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A B S T R A C T

We devote this paper to the axiomatization and the computability of $\mathbf{PDL}_0^{\Delta}$, a variant of iteration-free **PDL** with fork. Concerning the axiomatization, our results are based on the following: although the program operation of fork is not modally definable in the ordinary language of **PDL**, it becomes definable in a modal language strengthened by the introduction of propositional quantifiers. Instead of using axioms to define the program operation of fork in the language of **PDL** enlarged with propositional quantifiers, we add an unorthodox rule of proof that makes the canonical model standard for the program operation of fork and we use large programs for the proof of the Truth Lemma. Concerning the computability, we prove by a selection procedure that $\mathbf{PDL}_0^{\Delta}$ has a strong finite property, hence is decidable.

1. Introduction

Propositional dynamic logic (**PDL**) is an applied non-classical logic designed for reasoning about the behavior of programs [10]. The definition of its syntax is based on the idea of associating with each program α of some programming language the modal operator $[\alpha]$, formulas of the form $[\alpha]\phi$ being read "every execution of the program α from the present state leads to a state bearing the formula ϕ". Completeness and complexity results for the standard version of **PDL** in which programs are built up from program variables and tests by means of the operations of composition, union and iteration are given in [15,16]. A number of interesting variants have been obtained by extending or restricting the syntax or the semantics of **PDL** in different ways [7,9,14,19].

Some of these variants extend the ordinary semantics of **PDL** by considering sets W of states structured by means of a function $\star$ from the set of all pairs of states into the set of all states [5,11–13]: the state x is the result of applying the function $\star$ to the states y, z iff the information concerning x can be separated in a first part concerning y and a second part concerning z. The binary function $\star$ considered in [5,11] has its origin in the addition of an extra binary operation of fork denoted ∇ in relation algebras: in [5, Section 2], whenever x and y are related via R and z and t are related via S, states in $x \star z$ and states in $y \star t$ are related via $R\nabla S$ whereas in [11, Chapter 1], whenever x and y are related via R and x and z are related via S, x and states in $y \star z$ are related via $R\nabla S$.

This addition of fork in relation algebras gives rise to a variant of **PDL** which includes the program operation of fork denoted Δ. In this variant, for all programs α and β, one can use the modal operator $[\alpha\Delta\beta]$, formulas of the form $[\alpha\Delta\beta]\phi$ being read "every execution in parallel of the programs α and β from the present state leads to a state bearing the formula ϕ". The binary operation of fork ∇ considered in Benevides et al. [5, Section 2] gives rise to **PRSPDL**, a variant of **PDL** with

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fork whose axiomatization has been given in [2]. In this paper, we attack the problem of axiomatizing and deciding $\mathbf{PDL}_0^{\triangle}$, a variant of iteration-free **PDL** with fork whose semantics is based on the interpretation of the binary operation of fork ∇ considered in Frias [11, Chapter 1].

The difficulty in axiomatizing **PRSPDL** or $\mathbf{PDL}_0^{\triangle}$ originates in the fact that the program operations of fork considered above are not modally definable in the ordinary language of **PDL**. We overcome this difficulty by means of tools and techniques developed in [1,3,4]. Our results are based on the following: although fork is not modally definable, it becomes definable in a modal language strengthened by the introduction of propositional quantifiers. Instead of using axioms to define the program operation of fork in the language of **PDL** enlarged with propositional quantifiers, we add an unorthodox rule of proof that makes the canonical model standard for the program operation of fork and we use large programs for the proof of the Truth Lemma.

The difficulty in deciding **PRSPDL** or $\mathbf{PDL}_0^{\triangle}$ originates in the semantics of the fork. For instance, in a tableau method, some successors of the current state must be considered together, because they will later be composed by the binary function $\star$. Moreover, in $\mathbf{PDL}_0^{\triangle}$, an additional layer of complexity arises by the fact that the binary modalities $\circ$, $\rhd$ and $\lhd$ are somehow the *inverses* of each other. To overcome all these difficulties, we prove a strong finite model property using a selection procedure which, given a pointed model satisfying a formula, selects the states needed by the formula to be satisfied. We prove that this procedure terminates in a computable deterministic time.

We will first present the syntax (Section 2) and the semantics (Section 3) of $\mathbf{PDL}_0^{\triangle}$ and continue with results concerning the expressivity of $\mathbf{PDL}_0^{\triangle}$ (Section 4), the axiomatization/completeness of $\mathbf{PDL}_0^{\triangle}$ (Sections 5 and 6) and the decidability/complexity of $\mathbf{PDL}_0^{\triangle}$ (Section 7). We assume the reader is at home with tools and techniques in modal logic and dynamic logic. For more on this, see [6,15]. The proofs of some of our results can be found in the Annex.

2. Syntax

This section presents the syntax of $\mathbf{PDL}_0^{\triangle}$. As usual, we will follow the standard rules for omission of the parentheses.

Definition. The set **PRG** of all programs and the set **FRM** of all formulas are inductively defined as follows:

- $\alpha, \beta ::= a \mid (\alpha; \beta) \mid (\alpha \triangle \beta) \mid \phi?$;
- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [\alpha]\phi \mid (\phi \circ \psi) \mid (\phi \rhd \psi) \mid (\phi \lhd \psi)$;

where a ranges over a countably infinite set of program variables and p ranges over a countably infinite set of propositional variables.

We will use $\alpha, \beta, \dots$ for programs and $\phi, \psi, \dots$ for formulas. The other Boolean constructs for formulas are defined as usual. A number of other modal constructs for formulas can be defined in terms of the primitive ones as follows.

Definition. The modal constructs for formulas $\langle\cdot\rangle\cdot$, $(\cdot\bar{\circ}\cdot)$, $(\cdot\bar{\rhd}\cdot)$ and $(\cdot\bar{\lhd}\cdot)$ are defined as follows: $\langle\alpha\rangle\phi ::= \neg[\alpha]\neg\phi$; $(\phi\bar{\circ}\psi) ::= \neg(\neg\phi \circ \neg\psi)$; $(\phi\bar{\rhd}\psi) ::= \neg(\neg\phi \rhd \neg\psi)$; $(\phi\bar{\lhd}\psi) ::= \neg(\neg\phi \lhd \neg\psi)$. Moreover, for all formulas ϕ, let $\phi^0 ::= \neg\phi$ and $\phi^1 ::= \phi$.

It is well worth noting that programs and formulas are finite strings of symbols coming from a countable alphabet. It follows that there are countably many programs and countably many formulas. The construct $\cdot;\cdot$ comes from the class of algebras of binary relations [20]: the program $\alpha;\beta$ firstly executes α and secondly executes β. As for the construct $\cdot\triangle\cdot$, it comes from the class of proper fork algebras [11, Chapter 1]: the program $\alpha\triangle\beta$ performs a kind of parallel execution of α and β. The construct $[\cdot]\cdot$ comes from the language of **PDL** [10,15]: the formula $[\alpha]\phi$ says that "every execution of α from the present state leads to a state bearing the information ϕ". As for the constructs $\cdot\circ\cdot$, $\cdot\rhd\cdot$ and $\cdot\lhd\cdot$, they come from the language of conjugated arrow logic [8,18]: the formula $\phi\circ\psi$ says that "the present state is a combination of states bearing the information ϕ and ψ", the formula $\phi\rhd\psi$ says that "the present state can be combined to its left with a state bearing the information ϕ giving us a state bearing the information ψ" and the formula $\phi\lhd\psi$ says that "the present state can be combined to its right with a state bearing the information ψ giving us a state bearing the information ϕ".

Example. The formula $[a\triangle b](p\circ q)$ says that "the parallel execution of a and b from the present state always leads to a state resulting from the combination of states bearing the information p and q".

Obviously, programs are built up from program variables and tests by means of the constructs $\cdot;\cdot$ and $\cdot\triangle\cdot$. Let $\alpha(\phi_1?, \dots, \phi_n?)$ be a program with $(\phi_1?, \dots, \phi_n?)$ a sequence of some of its tests. The result of the replacement of $\phi_1?, \dots, \phi_n?$ in their places with other tests $\psi_1?, \dots, \psi_n?$ is another program which will be denoted $\alpha(\psi_1?, \dots, \psi_n?)$. Now, we introduce the function f from the set of all programs into itself defined as follows.

Definition. Let f be the function from the set of all programs into itself inductively defined as follows:

- $f(a) = a$;
- $f(\alpha; \beta) = f(\alpha); \top?; f(\beta)$;
- $f(\alpha \Delta \beta) = (f(\alpha); \top?) \Delta (f(\beta); \top?)$;
- $f(\phi?) = \phi?$.

Example. If $\alpha = a \Delta b$, $f(\alpha) = (a; \top?) \Delta (b; \top?)$.

The function f will be used later in our axiomatization and in our completeness proof of $\mathbf{PDL}_0^{\Delta}$. Now, we introduce parametrized actions and admissible forms.

