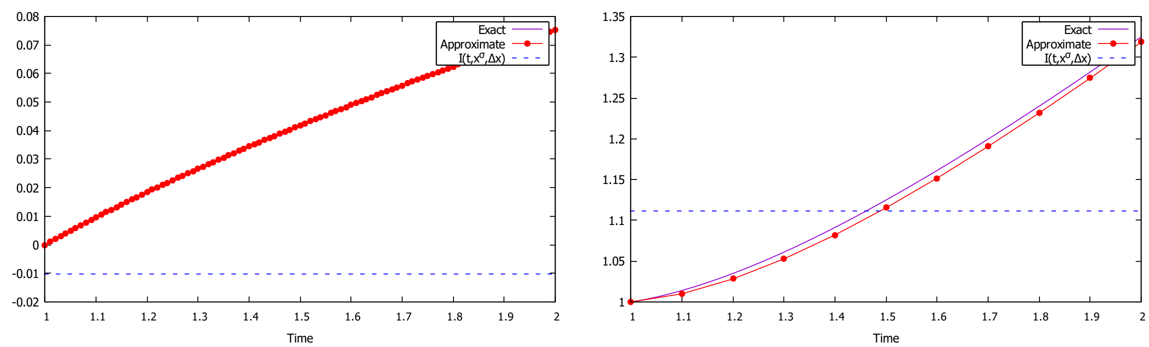
(B) $x_0 = 1, \Delta x_0 = 0.1, h = 0.01$

FIGURE 3. Numerical solution of (80) and the quantity (81) on time scales $\mathbb{T}_1$



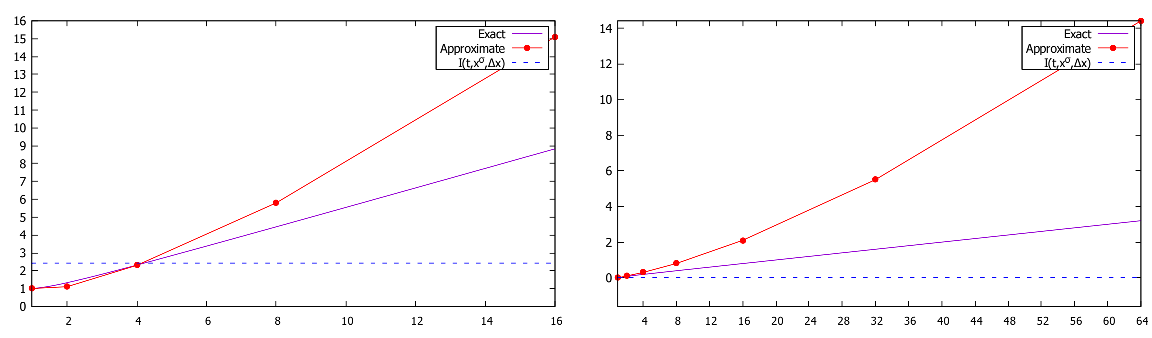
(A) $n = 5, x_0 = 0, \Delta x_0 = 0.1$ (B) $n = 3, x_0 = 1, \Delta x_0 = 0.1$

Figure 4. Numerical solution of (80) and the quantity (81) on time scales $\mathbb{T}_2$



(A) $x_0 = 0, \Delta x_0 = 0.1, h = 0.01$ (B) $x_0 = 1, \Delta x_0 = 0.1, h = 0.1$

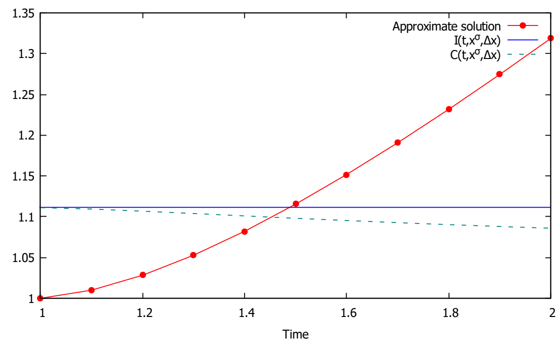
Figure 5. Numerical solution of (83) and the quantity $I(t, x^\sigma, v)$ on $\mathbb{T}_1$



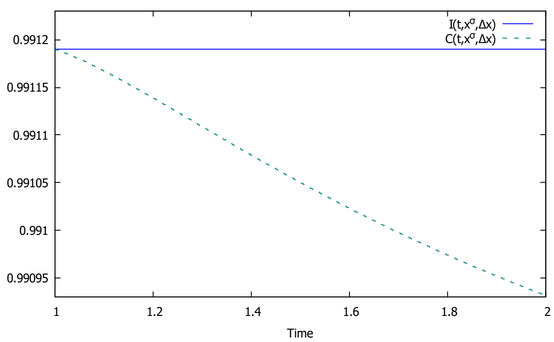
(A) $x_0 = 1, \Delta x_0 = 0.1, n = 4$ (B) $x_0 = 0, \Delta x_0 = 0.1, n = 6$

Figure 6. Numerical solution of (83) and the quantity $I(t, x^\sigma, v)$ on $\mathbb{T}_2$.

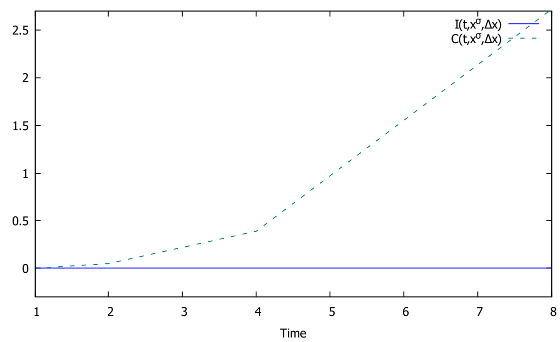
6.1.4. *Comparison between Torres's result and our result.* As we have seen, the quantity $I(t, x^\sigma, \Delta x)$ is a constant of motion over the solution of the time scales Euler–Lagrange equation (83). It is clearly not the case for the quantity $C(t, x^\sigma, \Delta x)$ provided by the Noether's theorem in [3].