Definition. The set of all parametrized actions and the set of all admissible forms are inductively defined as follows:

- $\breve{\alpha}, \breve{\beta} ::= (\breve{\alpha}; \beta) \mid (\alpha; \breve{\beta}) \mid (\breve{\alpha} \Delta \beta) \mid (\alpha \Delta \breve{\beta}) \mid \neg \breve{\phi}?$;
- $\breve{\phi}, \breve{\psi} ::= \sharp \mid [\breve{\alpha}]\bot \mid (\breve{\phi} \bar{\circ} \psi) \mid (\phi \bar{\circ} \breve{\psi}) \mid (\breve{\phi} \bar{\triangleright} \psi) \mid (\phi \bar{\triangleright} \breve{\psi}) \mid (\breve{\phi} \bar{\triangleleft} \psi) \mid (\phi \bar{\triangleleft} \breve{\psi})$;

where $\sharp$ is a new propositional variable, α, β range over **PRG** and ϕ, ψ range over **FRM**.

We will use $\breve{\alpha}, \breve{\beta}, \ldots$ for parametrized actions and $\breve{\phi}, \breve{\psi}, \ldots$ for admissible forms. It is well worth noting that parametrized actions and admissible forms are finite strings of symbols coming from a countable alphabet. It follows that there are countably many parametrized actions and countably many admissible forms. Remark that in each parametrized action $\breve{\alpha}$, $\sharp$ has a unique occurrence. The result of the replacement of $\sharp$ in its place in $\breve{\alpha}$ with a formula ψ is a program which will be denoted $\breve{\alpha}(\psi)$. As well, remark that in each admissible form $\breve{\phi}$, $\sharp$ has a unique occurrence. The result of the replacement of $\sharp$ in its place in $\breve{\phi}$ with a formula ψ is a formula which will be denoted $\breve{\phi}(\psi)$.

Example. For all programs α, $\alpha; \neg[\neg\sharp?]\bot?$ is a parametrized action whereas for all formulas ϕ, $\phi \bar{\circ} [\neg\sharp?]\bot$ is an admissible form. The result of the replacement of $\sharp$ in its place in $\alpha; \neg[\neg\sharp?]\bot?$ with a formula ψ is the program $\alpha; \neg[\neg\psi?]\bot?$. The result of the replacement of $\sharp$ in its place in $\phi \bar{\circ} [\neg\sharp?]\bot$ with a formula ψ is the formula $\phi \bar{\circ} [\neg\psi?]\bot$.

3. Semantics

Our task is now to present the semantics of $\mathbf{PDL}_0^{\Delta}$.

Definition. A frame is a 3-tuple $\mathcal{F} = (W, R, \star)$ where W is a nonempty set of states, R is a function from the set of all program variables into the set of all binary relations between states and $\star$ is a function from the set of all pairs of states into the set of all sets of states.

We will use $x, y, \ldots$ for states. The set W of states in a frame $\mathcal{F} = (W, R, \star)$ is to be regarded as the set of all possible states in a computation process. The function R from the set of all program variables into the set of all binary relations between states associates with each program variable a the binary relation $R(a)$ on W with $xR(a)y$ meaning that "y can be reached from x by performing program variable a". The function $\star$ from the set of all pairs of states into the set of all sets of states associates with each pair (x, y) of states the subset $x \star y$ of W with $z \in x \star y$ meaning that "z is a combination of x and y".

Definition. A model on the frame $\mathcal{F} = (W, R, \star)$ is a 4-tuple $\mathcal{M} = (W, R, \star, V)$ where V is a valuation on $\mathcal{F}$, i.e. a function from the set of all propositional variables into the set of all sets of states.

In the model $\mathcal{M} = (W, R, \star, V)$, the valuation V associates with each propositional variable p the subset $V(p)$ of W with $x \in V(p)$ meaning that "propositional variable p is true at state x in $\mathcal{M}$". We now define the property "state y can be reached from state x by performing program α in $\mathcal{M}$" — in symbols $xR_{\mathcal{M}}(\alpha)y$ — and the property "formula ϕ is true at state x in $\mathcal{M}$" — in symbols $x \in V_{\mathcal{M}}(\phi)$.

Definition. In model $\mathcal{M} = (W, R, \star, V)$, $R_{\mathcal{M}} : \alpha \mapsto R_{\mathcal{M}}(\alpha) \subseteq W \times W$ and $V_{\mathcal{M}} : \phi \mapsto V_{\mathcal{M}}(\phi) \subseteq W$ are inductively defined as follows:

- $xR_{\mathcal{M}}(a)y$ iff $xR(a)y$;
- $xR_{\mathcal{M}}(\alpha; \beta)y$ iff there exists $z \in W$ such that $xR_{\mathcal{M}}(\alpha)z$ and $zR_{\mathcal{M}}(\beta)y$;
- $xR_{\mathcal{M}}(\alpha \Delta \beta)y$ iff there exists $z, t \in W$ such that $xR_{\mathcal{M}}(\alpha)z$, $xR_{\mathcal{M}}(\beta)t$ and $y \in z \star t$;
- $xR_{\mathcal{M}}(\phi?)y$ iff $x = y$ and $y \in V_{\mathcal{M}}(\phi)$;
- $x \in V_{\mathcal{M}}(p)$ iff $x \in V(p)$;

- $x \notin V_{\mathcal{M}}(\bot)$;
- $x \in V_{\mathcal{M}}(\neg\phi)$ iff $x \notin V_{\mathcal{M}}(\phi)$;
- $x \in V_{\mathcal{M}}(\phi \vee \psi)$ iff either $x \in V_{\mathcal{M}}(\phi)$, or $x \in V_{\mathcal{M}}(\psi)$;
- $x \in V_{\mathcal{M}}([\alpha]\phi)$ iff for all $y \in W$, if $xR_{\mathcal{M}}(\alpha)y$, $y \in V_{\mathcal{M}}(\phi)$;
- $x \in V_{\mathcal{M}}(\phi \circ \psi)$ iff there exists $y, z \in W$ such that $x \in y \star z$, $y \in V_{\mathcal{M}}(\phi)$ and $z \in V_{\mathcal{M}}(\psi)$;
- $x \in V_{\mathcal{M}}(\phi \rhd \psi)$ iff there exists $y, z \in W$ such that $z \in y \star x$, $y \in V_{\mathcal{M}}(\phi)$ and $z \in V_{\mathcal{M}}(\psi)$;
- $x \in V_{\mathcal{M}}(\phi \lhd \psi)$ iff there exists $y, z \in W$ such that $y \in x \star z$, $y \in V_{\mathcal{M}}(\phi)$ and $z \in V_{\mathcal{M}}(\psi)$.

It follows that

Proposition 1. *Let $\mathcal{M} = (W, R, \star, V)$ be a model. For all $x \in W$, we have: $x \in V_{\mathcal{M}}(\langle\alpha\rangle\phi)$ iff there exists $y \in W$ such that $xR_{\mathcal{M}}(\alpha)y$ and $y \in V_{\mathcal{M}}(\phi)$; $x \in V_{\mathcal{M}}(\phi \bar{\circ} \psi)$ iff for all $y, z \in W$, if $x \in y \star z$, either $y \in V_{\mathcal{M}}(\phi)$, or $z \in V_{\mathcal{M}}(\psi)$; $x \in V_{\mathcal{M}}(\phi \bar{\rhd} \psi)$ iff for all $y, z \in W$, if $z \in y \star x$, either $y \in V_{\mathcal{M}}(\phi)$, or $z \in V_{\mathcal{M}}(\psi)$; $x \in V_{\mathcal{M}}(\phi \bar{\lhd} \psi)$ iff for all $y, z \in W$, if $y \in x \star z$, either $y \in V_{\mathcal{M}}(\phi)$, or $z \in V_{\mathcal{M}}(\psi)$.*

Example. Let $\mathcal{M} = (W, R, \star, V)$ be the model defined by:

- $W = \{x, y, z, t\}$;
- $R(a) = \{(x, y)\}$, $R(b) = \{(x, z)\}$, otherwise R is the empty function;
- $y \star z = \{t\}$, otherwise $\star$ is the empty function;
- $V(p) = \{y\}$, $V(q) = \{z\}$, otherwise V is the empty function.

Obviously, $xR_{\mathcal{M}}(a\Delta b)t$ and $t \in V_{\mathcal{M}}(p \circ q)$. Hence, $x \in V_{\mathcal{M}}(\langle a\Delta b\rangle(p \circ q))$.

We now define the property "state z can be reached from state x by performing parametrized action $\breve{\alpha}$ via state y in $\mathcal{M}$" — in symbols $x\breve{R}_{\mathcal{M}}(\breve{\alpha}, y)z$ — and the property "admissible form $\breve{\phi}$ is true at state x via state y in $\mathcal{M}$" — in symbols $x \in \breve{V}_{\mathcal{M}}(\breve{\phi}, y)$.

Definition. In model $\mathcal{M} = (W, R, \star, V)$, $\breve{R}_{\mathcal{M}} : (\breve{\alpha}, y) \mapsto \breve{R}_{\mathcal{M}}(\breve{\alpha}, y) \subseteq W \times W$ and $\breve{V}_{\mathcal{M}} : (\breve{\phi}, y) \mapsto \breve{V}_{\mathcal{M}}(\breve{\phi}, y) \subseteq W$ are inductively defined as follows:

- $x\breve{R}_{\mathcal{M}}(\breve{\alpha}; \beta, y)z$ iff there exists $t \in W$ such that $x\breve{R}_{\mathcal{M}}(\breve{\alpha}, y)t$ and $tR_{\mathcal{M}}(\beta)z$;
- $x\breve{R}_{\mathcal{M}}(\alpha; \breve{\beta}, y)z$ iff there exists $t \in W$ such that $xR_{\mathcal{M}}(\alpha)t$ and $t\breve{R}_{\mathcal{M}}(\breve{\beta}, y)z$;
- $x\breve{R}_{\mathcal{M}}(\breve{\alpha}\Delta\beta, y)z$ iff there exists $t, u \in W$ such that $x\breve{R}_{\mathcal{M}}(\breve{\alpha}, y)t$, $xR_{\mathcal{M}}(\beta)u$ and $z \in t \star u$;
- $x\breve{R}_{\mathcal{M}}(\alpha\Delta\breve{\beta}, y)z$ iff there exists $t, u \in W$ such that $xR_{\mathcal{M}}(\alpha)t$, $x\breve{R}_{\mathcal{M}}(\breve{\beta}, y)u$ and $z \in t \star u$;
- $x\breve{R}_{\mathcal{M}}(\neg\breve{\phi}?, y)z$ iff $x = z$ and $z \in \breve{V}_{\mathcal{M}}(\breve{\phi}, y)$;
- $x \in \breve{V}_{\mathcal{M}}(\sharp, y)$ iff $x = y$;
- $x \in \breve{V}_{\mathcal{M}}([\breve{\alpha}]\bot, y)$ iff there exists $z \in W$ such that $x\breve{R}_{\mathcal{M}}(\breve{\alpha}, y)z$;
- $x \in \breve{V}_{\mathcal{M}}(\breve{\phi} \bar{\circ} \psi, y)$ iff there exists $z, t \in W$ such that $x \in z \star t$, $z \in \breve{V}_{\mathcal{M}}(\breve{\phi}, y)$ and $t \notin V_{\mathcal{M}}(\psi)$;
- $x \in \breve{V}_{\mathcal{M}}(\phi \bar{\circ} \breve{\psi}, y)$ iff there exists $z, t \in W$ such that $x \in z \star t$, $z \notin V_{\mathcal{M}}(\phi)$ and $t \in \breve{V}_{\mathcal{M}}(\breve{\psi}, y)$;
- $x \in \breve{V}_{\mathcal{M}}(\breve{\phi} \bar{\rhd} \psi, y)$ iff there exists $z, t \in W$ such that $t \in z \star x$, $z \in \breve{V}_{\mathcal{M}}(\breve{\phi}, y)$ and $t \notin V_{\mathcal{M}}(\psi)$;
- $x \in \breve{V}_{\mathcal{M}}(\phi \bar{\rhd} \breve{\psi}, y)$ iff there exists $z, t \in W$ such that $t \in z \star x$, $z \notin V_{\mathcal{M}}(\phi)$ and $t \in \breve{V}_{\mathcal{M}}(\breve{\psi}, y)$;
- $x \in \breve{V}_{\mathcal{M}}(\breve{\phi} \bar{\lhd} \psi, y)$ iff there exists $z, t \in W$ such that $z \in x \star t$, $z \in \breve{V}_{\mathcal{M}}(\breve{\phi}, y)$ and $t \notin V_{\mathcal{M}}(\psi)$;
- $x \in \breve{V}_{\mathcal{M}}(\phi \bar{\lhd} \breve{\psi}, y)$ iff there exists $z, t \in W$ such that $z \in x \star t$, $z \notin V_{\mathcal{M}}(\phi)$ and $t \in \breve{V}_{\mathcal{M}}(\breve{\psi}, y)$.

It follows that

Proposition 2. *Let $\mathcal{M} = (W, R, \star, V)$ be a model. Let ψ be a formula. Let $\breve{\alpha}$ be a parametrized action. For all $x, z \in W$, the following conditions are equivalent: $xR_{\mathcal{M}}(\breve{\alpha}(\psi))z$; there exists $y \in W$ such that $x\breve{R}_{\mathcal{M}}(\breve{\alpha}, y)z$ and $y \notin V_{\mathcal{M}}(\psi)$. Let $\breve{\phi}$ be an admissible form. For all $x \in W$, the following conditions are equivalent: $x \in V_{\mathcal{M}}(\breve{\phi}(\psi))$; for all $y \in W$, if $x \in V_{\mathcal{M}}(\breve{\phi}, y)$, $y \in V_{\mathcal{M}}(\psi)$.*

The concept of validity is defined in the usual way as follows.

Definition. We shall say that a formula ϕ is valid in a model $\mathcal{M}$, in symbols $\mathcal{M} \models \phi$, iff $V_{\mathcal{M}}(\phi) = W$. A formula ϕ is said to be valid in a frame $\mathcal{F}$, in symbols $\mathcal{F} \models \phi$, iff for all models $\mathcal{M}$ on $\mathcal{F}$, $\mathcal{M} \models \phi$. We shall say that a formula ϕ is valid in a class $\mathcal{C}$ of frames, in symbols $\mathcal{C} \models \phi$, iff for all frames $\mathcal{F}$ in $\mathcal{C}$, $\mathcal{F} \models \phi$.

For technical reasons, we now consider three particular classes of frames.

Definition. A frame $\mathcal{F}=(W,R,\star)$ is said to be separated iff for all $x,y,z,t,u\in W$, if $u\in x\star y$ and $u\in z\star t$, $x=z$ and $y=t$. We shall say that a frame $\mathcal{F}=(W,R,\star)$ is deterministic iff for all $x,y,z,t\in W$, if $z\in x\star y$ and $t\in x\star y$, $z=t$. A frame $\mathcal{F}=(W,R,\star)$ is said to be serial iff for all $x,y\in W$, there exists $z\in W$ such that $z\in x\star y$.

In separated frames, there is at most one way to decompose a given state; in deterministic frames, there is at most one way to combine two given states; in serial frames, it is always possible to combine two given states. Frias [11, Chapter 1] only considers separated, deterministic and serial frames. Here are some valid formulas and admissible rules of proof.

Proposition 3 *(Validity). The following formulas are valid in the class of all frames:*

(**A**1) $[\alpha](\phi\to\psi)\to([\alpha]\phi\to[\alpha]\psi)$;
(**A**2) $\langle\alpha;\beta\rangle\phi\leftrightarrow\langle\alpha\rangle\langle\beta\rangle\phi$;
(**A**3) $\langle\alpha\Delta\beta\rangle\phi\to\langle\alpha\rangle((\phi\wedge\psi)\lhd\top)\vee\langle\beta\rangle(\top\rhd(\phi\wedge\neg\psi))$;
(**A**4) $\langle\phi?\rangle\psi\leftrightarrow\phi\wedge\psi$;
(**A**5) $(\phi\to\psi)\bar{\circ}\chi\to(\phi\bar{\circ}\chi\to\psi\bar{\circ}\chi)$;
(**A**6) $\phi\bar{\circ}(\psi\to\chi)\to(\phi\bar{\circ}\psi\to\phi\bar{\circ}\chi)$;
(**A**7) $(\phi\to\psi)\bar{\rhd}\chi\to(\phi\bar{\rhd}\chi\to\psi\bar{\rhd}\chi)$;
(**A**8) $\phi\bar{\rhd}(\psi\to\chi)\to(\phi\bar{\rhd}\psi\to\phi\bar{\rhd}\chi)$;
(**A**9) $(\phi\to\psi)\bar{\lhd}\chi\to(\phi\bar{\lhd}\chi\to\psi\bar{\lhd}\chi)$;
(**A**10) $\phi\bar{\lhd}(\psi\to\chi)\to(\phi\bar{\lhd}\psi\to\phi\bar{\lhd}\chi)$;
(**A**11) $\phi\circ\neg(\phi\rhd\neg\psi)\to\psi$;
(**A**12) $\phi\rhd\neg(\phi\circ\neg\psi)\to\psi$;
(**A**13) $\neg(\neg\phi\lhd\psi)\circ\psi\to\phi$;
(**A**14) $\neg(\neg\phi\circ\psi)\lhd\psi\to\phi$;
(**A**15) $[(\alpha;\phi?)\Delta(\beta;\psi?)](\phi\circ\psi)$;
(**A**16) $\langle\alpha(\phi?)\rangle\psi\to\langle\alpha((\phi\wedge\chi)?)\rangle\psi\vee\langle\alpha((\phi\wedge\neg\chi)?)\rangle\psi$;
(**A**17) $\langle f(\alpha)\rangle\phi\leftrightarrow\langle\alpha\rangle\phi$.

Proposition 4 *(Validity). The following formula is valid in the class of all separated frames:*

(**A**18) $p\circ q\to(p\bar{\circ}\bot)\wedge(\bot\bar{\circ}q)$.