(A) On $\mathbb{T}_1$, $x_0 = 1, \Delta x_0 = 0.1, h = 0.1$



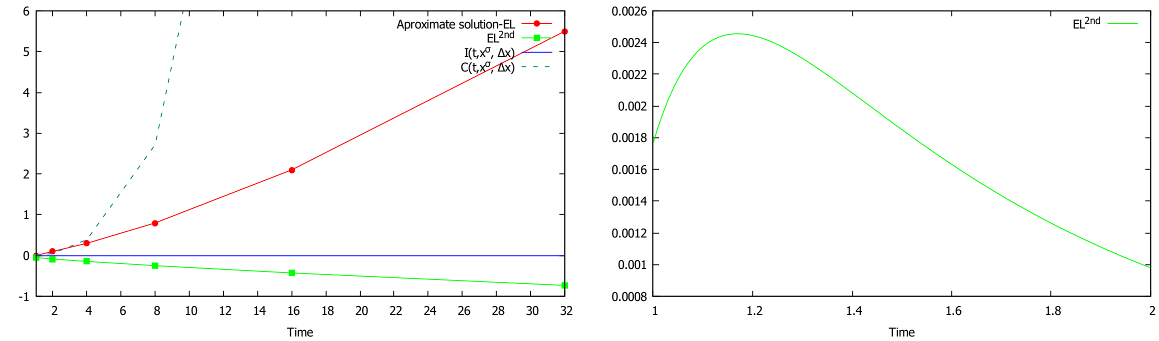
(B) On $\mathbb{T}_1$, $x_0 = 1, \Delta x_0 = 0.1, h = 0.001$



(C) On $\mathbb{T}_2$, $x_0 = 0, \Delta x_0 = 0.1, n = 3$

FIGURE 7. The trace of $I(t, x^\sigma, v)$ and $C(t, x^\sigma, v)$ on time scales $\mathbb{T}_1$ and $\mathbb{T}_2$

6.1.5. *Simulation of the second Euler–Lagrange equation.* In [3], the authors require for the quantity $C(t, x^\sigma, v)$ to be a constant of motion over the solution of (83) that the second Euler–Lagrange equation must be satisfied. We then test the equality to zero of the left-hand side of the equatio. We obtain the following green lines for the time-scales $\mathbb{T}_1$ and $\mathbb{T}_2$



(A) On $\mathbb{T}_2$, $x_0 = 0, \Delta x_0 = 0.1, n = 5$ (B) On $\mathbb{T}_1$, $x_0 = 1, \Delta x_0 = 0.1, h = 0.01$

FIGURE 8. Behavior of the second Euler–Lagrange, $I(t, x^\sigma, v)$ and $C(t, x^\sigma, v)$.

proving that the second Euler-Lagrange equation is not satisfied.

6.2. **The Kepler problem in the plane and a result of X.H. Zhai and L.Y. Zhang.** We consider the time scales analogue of the *Kepler problem* in the plane already studied by X.H. Zhai and L.Y. Zhang in ([29],Example 1).

We consider the Lagrangian defined on $\left(\mathbb{R}^2 \setminus \{0\}\right) \times \mathbb{R}^2$ by

$$L(x_1, x_2, v_1, v_2) = \frac{1}{2}(v_1^2 + v_2^2) + \frac{1}{\sqrt{x_1^2 + x_2^2}}, \tag{85}$$

which corresponds to the Lagrangian of the Kepler problem of two interacting particle with one of mass one under the gravitational field in the plane where one of the particle is positioned at the origin.

A time scales analogue of the Kepler problem in the shifted calculus of variation setting is then associated to the functional

$$\mathcal{L}_{L,[a,b],\mathbb{T}}(x) = \int_a^b \left[\frac{1}{2}(\Delta[x_1])^2 + (\Delta[x_2])^2) + \frac{1}{\sqrt{(x_1^\sigma)^2 + (x_2^\sigma)^2}} \right] \Delta t. \tag{86}$$

The Euler–Lagrange equations are given by

$$\left\{ \begin{array}{rcl} \Delta \circ \Delta[x_1] & = & -\dfrac{x_1^\sigma}{((x_1^\sigma)^2 + (x_2^\sigma)^2)^{3/2}}, \\ \Delta \circ \Delta[x_2] & = & -\dfrac{x_2^\sigma}{((x_1^\sigma)^2 + (x_2^\sigma)^2)^{3/2}}. \end{array} \right. \tag{87}$$

Moreover the Hamiltonian function associated to (87) is given by

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2}\left(p_1^2 + p_2^2\right) - \frac{1}{\sqrt{(x_1^\sigma)^2 + (x_2^\sigma)^2}}. \tag{88}$$

One easily shows that the group of rotations

$$g_s(x_1, x_2) = (x_1 \cos(s) - x_2 \sin(s), x_1 \sin(s) + x_2 \cos(s)), \tag{89}$$

for $s \in \mathbb{R}$, $(x_1, x_2) \in \mathbb{R}^2$ is a variational symmetry of the functional on any time scales $\mathbb{T}$. Indeed, we have for all $s \in \mathbb{R}$, $x = (x_1, x_2) \in C^{1,\Delta}_{rd}(\mathbb{T})$ and $t \in \mathbb{T}^\kappa$

$$L(x, \Delta x) = L(g_s(x), \Delta\,[g_s(x)]), \tag{90}$$

as $\Delta\,[g_s(x)]) = g_s\,(\Delta[x])$ by linearity and continuity of g_s with respect to x, and the fact that g_s is an isometry. The invariance of the functional then follows.

As $\dfrac{\partial g_s}{\partial s}(x_1, x_2)|_{s=0} = (-x_2, x_1)$, the Noether theorem on time scales then ensure that the function

$$I_1(\cdot, x(\cdot)) = -x_2\,\Delta[x_1] + x_1\,\Delta[x_2], \tag{91}$$

is a first integral of the time scales equation (87). This result coincide with the one given by X.H. Zhai and L.Y. Zhang in ([29],equation (45)).

It is clear that the group of time translations is a variational symmetry of (86), since this functional does not depend on the time. Then, our Noether theorem on time scales produces the following first integral

$$I_2(\cdot, x(\cdot)) = -H(x_1^\sigma, x_2^\sigma, \Delta x_1, \Delta x_2) + \int_a^t \Delta H(x_1^\sigma, x_2^\sigma, \Delta x_1, \Delta x_2)\Delta t \tag{92}$$

Indeed, if we consider the uniform time scales $\mathbb{T} = \{t_k = a + kh,\ k \in \mathbb{N}\}$ on the interval $[0; 3.5]$ with $h = 0.1$ and the initial conditions are $x_1 = 1, x_2 = 0, v_1 = v_2 = 1$, we obtain the following simulation

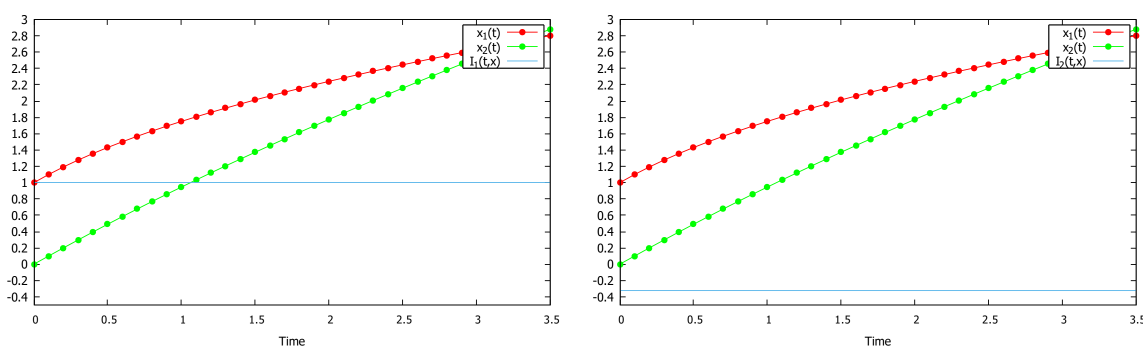
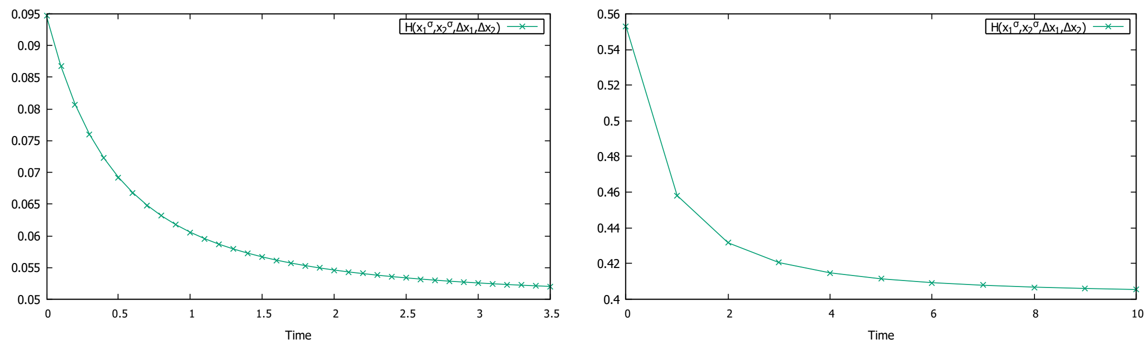


FIGURE 9. Simulation of the quantities $I_1(t, x)$ and $I_2(t, x)$.

However, as for the Z. Bartosiewicsz and D.F.M. Torres example [3], a problem occurs with time dependent group of transformations. Namely, X.H. Zhai and L.Y. Zhang asserts that the Hamiltonian is a constant of motion on the solutions of (87), i.e. that the quantity

$$H(t, x_1^\sigma, x_2^\sigma, \Delta x_1, \Delta x_2) = \frac{1}{2}\left((\Delta[x_1])^2 + (\Delta[x_2])^2\right) - \frac{1}{\sqrt{(x_1^\sigma)^2 + (x_2^\sigma)^2}}, \tag{93}$$

is a constant on the solution of the equation for an arbitrary time scale. This is of course the case for any continuous time scales $\mathbb{T} = [a, b]$ but not the case for other time scales like $\mathbb{T} = \{t_k = a + kh,\ k \in \mathbb{N}\}$. Indeed, in this case, one obtain the following simulation



(A) Uniform time scales on $[0; 3.5]$ with $h = 0.1$ (B) Uniform time scales on $[0, 10]$ with $h = 1$

FIGURE 10. Simulation of the Hamiltonian function (93) on a uniform time scales over the solutions (87).

7. Caputo duality principle and a time scales Noether's Theorem for the nabla calculus of variations

In this section, some properties, basic definitions about *Caputo's duality principle* are presented and such principle was also applied to the calculus of variations on time scales. We refer to [11] which contain more details and proofs on *Caputo's duality principle*.

7.1. Reminder about Caputo duality principle.

Definition 9. *Let $\mathbb{T}$ be a time scale. The dual time scales of $\mathbb{T}$ is a new time scales defined by $\mathbb{T}^* := \{\tau \in \mathbb{R} : -\tau \in \mathbb{T}\}$.*

Definition 10. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function defined on a time scales $\mathbb{T}$. The dual function $f : \mathbb{T}^* \to \mathbb{R}$ is defined by $f^*(\tau) = f(-\tau)$ for all $\tau \in \mathbb{T}^*$. The dual time scales of $\mathbb{T}$ is a new time scales defined by $\mathbb{T}^* := \{\tau \in \mathbb{R} : -\tau \in \mathbb{T}\}$.*

Let $\mathbb{T}$ be a time scale. If $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ denote, respectively, the forward and backward jump operators on $\mathbb{T}$, then we denote to the forward and backward jump operators on $\mathbb{T}^*$, respectively, by $\hat{\sigma}, \hat{\rho} : \mathbb{T}^* \to \mathbb{T}^*$.

Let μ (resp. ν) the forward (resp. the backward) graininess on $\mathbb{T}$, we denote by $\hat{\mu}$ (resp. $\hat{\nu}$), the forward (resp. the backward) graininess on $\mathbb{T}^*$.

Let Δ (resp. ∇) the delta (resp. the nabla) derivative on $\mathbb{T}$, we denote by $\hat{\Delta}$ (resp. $\hat{\nabla}$) the delta (resp. the nabla) derivative on $\mathbb{T}^*$.