Proposition 5 *(Admissibility). The following rules of proof preserve validity in the class of all frames:*

(**MP**) *from ϕ and $\phi\to\psi$, infer ψ;*
(**N**) *from ϕ, infer $[\alpha]\phi$; from ϕ, infer $\phi\bar{\circ}\psi$; from ϕ, infer $\psi\bar{\circ}\phi$.*

(**A**1) is the distribution axiom of **PDL**, (**A**2) is the composition axiom, (**A**4) is the test axiom, (**A**5)–(**A**10) are the distribution axioms of conjugated arrow logic and (**A**11)–(**A**14) are the tense axioms of conjugated arrow logic whereas (**A**3) and (**A**15)–(**A**18) are axioms concerning specific properties of the program operation of fork or the constructs $\cdot\circ\cdot$, $\cdot\rhd\cdot$ and $\cdot\lhd\cdot$. (**MP**) is the modus ponens rule of proof and (**N**) is the necessitation rule of proof. They are probably familiar to the reader. As for the following rule of proof, it concerns specific properties of the program operation of fork and the constructs $\cdot\rhd\cdot$ and $\cdot\lhd\cdot$.

Proposition 6 *(Admissibility). The following rule of proof preserves validity in the class of all separated frames:*

(**FOR**) *from $\{\breve{\phi}(\langle\alpha\rangle((\psi\wedge p)\lhd\top)\vee\langle\beta\rangle(\top\rhd(\psi\wedge\neg p))):\ p$ is a propositional variable$\}$, infer $\breve{\phi}(\langle\alpha\Delta\beta\rangle\psi)$.*

Proof. Suppose that for all propositional variables p, $\breve{\phi}(\langle\alpha\rangle((\psi\wedge p)\lhd\top)\vee\langle\beta\rangle(\top\rhd(\psi\wedge\neg p)))$ is valid in the class of all separated frames. Suppose $\breve{\phi}(\langle\alpha\Delta\beta\rangle\psi)$ is not valid in the class of all separated frames. Hence, there exists a separated model $\mathcal{M}=(W,R,\star,V)$ and there exists $x\in W$ such that $x\notin V_{\mathcal{M}}(\breve{\phi}(\langle\alpha\Delta\beta\rangle\psi))$. By Proposition 2, there exists $y\in W$ such that $x\in\breve{V}_{\mathcal{M}}(\breve{\phi},y)$ and $y\notin V_{\mathcal{M}}(\langle\alpha\Delta\beta\rangle\psi)$. Let p be a propositional variable not occurring in $\breve{\phi},\alpha,\beta,\psi$ and $V':\ q\mapsto V'(q)\subseteq W$ be such that $V'\sim_p V$ and $V'(p)=\{z$: there exists $t,u\in W$ such that $yR_{\mathcal{M}}(\beta)u$ and $z\in t\star u\}$. Since $x\in\breve{V}_{\mathcal{M}}(\breve{\phi},y)$, $x\in\breve{V}_{(W,R,\star,V')}(\breve{\phi},y)$. Since for all propositional variables p, $\breve{\phi}(\langle\alpha\rangle((\psi\wedge p)\lhd\top)\vee\langle\beta\rangle(\top\rhd(\psi\wedge\neg p)))$ is valid in the class of all separated frames and $\mathcal{M}$ is separated, $x\in V_{(W,R,\star,V')}(\breve{\phi}(\langle\alpha\rangle((\psi\wedge p)\lhd\top)\vee\langle\beta\rangle(\top\rhd(\psi\wedge\neg p))))$. By Proposition 2, since $x\in\breve{V}_{(W,R,\star,V')}(\breve{\phi},y)$, $y\in V_{(W,R,\star,V')}(\langle\alpha\rangle((\psi\wedge p)\lhd\top)\vee\langle\beta\rangle(\top\rhd(\psi\wedge\neg p)))$. Thus, either $y\in V_{(W,R,\star,V')}(\langle\alpha\rangle((\psi\wedge p)\lhd\top))$, or $y\in V_{(W,R,\star,V')}(\langle\beta\rangle(\top\rhd(\psi\wedge\neg p)))$.

Case $y\in V_{(W,R,\star,V')}(\langle\alpha\rangle((\psi\wedge p)\lhd\top))$. Hence, there exists $z\in W$ such that $yR_{(W,R,\star,V')}(\alpha)z$ and $z\in V_{(W,R,\star,V')}((\psi\wedge p)\lhd\top)$. Thus, there exists $t,u\in W$ such that $t\in z\star u$ and $t\in V_{(W,R,\star,V')}(\psi\wedge p)$. Therefore, $t\in V_{(W,R,\star,V')}(\psi)$ and there exists $v, w \in W$ such that $yR_{\mathcal{M}}(\beta)w$ and $t \in v \star w$. Since $t \in z \star u$ and $\mathcal{M}$ is separated, $w = u$. Since $yR_{\mathcal{M}}(\beta)w$, $yR_{\mathcal{M}}(\beta)u$. Since p does not occur in α, $V' \sim_p V$ and $yR_{(W,R,\star,V')}(\alpha)z$, $yR_{\mathcal{M}}(\alpha)z$. Since $yR_{\mathcal{M}}(\beta)u$ and $t \in z \star u$, $yR_{\mathcal{M}}(\alpha\Delta\beta)t$. Since p does not occur in ψ, $V' \sim_p V$ and $t \in V_{(W,R,\star,V')}(\psi)$, $t \in V_{\mathcal{M}}(\psi)$. Since $yR_{\mathcal{M}}(\alpha\Delta\beta)t$, $y \in V_{\mathcal{M}}(\langle\alpha\Delta\beta\rangle\psi)$: a contradiction.

Case $y \in V_{(W,R,\star,V')}(\langle\beta\rangle(\top \rhd (\psi \wedge \neg p)))$. Hence, there exists $z \in W$ such that $yR_{(W,R,\star,V')}(\beta)z$ and $z \in V_{(W,R,\star,V')}(\top \rhd (\psi \wedge \neg p))$. Thus, there exists $t, u \in W$ such that $u \in t \star z$ and $u \in V_{(W,R,\star,V')}(\psi \wedge \neg p)$. Therefore, for all $v, w \in W$, if $yR_{\mathcal{M}}(\beta)w$, $u \notin v \star w$. Since $u \in t \star z$, not $yR_{\mathcal{M}}(\beta)z$. Since p does not occur in β and $V' \sim_p V$, not $yR_{(W,R,\star,V')}(\beta)z$: a contradiction. $\dashv$

There is an important point we should make: (**FOR**) is an infinitary rule of proof, i.e. it has an infinite set of formulas as preconditions. In some ways, it is similar to the rule for intersection from [3,4].

4. Expressivity

This section studies the expressivity of $\mathbf{PDL}_0^{\Delta}$.

Definition. Let $\mathcal{C}$ be a class of frames. We shall say that $\mathcal{C}$ is modally definable by the formula ϕ iff for all frames $\mathcal{F}$, $\mathcal{F}$ is in $\mathcal{C}$ iff $\mathcal{F} \models \phi$.

The following propositions show elementary classes of frames that are modally definable.

Proposition 7. *The elementary classes of frames defined by the first-order sentences in the hereunder table are modally definable by the associated formulas.*

1.	$\forall x\, \exists y\; y \in x \star x$	$\langle\top?\Delta\top?\rangle\top$
2.	$\forall x\, \forall y\, \forall z\; (y \in x \star x \wedge z \in x \star x \to y = z)$	$\langle\top?\Delta\top?\rangle p \to [\top?\Delta\top?]p$
3.	$\forall x\, \forall y\; (y \in x \star x \to x \in x \star y)$	$p \to [\top?\Delta\top?](p \rhd p)$
4.	$\forall x\, \forall y\; (y \in x \star x \to x \in y \star x)$	$p \to [\top?\Delta\top?](p \lhd p)$
5.	$\forall x\, \forall y\, \forall z\; (z \in x \star y \leftrightarrow z \in y \star x)$	$p \circ q \leftrightarrow q \circ p$
6.	$\forall x\, \exists y\, \exists z\; x \in y \star z$	$\top \circ \top$
7.	$\forall x\, \exists y\, \exists z\; y \in z \star x$	$\top \rhd \top$
8.	$\forall x\, \exists y\, \exists z\; z \in x \star y$	$\top \lhd \top$
9.	$\forall x\, \forall y\, \forall z\, \forall t\; (t \in (x \star y) \star z \leftrightarrow t \in x \star (y \star z))$	$(p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$
10.	$\forall x\, \forall y\, \forall z\; x \notin y \star z$	$\bot \bar{\circ} \bot$

Proposition 8. *The class of all separated frames is modally definable by the formula $p \circ q \to (p \bar{\circ} \bot) \wedge (\bot \bar{\circ} q)$.*

The following proposition shows an elementary class of frames that is not modally definable.

Proposition 9. *The class of all deterministic frames is not modally definable.*

As for the class of all serial frames,

Proposition 10. *The class of all serial frames is not modally definable.*

In other respect, the formula $\langle\phi?\rangle\psi \leftrightarrow \phi \wedge \psi$, being valid in the class of all frames, seems to indicate that for all formulas, there exists an equivalent test-free formula. It is interesting to observe that this assertion is false.

Proposition 11. *For all test-free formulas ϕ, $\langle\top?\Delta\top?\rangle\top \leftrightarrow \phi$ is not valid in the class of all separated deterministic frames.*

The following proposition illustrates the fact that the program operation of fork cannot be defined from the fork-free fragment of the language.