Proposition 3. *Let $\mathbb{T}$ be a time scales with $a, b \in \mathbb{T}$, $a < b$ and let $f : \mathbb{T} \to \mathbb{R}$ a function. We have the following:*

- $(\mathbb{T}^\kappa)^* = (\mathbb{T}^*)_\kappa$ *and* $(\mathbb{T}_\kappa)^* = (\mathbb{T}^*)^\kappa$
- $([a, b])^* = [-b, -a]$ *and* $([a, b]^\kappa)^* = [-b, -a]_\kappa \subseteq \mathbb{T}^*$.
- *For all* $\tau \in \mathbb{T}^*$, $\hat{\sigma}(\tau) = -\rho(-\tau) = -\rho^*(\tau)$ *and* $\hat{\rho}(\tau) = -\sigma(-\tau) = -\sigma^*(\tau)$.
- *For all* $\tau \in \mathbb{T}^*$, $\hat{\mu}(\tau) = \nu^*(\tau)$ *and* $\hat{\nu}(\tau) = \mu^*(\tau)$.
- *Given a function* $f : \mathbb{T} \to \mathbb{R}$ *and its dual* $f^* : \mathbb{T}^* \to \mathbb{R}$. *Then,* $f \in C^0_{\mathrm{rd}}(\mathbb{T})$ *(resp.* $f \in C^0_{\mathrm{ld}}(\mathbb{T})$*) if and only if* $f^* \in C^0_{\mathrm{ld}}(\mathbb{T})$ *(resp.* $f^* \in C^0_{\mathrm{rd}}(\mathbb{T})$*).*

- *If f is Δ (resp. ∇) differentiable at $t \in \mathbb{T}^\kappa$ (resp. at $t \in \mathbb{T}_\kappa$), then $f^* : \mathbb{T}^* \to \mathbb{R}$ is ∇ (resp. Δ) differentiable at $-t \in (\mathbb{T}^*)_\kappa$ (resp. $-t \in (\mathbb{T}^*)^\kappa$), and*

$$\begin{aligned}\Delta f(t) &= -\hat{\nabla} f^*(-t), \quad (resp. \nabla f(t) = -\hat{\Delta} f^*(-t)),\\ \Delta f(t) &= -\left(\hat{\nabla} f^*\right)^*(t), \quad (resp. \nabla f(t) = -\left(\hat{\Delta} f^*\right)^*(t)),\\ (\Delta f)^*(-t) &= -\hat{\nabla} f^*(-t), \quad (resp. (\nabla f)^*(-t) = -\hat{\Delta} f^*(-t)).\end{aligned}$$

- *If $f : [a, b] \to \mathbb{R}$ is rd-continuous, then*

$$\int_a^b f(t)\Delta t = \int_{-b}^{-a} f^*(\tau)\hat{\nabla}\tau.$$

- *If $f : [a, b] \to \mathbb{R}$ is ld-continuous, then*

$$\int_a^b f(t)\nabla t = \int_{-b}^{-a} f^*(\tau)\hat{\Delta}\tau.$$

Definition 11. *Let $L : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Lagrangian. Then, the corresponding dual lagrangian $L^* : \mathbb{T}^* \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by*

$$L^*(\tau, x, v) = L(-\tau, x, -v) \quad \text{for all} \quad (\tau, x, v) \in \mathbb{T}^* \times \mathbb{R}^n \times \mathbb{R}^n.$$

One can notice that,

$$\partial_t L^*(\tau, x, v) = -\partial_t L(-\tau, x, -v), \tag{94}$$

$$\partial_x L^*(\tau, x, v) = \partial_x L(-\tau, x, -v), \tag{95}$$

$$\partial_v L^*(\tau, x, v) = -\partial_v(-\tau, x, -v). \tag{96}$$

7.2. **A time scales Noether's theorem for the nabla nonshifted calculus of variations.** Consider the functional $\mathcal{L}_{L,[a,b],\mathbb{T}} : \mathrm{C}^{1,\nabla}_{\mathrm{ld}}(\mathbb{T}) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{L}_{L,[a,b],\mathbb{T}}(x) = \int_a^b L\left(t, x(t), \nabla x(t)\right)\nabla t \tag{97}$$

where $L : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Lagrangian on the time scales $\mathbb{T}$.

Theorem 9 (Euler–Lagrange equation [9])**.** *Assume that ρ is Δ-differentiable on $\mathbb{T}^\kappa$. Then, the critical points of the functionl* (97) *are solutions of the following Euler–Lagrange equation*

$$\Delta\left[\frac{\partial L}{\partial v}(t, x(t), \nabla x(\tau))\right] = \Delta\rho(t)\frac{\partial L}{\partial x}(t, x(t), \nabla x(t)), \tag{$\mathrm{EL}^{\Delta\circ\nabla}$}$$

for every $t \in \mathbb{T}^\kappa_\kappa$.

Theorem 10 (Noether's Theorem - Nonshifted case)**.** *Let $\mathbb{T}$ be a time scales such that ρ is Δ-differentiable on $\mathbb{T}^\kappa$. Let $G = \{g_s(t, x) = (g^0_s(t), g^1_s(x))\}_{s\in\mathbb{R}}$ a $(\nabla, \mathbb{T})$-variational symmetry of the functional* (97) *with the corresponding infinitesimal generator given by*

$$X = \zeta(t)\frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x}. \tag{98}$$

Then, the function

$$\bar{I}(t, x, v) = -\zeta^\rho(t) \cdot \mathcal{H}(\ast) + \xi^\rho(x) \cdot \partial_v L(\ast) + \int_a^t \zeta(t)\Big[\Delta\rho(t)\partial_t L(\ast) + \Delta\big(\mathcal{H}(\ast)\big)\Big]\Delta t, \tag{99}$$

where $\mathcal{H}$ *is defined in* (21) *and* $(\ast) = (t, x(t), \nabla x(t))$, *is a constant of motion over the solution of the time scales Euler–Lagrange equation* $(\mathrm{EL}^{\Delta\circ\nabla})$, *i.e., that*

$$\Delta\left[I\left(\cdot\,, x(\cdot), \nabla x(\cdot)\right)\right](t) = 0, \tag{100}$$

for all solutions x *of* $(\mathrm{EL}^{\Delta\circ\nabla})$ *and any* $t \in \mathbb{T}_\kappa^\kappa$. .

7.3. **A time scales Noether's theorem for the nabla shifted calculus of variations.** Consider the following functional $\mathcal{L}^\rho_{L,[a,b],\mathbb{T}}, \mathbb{T} : \mathrm{C}^{1,\nabla}_{\mathrm{ld}}(\mathbb{T}) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{L}^\rho_{L,[a,b],\mathbb{T}}(x) = \int_a^b L\left(t, x^\rho(t), \nabla x(t)\right) \nabla t \tag{101}$$

where $L : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Lagrangian on the time scales $\mathbb{T}$.

Theorem 11 (Euler–Lagrange equation)**.** *The critical points of* $\mathcal{L}^\rho_{L,[a,b],\mathbb{T}}$ *are solutions of the following Euler–Lagrange equation*

$$\nabla\left[\frac{\partial L}{\partial v}(t, x^\rho(t), \nabla x(t))\right] = \frac{\partial L}{\partial x}(t, x^\rho(t), \nabla x(t)), \tag{$\mathrm{EL}^{\nabla\circ\nabla}$}$$

for every $t \in \mathbb{T}_\kappa$.

Theorem 12 (Noether's Theorem - ρ-shifted case)**.** *Let* $\mathbb{T}$ *be a time scales and let* $G = \{g_s(t,x) = (g_s^0(t), g_s^1(x))\}_{s\in\mathbb{R}}$ *a* $(\nabla, \mathbb{T})$*-admissible projectable group of transformations be a variational symmetry of* $\mathcal{L}^\rho_{L,[a,b],\mathbb{T}}$ *and let the corresponding infinitesimal generator given by*

$$X = \zeta(t)\frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x}. \tag{102}$$

Then, the function

$$\bar{I}(t, x^\rho, v) = -\zeta(t)\cdot\bar{\mathscr{H}}(\ast_\rho) + \xi(x)\cdot\partial_v L(\ast_\rho) + \int_a^t \zeta^\rho(t)\Big[\partial_t L(\ast_\rho) + \nabla\big(\bar{\mathscr{H}}(\ast_\rho)\big)\Big]\nabla t, \tag{103}$$

where $\bar{\mathscr{H}} : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ *is defined by* $\bar{\mathscr{H}}(t,x,v) = \mathcal{H}(t,x,v) - \partial_t L(t,x,v)\nu(t)$ *and* $(\ast_\rho) = (t, x^\rho(t), \nabla x(t))$, *is a constant of motion over the solution of the time scales Euler–Lagrange equation, i.e., that*

$$\nabla\left[I\left(\cdot\,, x(\cdot), \nabla x(\cdot)\right)\right](t) = 0, \tag{104}$$

for all solutions x *of* $(\mathrm{EL}^{\nabla\circ\nabla})$ *and any* $t \in \mathbb{T}_\kappa$.

7.4. **Example and simulations.** Consider the time scales $\mathbb{T} = \{t_k = a + kh,\ k \in \mathbb{N}\}$ and the following Lagrangian [22]

$$L(t,x,v) = L(t,x,v) = \frac{t}{2}\left(v^2 - 2e^x\right),$$

then the corresponding Euler–Lagrange equation is given by

$$\Delta(t\nabla x) = -te^x.$$

The family of transformation $G = \{g_s(t,x) = (te^s, x-2s)\}_{s\in\mathbb{R}}$ where its infinitesimal generator is given by

$$X = t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}$$

is a variational symmetry of L. Indeed, we have

$$L\left(e^s t, x-2s, \frac{\nabla x}{e^s}\right)\cdot e^s = \frac{e^s t}{2}\left[\left(\frac{\nabla x}{e^s}\right)^2 - 2e^{x-2s}\right]e^s = L\left(t,x,\nabla x\right).$$

Therefore, Noether's theorem gives the following conservation law

$$\bar{I}(t,x,v) = -\rho(t)\cdot\frac{t}{2}\left(v^2+2e^x\right) - 2tv + \int_a^t \frac{t}{2}\left[\left(v^2-2e^x\right)+\Delta\left(t\left(v^2+2e^x\right)\right)\right]\Delta t. \tag{105}$$

In a shifted case, consider the following Lagrangian

$$L(t,x^\rho,v) = \frac{t}{2}\left(v^2-2e^{x^\rho}\right), \tag{106}$$

with the (shifted) Euler–Lagrange equation is given by

$$\nabla(\nabla x) = -te^{x^\rho}. \tag{107}$$

Using the invariance criterion of the functional $\mathcal{L}^\rho_{L,[a,b],\mathbb{T}}$ given by [27]

$$\zeta\frac{\partial L}{\partial t} + \xi^\rho\frac{\partial L}{\partial x} + (\nabla\xi - \nabla x\nabla\zeta)\frac{\partial L}{\partial v} + \nabla\zeta\cdot L = 0, \tag{108}$$

one check that the family of transformation $G = \{g_s(t,x) = (te^s, x-2s)\}_{s\in\mathbb{R}}$ is also a variational symmetry of (106). The Noether's theorem gives the following conservation law:

$$\bar{I}(t,x^\rho,v) = -t\cdot\bar{\mathscr{H}}(t,x^\rho,v) - 2tv + \int_a^t \rho(t)\left[\frac{1}{2}\left(v^2-2e^x\right)+\nabla\bar{\mathscr{H}}(t,x^\rho,v)\right]\nabla t, \tag{109}$$

where $\bar{\mathscr{H}}(t,x^\rho,v) = \frac{\rho(t)}{2}\left(v^2-2e^{x^\rho}\right)+2te^{x^\rho}$.

Simulations of the quantities $\bar{I}(t,x,v)$ and $\bar{I}(t,x^\rho,v)$ over $\mathbb{T}$ with $x_0 = 1, v_0 = 0.1$ and $h = 10$ give:

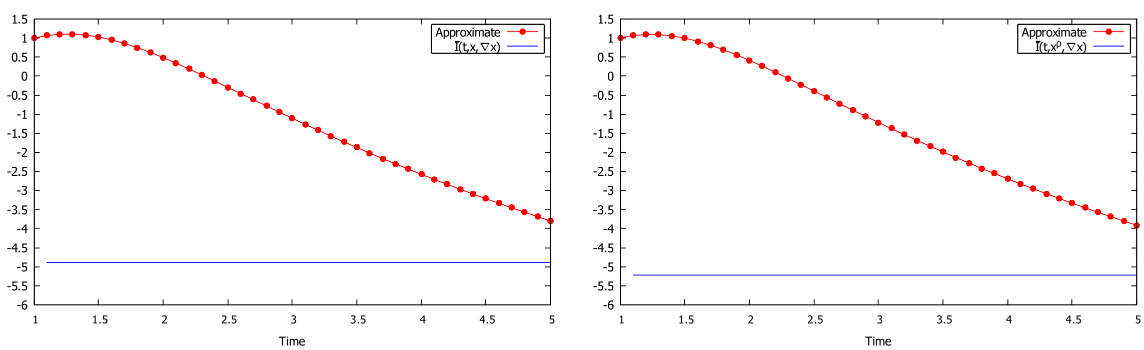


FIGURE 11. The simulation of $\bar{I}(t,x,v)$ and $\bar{I}(t,x^\rho,v)$.