Proposition 12. *Let a be a program variable. For all fork-free formulas ϕ, $\langle a\Delta a\rangle\top \leftrightarrow \phi$ is not valid in the class of all separated deterministic frames.*

The following proposition illustrates the fact that, in the presence of propositional quantifiers, the program operation of fork becomes definable from the fork-free fragment of the language in the class of all separated frames.

Proposition 13. *Let $\mathcal{M} = (W, R, \star, V)$ be a separated model and $x \in W$. For all admissible forms $\breve{\phi}$, for all programs α, β, for all formulas ψ and for all propositional variables p, if p does not occur in $\breve{\phi}, \alpha, \beta, \psi$, the following conditions are equivalent:* (1) $x \in V_{\mathcal{M}}(\breve{\phi}(\langle \alpha \Delta \beta \rangle \psi))$*;* (2) *for all $V' : q \mapsto V'(q) \subseteq W$, if $V' \sim_p V$, $x \in V_{(W,R,\star,V')}(\breve{\phi}(\langle \alpha \rangle ((\psi \wedge p) \lhd \top) \vee \langle \beta \rangle (\top \rhd (\psi \wedge \neg p))))$.*

Proof. (1) → (2). By Proposition 2. Left to the reader.

(2) → (1). Similar to the proof of Proposition 6. ⊣

More precisely, in the presence of propositional quantifiers, the formulas $\langle \alpha \Delta \beta \rangle \phi$ and $\forall p(\langle \alpha \rangle ((\phi \wedge p) \lhd \top) \vee \langle \beta \rangle (\top \rhd (\phi \wedge \neg p)))$ are logically equivalent in the class of all separated frames. The implication $\langle \alpha \Delta \beta \rangle \phi \to \forall p(\langle \alpha \rangle ((\phi \wedge p) \lhd \top) \vee \langle \beta \rangle (\top \rhd (\phi \wedge \neg p)))$ can be expressed without propositional quantifiers by formulas: $\langle \alpha \Delta \beta \rangle \phi \to \langle \alpha \rangle ((\phi \wedge \psi) \lhd \top) \vee \langle \beta \rangle (\top \rhd (\phi \wedge \neg \psi)))$. See axiom (**A**3) in Proposition 3. As for the implication $\forall p(\langle \alpha \rangle ((\phi \wedge p) \lhd \top) \vee \langle \beta \rangle (\top \rhd (\phi \wedge \neg p))) \to \langle \alpha \Delta \beta \rangle \phi$, it can be expressed by a rule of proof. The simplest form of such a rule of proof is: from $\{\langle \alpha \rangle ((\phi \wedge p) \lhd \top) \vee \langle \beta \rangle (\top \rhd (\phi \wedge \neg p)) : p$ is a propositional variable$\}$, infer $\langle \alpha \Delta \beta \rangle \phi$. See Proposition 6.

In **PRSPDL**, the variant of **PDL** introduced by Benevides et al. [5], storing and recovering programs are considered. Within our context, let us momentarily add to the syntax the programs s_1, s_2, r_1 and r_2 with intended semantics in a model $\mathcal{M} = (W, R, \star, V)$ defined as follows:

- $xR_{\mathcal{M}}(s_1)y$ iff there exists $z \in W$ such that $y \in x \star z$;
- $xR_{\mathcal{M}}(s_2)y$ iff there exists $z \in W$ such that $y \in z \star x$;
- $xR_{\mathcal{M}}(r_1)y$ iff there exists $z \in W$ such that $x \in y \star z$;
- $xR_{\mathcal{M}}(r_2)y$ iff there exists $z \in W$ such that $x \in z \star y$.

The following propositions illustrate the fact that the programs s_1, s_2, r_1 and r_2 cannot be defined from our language.

Proposition 14. *Let $i \in \{1, 2\}$. For all s_i-free formulas ϕ, $\langle s_i \Delta s_i \rangle \top \leftrightarrow \phi$ is not valid in the class of all separated frames.*

Proof. We only consider the case $i = 1$. Suppose there exists a formula ϕ in our language such that $\langle s_1 \Delta s_1 \rangle \top \leftrightarrow \phi$ is valid in the class of all separated frames. Let $\mathcal{M} = (W, R, \star, V)$ and $\mathcal{M}' = (W', R', \star', V')$ be the models defined by

- $W = \{x, y_1, y_2, z_1, z_2, t_1, t_2\}$,
- R is the empty function,
- $x \star y_1 = \{z_1\}$, $x \star y_2 = \{z_2\}$, $z_1 \star z_2 = \{t_1\}$, $z_2 \star z_1 = \{t_2\}$, otherwise $\star$ is the empty function,
- V is the empty function,
- $W' = \{x'_1, x'_2, y'_1, y'_2, z'_1, z'_2, t'_1, t'_2\}$,
- R' is the empty function,
- $x'_1 \star' y'_1 = \{z'_1\}$, $x'_2 \star' y'_2 = \{z'_2\}$, $z'_1 \star' z'_2 = \{t'_1\}$, $z'_2 \star' z'_1 = \{t'_2\}$, otherwise $\star'$ is the empty function,
- V' is the empty function.

Clearly, $x \in V_{\mathcal{M}}(\langle s_1 \Delta s_1 \rangle \top)$ but $x'_1 \notin V_{\mathcal{M}'}(\langle s_1 \Delta s_1 \rangle \top)$. Hence, since $\langle s_1 \Delta s_1 \rangle \top \leftrightarrow \phi$ is supposed to be valid, it must be the case that $x \in V_{\mathcal{M}}(\phi)$ and $x'_1 \notin V_{\mathcal{M}'}(\phi)$. But we will prove that if $x \in V_{\mathcal{M}}(\phi)$ then $x'_1 \in V_{\mathcal{M}}(\phi)$. First remark that for all s_1-free program α and all $w \in W$, if $xR_{\mathcal{M}}(\alpha)w$ then $w = x$. Then define the function r from W to W by $r(x) = x$, $r(y_1) = y_2$, $r(y_2) = y_1$, $r(z_1) = z_2$, $r(z_2) = z_1$, $r(t_1) = t_2$ and $r(t_2) = t_1$. It can easily be checked that for all $w_1, w_2 \in W$ and all s_1-free program α, $w_1 R_{\mathcal{M}}(\alpha) w_2$ iff $r(w_1) R_{\mathcal{M}}(\alpha) r(w_2)$. Now define the function f from W' to W by $f(x'_1) = f(x'_2) = x$, $f(y'_1) = y_1$, $f(y'_2) = y_2$, $f(z'_1) = z_1$, $f(z'_2) = z_2$, $f(t'_1) = t_1$ and $f(t'_2) = t_2$. Define also the binary relation Z between W and W' such that $(w, w') \in Z$ iff $w = f(w')$ or $r(w) = f(w')$. We prove that for all $n > 0$, all s_1-free formula ψ, all s_1-free program α, all $w_1 \in W$ and all $w'_1, w'_2 \in W'$:

1. if the number of occurrences of symbols in ψ is n and $(w_1, w'_1) \in Z$ then $w_1 \in V_{\mathcal{M}}(\psi)$ iff $w'_1 \in V_{\mathcal{M}'}(\psi)$;
2. if the number of occurrences of symbols in α is n then $w'_1 R_{\mathcal{M}'}(\alpha) w'_2$ iff $f(w'_1) R_{\mathcal{M}}(\alpha) f(w'_2)$.

The proof is by induction on n, left to the reader. ⊣

Proposition 15. *Let $i \in \{1, 2\}$. For all r_i-free formulas ϕ, $\langle (a; r_i) \Delta (r_i; a) \rangle \top \leftrightarrow \phi$ is not valid in the class of all separated frames.*

Proof. We only consider the case $i = 1$. Suppose there exists a formula ϕ in our language such that $\langle r_1 \Delta r_1 \rangle \top \leftrightarrow \phi$ is valid in the class of all separated frames. Let $\mathcal{M} = (W, R, \star, V)$ and $\mathcal{M}' = (W', R', \star', V')$ be the models defined by

- $W = \{x, y, z, s, t, u, v\}$,
- $R(a) = \{(x, s), (z, t)\}$, otherwise R is the empty function,
- $z \star y = \{x\}$, $u \star t = \{s, v\}$, otherwise $\star$ is the empty function,
- V is the empty function,
- $W' = \{x'_1, x'_2, y'_1, y'_2, z'_1, z'_2, s'_1, s'_2, t'_1, t'_2, u'_1, u'_2, v'_1, v'_2\}$,
- $R'(a) = \{(x'_1, s'_2), (x'_2, s'_1), (z'_1, t'_1), (z'_2, t'_2)\}$, otherwise R' is the empty function,
- for all $j \in \{1, 2\}$, $z'_j \star' y'_j = \{x'_j\}$ and $u'_j \star' t'_j = \{s'_j, v'_j\}$, otherwise $\star'$ is the empty function,
- V' is the empty function.

Clearly, $x \in V_{\mathcal{M}}(\langle(a; r_1)\Delta(r_1; a)\rangle\top)$ but $x'_1 \notin V_{\mathcal{M}'}(\langle(a; r_1)\Delta(r_1; a)\rangle\top)$. Hence, since $\langle(a; r_1)\Delta(r_1; a)\rangle\top \leftrightarrow \phi$ is supposed to be valid, it must be the case that $x \in V_{\mathcal{M}}(\phi)$ and $x'_1 \notin V_{\mathcal{M}'}(\phi)$. But we will prove that if $x \in V_{\mathcal{M}}(\phi)$ then $x'_1 \in V_{\mathcal{M}}(\phi)$. First remark that for all r_1-free program α and all $w_1, w_2 \in W$ such that $w_1 R_{\mathcal{M}}(\alpha) w_2$:

- if $w_2 = u$ then $w_1 = u$,
- if $w_2 \in \{x, y, z\}$ then $w_1 \in \{x, y, z\}$, and
- if $w_1 \in \{s, t, u, v\}$ then $w_2 \in \{s, t, u, v\}$.

Then, define the functions f_1 and f_2 from W to W' such that for all $j \in \{1, 2\}$, $f_j(x) = x'_j$, $f_j(y) = y'_j$, $f_j(z) = z'_j$, $f_j(s) = s'_j$, $f_j(t) = t'_j$, $f_j(u) = u'_j$ and $f_j(v) = v'_j$. For all $w' \in W'$, there is exactly one pair $(w, j) \in W \times \{1, 2\}$ such that $w' = f_i(w)$; hence we also define the function g from W' to W such that for all $w' \in W'$ there is $i \in \{1, 2\}$ such that $f_i(g(w')) = w'$. We prove that for all $n > 0$, all r_1-free formula ψ, all r_1-free program α, all $w_1, w_2 \in W$ and all $w'_1 \in W'$:

1. if the number of occurrences of symbols in ψ is n then $w_1 \in V_{\mathcal{M}}(\psi)$ iff $f_1(w_1) \in V_{\mathcal{M}'}(\psi)$ iff $f_2(w_1) \in V_{\mathcal{M}'}(\psi)$;
2. if the number of occurrences of symbols in α is $n - 1$ and $w_1 = g(w'_1)$ then $w_1 R_{\mathcal{M}}(\alpha) w_2$ iff there is $w'_2 \in W'$ such that $w_2 = g(w'_2)$ and $w'_1 R_{\mathcal{M}'}(\alpha) w'_2$;
3. if the number of occurrences of symbols in α is n and $w_1 \in \{s, t, u, v\}$ or $w_2 \in \{x, y, z\}$ then $w_1 R_{\mathcal{M}}(\alpha) w_2$ iff $f_1(w_1) R_{\mathcal{M}'}(\alpha) f_1(w_2)$ iff $f_2(w_1) R_{\mathcal{M}'}(\alpha) f_2(w_2)$.

The proof is by induction on n, left to the reader. $\dashv$

5. Axiom system

We now define $\mathbf{PDL}_0^{\Delta}$.

Definition. Let $\mathbf{PDL}_0^{\Delta}$ be the least set of formulas that contains all instances of propositional tautologies, that contains the formulas (**A**1)–(**A**18) considered in Propositions 3 and 4 and that is closed under the rules of proof (**MP**), (**N**) and (**FOR**) considered in Propositions 5 and 6.

It is easy to establish the soundness for $\mathbf{PDL}_0^{\Delta}$:

Proposition 16 *(Soundness for* $\mathbf{PDL}_0^{\Delta}$*). Let ϕ be a formula. If $\phi \in \mathbf{PDL}_0^{\Delta}$, ϕ is valid in the class of all separated frames.*

The completeness for $\mathbf{PDL}_0^{\Delta}$ is more difficult to establish and we defer proving it till next section. In the meantime, it is well worth noting that for all separated models $\mathcal{M} = (W, R, \star, V)$ and for all $x \in W$, $\{\phi : x \in V_{\mathcal{M}}(\phi)\}$ is a set of formulas that contains $\mathbf{PDL}_0^{\Delta}$ and that is closed under the rule of proof (**MP**). Now, we introduce theories.

Definition. A set S of formulas is said to be a theory iff $\mathbf{PDL}_0^{\Delta} \subseteq S$ and S is closed under the rules of proof (**MP**) and (**FOR**).

We will use $S, T, \ldots$ for theories. Obviously, the least theory is $\mathbf{PDL}_0^{\Delta}$ and the greatest theory is the set of all formulas. Not surprisingly, we have

Lemma 1. *Let S be a theory. The following conditions are equivalent: S is equal to the set of all formulas; there exists a formula ϕ such that $\phi \in S$ and $\neg\phi \in S$; $\bot \in S$.*

Referring to Lemma 1, we define what it means for a theory to be consistent.

Definition. We shall say that a theory S is consistent iff for all formulas ϕ, either $\phi \notin S$, or $\neg\phi \notin S$.

By Lemma 1, there is only one inconsistent theory: the set of all formulas. Now, we define what it means for a theory to be maximal.

Definition. A theory S is said to be maximal iff for all formulas ϕ, either $\phi \in S$, or $\neg\phi \in S$.

We will use the following lemma without explicit reference:

Lemma 2. *Let S be a maximal consistent theory. We have: $\bot \notin S$; for all formulas ϕ, $\neg\phi \in S$ iff $\phi \notin S$; for all formulas ϕ, ψ, $\phi \vee \psi \in S$ iff either $\phi \in S$, or $\psi \in S$.*

To know more about theories, we need yet another definition.

Definition. If α is a program, ϕ is a formula and S is a theory, let $[\alpha]S = \{\phi : [\alpha]\phi \in S\}$ and $S + \phi = \{\psi : \phi \to \psi \in S\}$.

In the next lemmas, we summarize some properties of theories.

Lemma 3. *Let S be a theory. For all programs α and for all formulas ϕ, we have:* (1) $[\phi?]S = S + \phi$; (2) $[\alpha]S$ *is a theory;* (3) $S + \phi$ *is a theory;* (4) ϕ, $S + \phi$ *is the least theory containing S and ϕ;* (5) $S + \phi$ *is consistent iff $\neg\phi \notin S$.*

Lemma 4. *Let S be a theory. If S is consistent, for all formulas ϕ, either $S + \phi$ is consistent, or there exists a formula ψ such that the following conditions are satisfied: $S + \psi$ is consistent; $\psi \to \neg\phi \in \mathbf{PDL}_0^\Delta$; if ϕ is in the form $\breve{\chi}(\langle\alpha\Delta\beta\rangle\theta)$ of a conclusion of the rule of proof* (**FOR**), *there exists a propositional variable p such that $\psi \to \neg\breve{\chi}(\langle\alpha\rangle((\theta \wedge p) \triangleleft \top) \vee \langle\beta\rangle(\top \triangleright (\theta \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$.*

Proof. Suppose S is consistent. Suppose $S + \phi$ is not consistent. By Lemma 3, $\neg\phi \in S$. Obviously, there are finitely many, say $k \geq 0$, representations of ϕ in the form of a conclusion of the rule of proof (**FOR**): $\breve{\chi}_1(\langle\alpha_1\Delta\beta_1\rangle\theta_1), \ldots, \breve{\chi}_k(\langle\alpha_k\Delta\beta_k\rangle\theta_k)$. We define by induction a sequence $(\psi_0, \ldots, \psi_k)$ of formulas such that for all $l \in \mathbb{N}$, if $l \leq k$, the following conditions are satisfied: $S + \psi_l$ is consistent; $\psi_l \to \neg\phi \in \mathbf{PDL}_0^\Delta$; for all $m \in \mathbb{N}$, if $1 \leq m \leq l$, there exists a propositional variable p such that $\psi_l \to \neg\breve{\chi}_m(\langle\alpha_m\rangle((\theta_m \wedge p) \triangleleft \top) \vee \langle\beta_m\rangle(\top \triangleright (\theta_m \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. First, let $\psi_0 = \neg\phi$. Obviously, the following conditions are satisfied: $S + \psi_0$ is consistent; $\psi_0 \to \neg\phi \in \mathbf{PDL}_0^\Delta$. Second, let $l \geq 1$ be such that $l \leq k$ and the formulas $\psi_0, \ldots, \psi_{l-1}$ have already been defined. Hence, $S + \psi_{l-1}$ is consistent; $\psi_{l-1} \to \neg\phi \in \mathbf{PDL}_0^\Delta$; for all $m \in \mathbb{N}$, if $1 \leq m \leq l-1$, there exists a propositional variable p such that $\psi_{l-1} \to \neg\breve{\chi}_m(\langle\alpha_m\rangle((\theta_m \wedge p) \triangleleft \top) \vee \langle\beta_m\rangle(\top \triangleright (\theta_m \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. Third, since $S + \psi_{l-1}$ is consistent and $\psi_{l-1} \to \neg\phi \in \mathbf{PDL}_0^\Delta$, $\phi \notin S + \psi_{l-1}$. Since $S + \psi_{l-1}$ is closed under the rule of proof (**FOR**), there exists a propositional variable p such that $\breve{\chi}_l(\langle\alpha_l\rangle((\theta_l \wedge p) \triangleleft \top) \vee \langle\beta_l\rangle(\top \triangleright (\theta_l \wedge \neg p))) \notin S + \psi_{l-1}$. Let $\psi_l = \psi_{l-1} \wedge \neg\breve{\chi}_l(\langle\alpha_l\rangle((\theta_l \wedge p) \triangleleft \top) \vee \langle\beta_l\rangle(\top \triangleright (\theta_l \wedge \neg p)))$. Obviously, the following conditions are satisfied: $S + \psi_l$ is consistent; $\psi_l \to \neg\phi \in \mathbf{PDL}_0^\Delta$; for all $m \in \mathbb{N}$, if $1 \leq m \leq l$, there exists a propositional variable p such that $\psi_l \to \neg\breve{\chi}_m(\langle\alpha_m\rangle((\theta_m \wedge p) \triangleleft \top) \vee \langle\beta_m\rangle(\top \triangleright (\theta_m \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. Finally, the reader may easily verify that the following conditions are satisfied: $S + \psi_k$ is consistent; $\psi_k \to \neg\phi \in \mathbf{PDL}_0^\Delta$; if ϕ is in the form $\breve{\chi}(\langle\alpha\Delta\beta\rangle\theta)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\psi_k \to \neg\breve{\chi}(\langle\alpha\rangle((\theta \wedge p) \triangleleft \top) \vee \langle\beta\rangle(\top \triangleright (\theta \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. $\dashv$

Now, we are ready for the Lindenbaum Lemma.