7.5. **Comparison with the work of N. Martins and D.F.M. Torres.** Applying the result of N. Martins and D.F.M. Torres in [27, Theorem 3.4] on our example, they assert that the quantity

$$M(t,x^\rho,v) = -t\cdot\frac{\rho(t)}{2}\left(v^2-2e^{x^\rho}\right)+2te^{x^\rho}-2tv \tag{110}$$

is constant of motion over the solutions of (107).

The simulations then gives the following results:

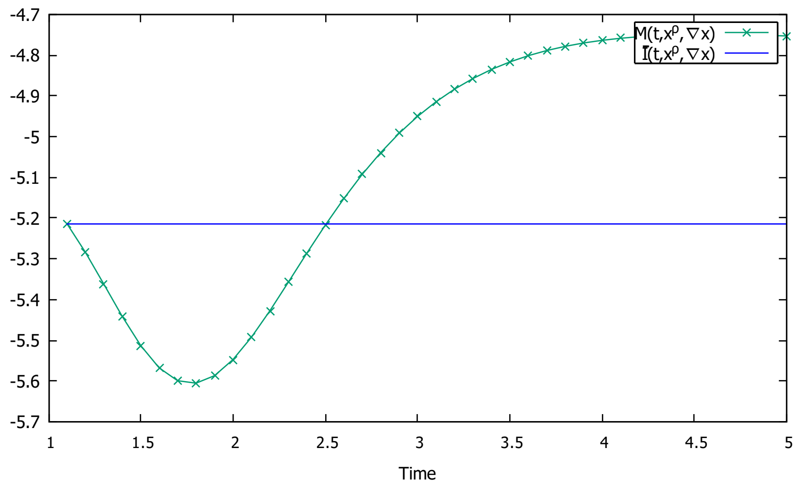


FIGURE 12. $x_0 = 1, v = 0.1, h = 0.1$

We clearly see that M is not constant.

8. PROOF OF THE MAIN RESULTS USING THE CAPUTO DUALITY PRINCIPLE

8.1. The nonshifted case.

Lemma 10. *Let $L : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous Lagrangian. Then*

$$\int_a^b L\left(t, x(t), \nabla x(t)\right) \nabla t = \int_\alpha^\beta L^* \left(\tau, x^*(\tau), \hat{\Delta} x^*(\tau)\right) \hat{\Delta}\tau, \tag{111}$$

for all function $x \in C^{1,\nabla}_{\mathrm{ld}}(\mathbb{T})$, where $\alpha = -b$ and $\beta = -a$.

The proof of this lemma is immediate from the last property of Proposition 3 and Definition 11 and the point of this lemma in the following (see [27]). $x \in C^{1,\nabla}_{\mathrm{ld}}(\mathbb{T})$ is a critical point of the functional (97) if and only if $x^* \in C^{1,\hat{\Delta}}_{\mathrm{rd}}(\mathbb{T}^*)$ is a critical point of the functional

$$\mathcal{L}_{[\alpha,\beta],\mathbb{T}^*}(y) = \int_\alpha^\beta L^* \left(\tau, y(\tau), \hat{\Delta} y(\tau)\right) \hat{\Delta}\tau, \quad \text{with} \quad \alpha = -b,\ \beta = -a. \tag{112}$$

8.1.1. *Proof of Euler–Lagrange equation.* The proof of follows from the previous lemma. Let $\hat{\sigma}$ is $\hat{\nabla}$-differentiable on $(\mathbb{T}^\kappa)^*$, Let $x^* \in C^{1,\hat{\Delta}}_{\mathrm{rd}}(\mathbb{T}^*)$ be a critical point of the functionl (112), then

$$\hat{\nabla} \left[\frac{\partial L^*}{\partial v} \left(\tau, x^*(\tau), \hat{\Delta} x^*(\tau)\right) \right] = \hat{\nabla}\hat{\sigma}(\tau) \frac{\partial L^*}{\partial x} \left(\tau, x^*(\tau), \hat{\Delta} x^*(\tau)\right), \tag{113}$$

for all $\tau \in (\mathbb{T}^\kappa_\kappa)^*$.
According to the relations (94) and (96), we have that:

$$\partial_x L^* \left(\tau, x^*(\tau), \hat{\Delta} x^*(\tau)\right) = \partial_x L\left(-\tau, x(-\tau), \nabla x(-\tau)\right) \tag{114}$$

$$\partial_v L^* \left(\tau, x^*(\tau), \hat{\Delta} x^*(\tau)\right) = -\partial_v L\left(-\tau, x(-\tau), \nabla x(-\tau)\right). \tag{115}$$

Now, let us take $P(\tau) = \partial_v L^*\left(\tau, x^*(\tau), \hat{\Delta}x^*(\tau)\right)$ and $Q(\tau) = \partial_v L\left(\tau, x(\tau), \nabla x(\tau)\right)$, then the equation (115) can be written as

$$P(\tau) = -Q^*(\tau),$$

so that,

$$\hat{\nabla}P(\tau) = -\hat{\nabla}Q^*(\tau) = \Delta Q(-\tau).$$

Since $\tau \in (\mathbb{T}^\kappa_\kappa)^*$, then we get by taking $t = -\tau$ that $t \in \mathbb{T}^\kappa_\kappa$ and with the help of Proposition 3 we deduce that ρ is Δ-differentiable at t. Finally, using the relation $\hat{\nabla}\hat{\sigma}(\tau) = \Delta\rho(-\tau)$ and (113) we obtain

$$\Delta Q(t) = \Delta\rho(t)\frac{\partial L}{\partial x}(t, x(t), \nabla x(t)).$$

This complete the proof.