Lemma 5 *(Lindenbaum Lemma). Let S be a theory. If S is consistent, there exists a maximal consistent theory containing S.*

Proof. Suppose S is consistent. Since there are countably many formulas, there exists an enumeration $\phi_1, \phi_2, \ldots$ of the set of all formulas. Let $T_0, T_1, \ldots$ be the sequence of consistent theories inductively defined as follows. First, let $T_0 = S$. Obviously, T_0 is consistent. Second, let $n \geq 1$ be such that consistent theories $T_0, \ldots, T_{n-1}$ have already been defined. Third, by Lemma 4, either $T_{n-1} + \phi_n$ is consistent, or there exists a formula ψ such that the following conditions are satisfied: $T_{n-1} + \psi$ is consistent; $\psi \to \neg\phi_n \in \mathbf{PDL}_0^\Delta$; if ϕ_n is in the form $\breve{\chi}(\langle\alpha\Delta\beta\rangle\theta)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\psi \to \neg\breve{\chi}(\langle\alpha\rangle((\theta \wedge p) \triangleleft \top) \vee \langle\beta\rangle(\top \triangleright (\theta \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. In the former case, let $T_n = T_{n-1} + \phi_n$. In the latter case, let $T_n = T_{n-1} + \psi$. Obviously, T_n is consistent. Finally, the reader may easily verify that $T_0 \cup T_1 \cup \ldots$ is a maximal consistent theory containing S. $\dashv$

To define the canonical frame of $\mathbf{PDL}_0^\Delta$ in next section, we need yet another definition.

Definition. If S and T are theory, let $S \circ T = \{\phi \circ \psi : \phi \in S \text{ and } \psi \in T\}$.

To end this section, we present useful results.

Lemma 6. *Let ϕ, ψ be formulas and $\otimes \in \{\circ, \rhd, \rhd\}$. For all maximal consistent theories S, if $\phi \otimes \psi \in S$, for all formulas χ, we have: (1) either $(\phi \wedge \chi) \otimes \psi \in S$, or there exists a formula θ such that the following conditions are satisfied: $(\phi \wedge \theta) \otimes \psi \in S$; $\theta \to \neg\chi \in$ $\mathbf{PDL}_0^{\Delta}$; if χ is in the form $\breve{\tau}(\langle\alpha\Delta\beta\rangle\mu)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\theta \to \neg\breve{\tau}(\langle\alpha\rangle((\mu \wedge p) \lhd \top) \vee \langle\beta\rangle(\top \rhd (\mu \wedge \neg p))) \in \mathbf{PDL}_0^{\Delta}$; (2) either $\phi \otimes (\psi \wedge \chi) \in S$, or there exists a formula θ such that the following conditions are satisfied: $\phi \otimes (\psi \wedge \theta) \in S$; $\theta \to \neg\chi \in \mathbf{PDL}_0^{\Delta}$; if χ is in the form $\breve{\tau}(\langle\alpha\Delta\beta\rangle\mu)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\theta \to \neg\breve{\tau}(\langle\alpha\rangle((\mu \wedge p) \lhd \top) \vee \langle\beta\rangle(\top \rhd (\mu \wedge \neg p))) \in \mathbf{PDL}_0^{\Delta}$.*

Lemma 7. *Let ϕ, ψ be formulas. For all maximal consistent theories S, we have: (1) if $\phi \circ \psi \in S$, there exist maximal consistent theories T, U such that $T \circ U \subseteq S$, $\phi \in T$ and $\psi \in U$; (2) if $\phi \rhd \psi \in S$, there exist maximal consistent theories T, U such that $T \circ S \subseteq U$, $\phi \in T$ and $\psi \in U$; (3) if $\phi \lhd \psi \in S$, there exist maximal consistent theories T, U such that $S \circ U \subseteq T$, $\phi \in T$ and $\psi \in U$.*

6. Completeness

Now, for the canonical frame of $\mathbf{PDL}_0^{\Delta}$.

Definition. The canonical frame of $\mathbf{PDL}_0^{\Delta}$ is the 3-tuple $\mathcal{F}_c = (W_c, R_c, \star_c)$ where W_c is the set of all maximal consistent theories, R_c is the function from the set of all program variables into the set of all binary relations between maximal consistent theories defined by $SR_c(a)T$ iff $[a]S \subseteq T$ and $\star_c$ is the function from the set of all pairs of maximal consistent theories into the set of all sets of maximal consistent theories defined by $U \in S \star_c T$ iff $S \circ T \subseteq U$.

We show first that

Lemma 8. *$\mathcal{F}_c$ is separated.*

Now, for the canonical valuation of $\mathbf{PDL}_0^{\Delta}$ and the canonical model of $\mathbf{PDL}_0^{\Delta}$.

Definition. The canonical model of $\mathbf{PDL}_0^{\Delta}$ is the 4-tuple $\mathcal{M}_c = (W_c, R_c, \star_c, V_c)$ where V_c is the canonical valuation of $\mathbf{PDL}_0^{\Delta}$, i.e. the function from the set of all propositional variables into the set of all sets of maximal consistent theories defined by $S \in V_c(p)$ iff $p \in S$.

For the proof of the Truth Lemma, we have to consider large programs.

Definition. The set of all large programs is inductively defined as follows:

- $A ::= a \mid (A; B) \mid (A\Delta B) \mid \bar{S}?$;

where for all consistent theories S, $\bar{S}$ is a new symbol.

We will use $A, B, \dots$ for large programs. Let us be clear that each large program is a finite string of symbols coming from an uncountable alphabet. It follows that there are uncountably many large programs. For convenience, we omit the parentheses in accordance with the standard rules. It is essential that large programs are built up from program variables and symbols for consistent theories by means of the operations ; and Δ. Let $A(\bar{S_1}?, \dots, \bar{S_n}?)$ be a large program with $(\bar{S_1}, \dots, \bar{S_n})$ a sequence of some of its symbols for consistent theories. The result of the replacement of $\bar{S_1}, \dots, \bar{S_n}$ in their places with other symbols $\bar{T_1}, \dots, \bar{T_n}$ for consistent theories is another large program which will be denoted $A(\bar{T_1}?, \dots, \bar{T_n}?)$.

Definition. A large program $A(\bar{S_1}?, \dots, \bar{S_n}?)$ with $(\bar{S_1}, \dots, \bar{S_n})$ the sequence of all its symbols for consistent theories will be defined to be maximal if the theories $S_1, \dots, S_n$ are maximal.

It appears that large programs, either maximal, or not, can be associated with sets of programs.

Definition. The kernel function $\ker: A \mapsto \ker(A) \subseteq \mathbf{PRG}$ is inductively defined as follows:

- $\ker(a) = \{a\}$;
- $\ker(A; B) = \{\alpha;\beta :\ \alpha \in \ker(A) \text{ and } \beta \in \ker(B)\}$;
- $\ker(A\Delta B) = \{\alpha\Delta\beta :\ \alpha \in \ker(A) \text{ and } \beta \in \ker(B)\}$;
- $\ker(\bar{S}) = \{\phi? :\ \phi \in S\}$.

The following lemmas play an important role in the proof of the completeness for $\mathbf{PDL}_0^{\Delta}$.

Lemma 9. *Let $\alpha(\phi?)$ be a program. For all maximal consistent theories S, if $\langle\alpha(\phi?)\rangle\top \in S$, for all formulas ψ, we have: either $\langle\alpha((\phi \wedge \psi)?)\rangle\top \in S$, or there exists a formula χ such that the following conditions are satisfied: $\langle\alpha((\phi \wedge \chi)?)\rangle\top \in S$; $\chi \to \neg\psi \in \mathbf{PDL}_0^\Delta$; if ψ is in the form $\breve{\theta}(\langle\beta\Delta\gamma\rangle\tau)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\chi \to \neg\breve{\theta}(\langle\beta\rangle((\tau \wedge p) \lhd \top) \vee \langle\gamma\rangle(\top \rhd (\tau \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$.*

Lemma 10 *(Diamond Lemma). Let α be a program and ϕ be a formula. For all maximal consistent theories S, if $[\alpha]\phi \notin S$, there exists a maximal program A and there exists a maximal consistent theory T such that $f(\alpha) \in \ker(A)$, for all programs β, if $\beta \in \ker(A)$, $[\beta]S \subseteq T$ and $\phi \notin T$.*

With this established, we are ready for the Truth Lemma.