8.1.2. *Proof of Noether's theorem.* It follows from [27] that, if G is a variational symmetry of the functional (97) with the corresponding infinitesimal generator $X = \zeta\partial_t + \xi\partial_x$, then the group G^* defined by

$$\begin{cases} (g^*)^0_s(\tau) = \tau - s\zeta^*(\tau), \\ (g^*)^1_s(x) = y + s\xi^*(y), \end{cases} \tag{116}$$

where, $\zeta^*(\tau) = \zeta(-\tau)$ and $\xi^*(y) = \xi(y)$ is a variational symmetry of the functional (112). Then, applying Theorem 7 to the functional (112), we have from (20) that the function

$$I^*(\tau, x^*) = (\zeta^*)^{\hat{\sigma}}(\tau)\bar{\mathcal{H}}^*[x^*](\tau) + \xi^{\hat{\sigma}}(x^*)\cdot\partial_v L^*[x^*](\tau) - \int_a^\tau \zeta^*\left[\hat{\nabla}\hat{\sigma}\partial_t L^*[x^*](\tau) + \hat{\nabla}\left(\bar{\mathcal{H}}^*[x^*](\tau)\right)\right]\hat{\nabla}\tau, \tag{117}$$

is constant over the solution of (113), i.e.,

$$\hat{\nabla}\left[(\zeta^*)^{\hat{\sigma}}(\tau)\bar{\mathcal{H}}^*[x^*](\tau) + \xi^{\hat{\sigma}}(x^*)\cdot\partial_v L^*[x^*](\tau)\right] - \zeta^*\hat{\nabla}\hat{\sigma}\partial_t L^*[x^*](\tau) - \zeta^*\hat{\nabla}\left(\bar{\mathcal{H}}^*[x^*](\tau)\right) = 0, \tag{118}$$

where $[x^*](\tau) = \left(\tau, x^*(\tau), \hat{\Delta}x^*(\tau)\right)$ and $\bar{\mathcal{H}}^*[x^*](\tau) = L^*[x^*](\tau) - \partial_v L^*[x^*](\tau)\cdot\hat{\Delta}x^*(\tau)$.
For simplicity, let $[x](\tau) = (\tau, x(\tau), \nabla x(\tau))$, $Q(\tau) = \partial_v L[x](\tau)$, $T(\tau) = \partial_t L[x](\tau)$ and $Z(\tau) = Q(\tau)\cdot\nabla x(\tau) - L[x](\tau)$. Taking in your maid the relations:

$$\begin{aligned}
(\zeta^*)^{\hat{\sigma}}(\tau) &= (\zeta^\rho)^*(\tau), \quad \xi^{\hat{\sigma}}(x^*(\tau)) = (\xi^\rho\circ x)^*(\tau), \\
\hat{\Delta}x^*(\tau) &= -\nabla x(-\tau), \quad \hat{\nabla}\hat{\sigma}(\tau) = (\Delta\rho)^*(\tau) = \Delta\rho(-\tau) \\
\partial_v L^*[x^*](\tau) &= -\partial_v L(-\tau, x(-\tau), \nabla x(-\tau)) = -\partial_v L[x](-\tau) = -Q^*(\tau) \\
\partial_t L^*[x^*](\tau) &= -\partial_t L(-\tau, x(-\tau), \nabla x(-\tau)) = -\partial_t L[x](-\tau) = -T^*(\tau) \\
\bar{\mathcal{H}}^*[x^*](\tau) &= Q(-\tau)\cdot\nabla x(-\tau) - L[x](-\tau) = Z^*(\tau).
\end{aligned}$$

we have the term $\hat{\nabla}[\cdots]$ in (118) becomes

$$\begin{aligned}
\hat{\nabla}\left((\zeta^*)^{\hat{\sigma}}(\tau)Z^*(\tau) + \xi^{\hat{\sigma}}(x^*)\cdot\partial_v L^*[x^*](\tau)\right) &= \hat{\nabla}(\zeta^\rho\cdot Z)^*(\tau) - \hat{\nabla}\Big((\xi^\rho\circ x)^*\cdot Q^*\Big)(\tau) \\
&= -\Delta(\zeta^\rho\cdot Z)(-\tau) + \Delta(\xi^\rho(x)\cdot Q)(-\tau),
\end{aligned}$$

and the rest terms, we have

$$\zeta^*(\tau)\hat{\nabla}\hat{\sigma}(\tau)\partial_t L^*[x^*](\tau) = -\left(\zeta \cdot \Delta\rho \cdot T\right)^*(\tau),$$
$$\zeta^*(\tau)\hat{\nabla}\left(\bar{\mathcal{H}}^*[x^*](\tau)\right) = -\left(\zeta \cdot \Delta Z\right)^*(\tau).$$

Substituting all of these formulas into (118) and replacing $-\tau$ by $t \in \mathbb{T}_\kappa^\kappa$ gives

$$\Delta\Big(-\zeta^\rho(t)\cdot Z(t) + \xi^\rho(x(t))\cdot Q(t)\Big) + \zeta(t)\Big(\Delta\rho(t)\cdot T(t) + \Delta Z(t)\Big) = 0.$$

We complete the proof by taking the Δ-antiderivative of the latter expression.

9. Proof of the technical Lemma

Proof of Lemma 3. Using the time scales chain rule, we obtain

$$\Delta_{\widetilde{\mathbb{T}}_s}\left(g_s^1 \circ x \circ (g_s^0)^{-1}\right)(\tau) = \Delta\left(g_s^1 \circ x\right)(t)\Delta_{\widetilde{\mathbb{T}}_s}\left(g_s^0\right)^{-1}(\tau).$$

Then, using the time scales derivative formula for inverse function, we obtain

$$\Delta_{\widetilde{\mathbb{T}}_s}\left(g_s^1 \circ x \circ (g_s^0)^{-1}\right)(\tau) = \Delta\left(g_s^1 \circ x\right)(t)\frac{1}{\Delta g_s^0(t)}. \tag{119}$$

Using the change of variable formula for time scales integrals, we obtain

$$\begin{aligned}\int_{\tau_a}^{\tau_b} L_s\left(\tau, g_s^1 \circ x \circ (g_s^0)^{-1}(\tau), \Delta_{\widetilde{\mathbb{T}}_s}\left(g_s^1 \circ x \circ (g_s^0)^{-1}\right)(\tau)\right)\Delta_{\widetilde{\mathbb{T}}_s}\tau \\ = \int_a^b L_s\left(g_s^0(t), (g_s^1 \circ x)(t), \Delta\left(g_s^1 \circ x\right)(t)\frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t)\Delta t.\end{aligned}$$

Finally, using the invariance condition in Equation (18), we obtain the result. □

Proof of Lemma 5. For the necessary condition, let $\gamma = (t, x) \in \mathcal{F}$ be a critical point of $\mathcal{L}_{\mathbb{L}}$. Then, from Equation (EL$^{\nabla\circ\Delta}$), it satisfies the following Euler–Lagrange equations

$$(\mathrm{EL}^{\nabla\circ\Delta})_{\mathbb{L}} \begin{cases} \nabla\left[\dfrac{\partial \mathbb{L}}{\partial v}(\bar{\star}_\tau)\right] = \nabla\sigma(\tau)\dfrac{\partial \mathbb{L}}{\partial x}(\bar{\star}_\tau), \\ \nabla\left[\dfrac{\partial \mathbb{L}}{\partial w}(\bar{\star}_\tau)\right] = \nabla\sigma(\tau)\dfrac{\partial \mathbb{L}}{\partial t}(\bar{\star}_\tau), \end{cases} \tag{120}$$

for all $\tau \in \mathbb{T}_\kappa^\kappa$, where $\star_\tau = (t(\tau), (x \circ t)(\tau), \Delta[t](\tau), \Delta[x \circ t](\tau))$.

By definition, we have

$$\frac{\partial \mathbb{L}}{\partial t}(\star_\tau) = \frac{\partial L}{\partial t}(\star_\tau)\Delta[t](\tau), \qquad \frac{\partial \mathbb{L}}{\partial w}(\star_\tau) = L(\star_\tau) - \Delta[x \circ t](\tau)\frac{1}{\Delta[t](\tau)}\frac{\partial L}{\partial v}(\star_\tau), \tag{121}$$

$$\frac{\partial \mathbb{L}}{\partial x}(\star_\tau) = \frac{\partial L}{\partial x}(\star_\tau)\Delta[t](\tau), \qquad \frac{\partial \mathbb{L}}{\partial v}(\star_\tau) = \frac{\partial L}{\partial v}(\star_\tau). \tag{122}$$

As $\gamma \in \mathcal{F}$, we have $(\star_\tau) = (\tau, x(\tau), \Delta x(\tau))$. As a consequence, the first Euler–Lagrange equation is equivalent to

$$\nabla\left[\frac{\partial L}{\partial v}(\star_\tau)\right] = \nabla\sigma(\tau)\frac{\partial L}{\partial x}(\star_\tau). \tag{123}$$

for all $\tau \in \mathbb{T}^\kappa_\kappa$ and the second Euler–Lagrange equation is equivalent to

$$\nabla\sigma(\tau)\frac{\partial L}{\partial t}(\star_\tau) + \nabla\left(\Delta x(\tau)\frac{\partial L}{\partial v}(\star_\tau) - L(\star_\tau)\right) = 0, \tag{124}$$

for all $\tau \in \mathbb{T}^\kappa_\kappa$, which corresponds to the condition ($\circledast$). As Equation (123) is the Euler–Lagrange equation associated with the Lagrangian functional $\mathcal{L}_{L,[a,b],\mathbb{T}}$, we obtain that x is a critical point of $\mathcal{L}_{L,[a,b],\mathbb{T}}$ and ($\circledast$) is satisfied.

For the sufficient condition, let us assume that ($\circledast$) is satisfied and let x be a critical point of $\mathcal{L}_{L,[a,b],\mathbb{T}}$ and let γ be the path such that $(t,x) \in \mathcal{F}$. The previous computations show that γ satisfies equation (123) by assumption on x and equation (124) by hypothesis. As a consequence, γ is a critical point of $\mathcal{L}_{\mathbb{L}}$. This concludes the proof. □

Proof of Lemma 4. Let $\gamma = (t,x) \in \mathcal{F}$. By definition, we have

$$\mathcal{L}_{\mathbb{L}}(g_s(\gamma)) = \int_a^b \mathbb{L}\left(g_s^0(t(\tau)), (g_s^1\circ x)(t(\tau)), \Delta_{\widetilde{\mathbb{T}}_s} g_s^0(t(\tau)), \Delta_{\widetilde{\mathbb{T}}_s}\left(g_s^1\circ x\right)(t(\tau))\right)\Delta_{\widetilde{\mathbb{T}}_s}\tau. \tag{125}$$

Using the definition of $\mathbb{L}$ and the fact that $t(\tau) = \tau$ and $\Delta g_s^0(\tau) \neq 0$ for all $\tau \in \mathbb{T}^\kappa$, we obtain

$$\mathcal{L}_{\mathbb{L}}(g_s(\gamma)) = \int_a^b L_s\left(g_s^0(\tau), (g_s^1\circ x)(\tau), \Delta\left(g_s^1\circ x\right)(\tau)\frac{1}{\Delta g_s^0(\tau)}\right)\Delta g_s^0(\tau)\Delta\tau. \tag{126}$$

Using the invariance of $\mathcal{L}_{L,[a,b],\mathbb{T}}$ with the Lemma 3, we obtain

$$\mathcal{L}_{\mathbb{L}}(g_s(\gamma)) = \int_a^b L\left(\tau, x(\tau), \Delta x(\tau)\right)\Delta\tau. \tag{127}$$

In consequence, as $\Delta t(\tau) = 1$, we obtain

$$\mathcal{L}_{\mathbb{L}}(g_s(\gamma)) = \int_a^b \mathbb{L}\left(\tau, x(\tau), 1, \Delta x(\tau)\right)d\tau = \mathcal{L}_{\mathbb{L}}(\gamma). \tag{128}$$

This concludes the proof. □

Proof of Lemma 6. Let $s \in \mathbb{R}$. Using the formula $g_s^0 \circ \sigma = \tilde{\sigma}_s \circ g_s^0$, we have that

$$\left[g_s^1\circ x\circ (g_s^0)^{-1}\right]^{\tilde{\sigma}_s}(\tau) = \left[g_s^1\circ x\circ (g_s^0)^{-1}\circ\tilde{\sigma}_s\circ g_s^0\right](t) = \left[g_s^1\circ x\circ\sigma\right](t)$$

Using the formula (119) and the change of variable formula for time scales integrals, we obtain

$$\int_{\tau_a}^{\tau_b} L_s\left(\tau, \left[g_s^1\circ x\circ (g_s^0)^{-1}\right]^{\tilde{\sigma}_s}(\tau), \Delta_{\widetilde{\mathbb{T}}_s}\left[g_s^1\circ x\circ (g_s^0)^{-1}\right](\tau)\right)\Delta_{\widetilde{\mathbb{T}}_s}\tau = \int_{t_a}^{t_b} L_s\left(g_s^0(t), \left[g_s^1\circ x\right]^\sigma(t), \Delta\left[g_s^1\circ x\right](t)\cdot\frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t)\Delta t.$$

□

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