Lemma 11 *(Truth Lemma). Let α be a program. For all maximal consistent theories S, T, the following conditions are equivalent: $SR_{\mathcal{M}_c}(\alpha)T$; there exists a maximal program A such that $f(\alpha) \in \ker(A)$ and for all programs β, if $\beta \in \ker(A)$, $[\beta]S \subseteq T$. Let ϕ be a formula. For all maximal consistent theories S, the following conditions are equivalent: $S \in V_{\mathcal{M}_c}(\phi)$; $\phi \in S$.*

Proof. Let $P(\cdot)$ be the property about programs and formulas defined as follows:

- for all programs α, $P(\alpha)$ iff for all maximal consistent theories S, T, $SR_{\mathcal{M}_c}(\alpha)T$ iff there exists a maximal program A such that $f(\alpha) \in \ker(A)$ and for all programs β, if $\beta \in \ker(A)$, $[\beta]S \subseteq T$;
- for all formulas ϕ, $P(\phi)$ iff for all maximal consistent theories S, $S \in V_{\mathcal{M}_c}(\phi)$ iff $\phi \in S$.

The proof that $P(\cdot)$ holds for all programs and for all formulas will be done by induction on the formation of programs and formulas.

Hypothesis. Let α be a program such that for all expressions *exp* (either a program, or a formula), if *exp* is an expression strictly occurring in α, $P(exp)$ holds.

Step. We demonstrate $P(\alpha)$ holds.

Case $\alpha = a$. Left to the reader.

Case $\alpha = \beta;\gamma$. Let S, T be maximal consistent theories.

- Suppose $SR_{\mathcal{M}_c}(\beta;\gamma)T$. We demonstrate there exists a maximal program A such that $f(\beta); \top?; f(\gamma) \in \ker(A)$ and for all programs δ, if $\delta \in \ker(A)$, $[\delta]S \subseteq T$. Since $SR_{\mathcal{M}_c}(\beta;\gamma)T$, there exists a maximal consistent theory U such that $SR_{\mathcal{M}_c}(\beta)U$ and $UR_{\mathcal{M}_c}(\gamma)T$. Since $P(\beta)$ and $P(\gamma)$, there exists a maximal program A' such that $f(\beta) \in \ker(A')$ and for all programs δ', if $\delta' \in \ker(A')$, $[\delta']S \subseteq U$ and there exists a maximal program A'' such that $f(\gamma) \in \ker(A'')$ and for all programs δ'', if $\delta'' \in \ker(A'')$, $[\delta'']U \subseteq T$. Since $\top \in U$, $f(\beta); \top?; f(\gamma) \in \ker(A'; \bar{U}; A'')$. Now, let $\delta'; \phi?; \delta'' \in \ker(A'; \bar{U}; A'')$ and $\psi \in [\delta'; \phi?; \delta'']S$. Hence, $\delta' \in \ker(A')$, $\phi \in U$, $\delta'' \in \ker(A'')$ and $[\delta'; \phi?; \delta'']\psi \in S$. Thus, $[\delta'](\phi \to [\delta'']\psi) \in S$. Therefore, $\phi \to [\delta'']\psi \in [\delta']S$. Since $\delta' \in \ker(A')$, $[\delta']S \subseteq U$. Since $\phi \to [\delta'']\psi \in [\delta']S$, $\phi \to [\delta'']\psi \in U$. Since $\phi \in U$, $[\delta'']\psi \in U$. Consequently, $\psi \in [\delta'']U$. Since $\delta'' \in \ker(A'')$, $[\delta'']U \subseteq T$. Since $\psi \in [\delta'']U$, $\psi \in T$. Hence, for all programs δ, if $\delta \in \ker(A'; \bar{U}; A'')$, $[\delta]S \subseteq T$. Since $f(\beta); \top?; f(\gamma) \in \ker(A'; \bar{U}; A'')$, it suffices to take $A = A'; \bar{U}; A''$.
- Suppose there exists a maximal program A such that $f(\beta); \top?; f(\gamma) \in \ker(A)$ and for all programs δ, if $\delta \in \ker(A)$, $[\delta]S \subseteq T$. We demonstrate $SR_{\mathcal{M}_c}(\beta;\gamma)T$. Since $f(\beta); \top?; f(\gamma) \in \ker(A)$, there exists a maximal program A', there exists a maximal consistent theory U and there exists a maximal program A'' such that $f(\beta) \in \ker(A')$, $f(\gamma) \in \ker(A'')$ and $A = A'; \bar{U}; A''$. Now, let $\delta' \in \ker(A')$ and $\phi \in [\delta']S$. Hence, $[\delta']\phi \in S$. Let $\delta'' \in \ker(A'')$. Since $[\delta']\phi \in S$, $[\delta'](\neg\phi \to [\delta'']\bot) \in S$. Thus, $[\delta'; \neg\phi?; \delta'']\bot \in S$. Therefore, $\bot \in [\delta'; \neg\phi?; \delta'']S$. Since T is consistent, by Lemma 1, $\bot \notin T$. Since for all programs δ, if $\delta \in \ker(A)$, $[\delta]S \subseteq T$ and $\bot \in [\delta'; \neg\phi?; \delta'']S$, $\delta'; \neg\phi?; \delta'' \notin \ker(A)$. Since $A = A'; \bar{U}; A''$, $\delta' \in \ker(A')$ and $\delta'' \in \ker(A'')$, $\neg\phi \notin U$. Since U is maximal, $\phi \in U$. Consequently, for all $\delta' \in \ker(A')$, $[\delta']S \subseteq U$. Since $f(\beta) \in \ker(A')$ and $P(\beta)$, $SR_{\mathcal{M}_c}(\beta)U$. Now, let $\delta'' \in \ker(A'')$ and $\phi \in [\delta'']U$. Hence, $[\delta'']\phi \in U$. Let $\delta' \in \ker(A')$. Thus, $[\delta']([\delta'']\phi \to [\delta'']\phi) \in S$. Therefore, $[\delta'; [\delta'']\phi?; \delta'']\phi \in S$. Consequently, $\phi \in [\delta'; [\delta'']\phi?; \delta'']S$. Since $\delta' \in \ker(A')$, $[\delta'']\phi \in U$ and $\delta'' \in \ker(A'')$, $\delta'; [\delta'']\phi?; \delta'' \in \ker(A'; \bar{U}; A'')$. Since $A = A'; \bar{U}; A''$, $\delta'; [\delta'']\phi?; \delta'' \in \ker(A)$. Since for all programs δ, if $\delta \in \ker(A)$, $[\delta]S \subseteq T$, $\delta'; [\delta'']\phi?; \delta'' \in \ker(A)$ and $\phi \in [\delta'; [\delta'']\phi?; \delta'']S$, $\phi \in T$. Hence, for all $\delta'' \in \ker(A'')$, $[\delta'']U \subseteq T$. Since $f(\gamma) \in \ker(A'')$ and $P(\gamma)$, $UR_{\mathcal{M}_c}(\gamma)T$. Since $SR_{\mathcal{M}_c}(\beta)U$, $SR_{\mathcal{M}_c}(\beta;\gamma)T$.

Case $\alpha = \beta\Delta\gamma$. Let S, T be maximal consistent theories.

- Suppose $SR_{\mathcal{M}_c}(\beta\Delta\gamma)T$. We demonstrate there exists a maximal program A such that $(f(\beta); \top?)\Delta(f(\gamma); \top?) \in \ker(A)$ and for all programs δ, if $\delta \in \ker(A)$, $[\delta]S \subseteq T$. Since $SR_{\mathcal{M}_c}(\beta\Delta\gamma)T$, there exist maximal consistent theories U, V such that $SR_{\mathcal{M}_c}(\beta)U$, $SR_{\mathcal{M}_c}(\gamma)V$ and $T \in U \star_c V$. Since $P(\beta)$ and $P(\gamma)$, there exists a maximal program A' such that $f(\beta) \in \ker(A')$ and for all programs δ', if $\delta' \in \ker(A')$, $[\delta']S \subseteq U$ and there exists a maximal program A'' such that $f(\gamma) \in \ker(A'')$ and for all programs δ'', if $\delta'' \in \ker(A'')$, $[\delta'']S \subseteq V$. Since $\top \in U$ and $\top \in V$, $(f(\beta); \top?)\Delta(f(\gamma); \top?) \in$

$\ker((A'; \bar{U})\Delta(A''; \bar{V}))$. Now, let $(\delta'; \phi?)\Delta(\delta''; \psi?) \in \ker((A'; \bar{U})\Delta(A''; \bar{V}))$ and $\chi \in [(\delta'; \phi?)\Delta(\delta''; \psi?)]S$. Hence, $\delta' \in \ker(A')$, $\phi \in U$, $\delta'' \in \ker(A'')$, $\psi \in V$ and $[(\delta'; \phi?)\Delta(\delta''; \psi?)]\chi \in S$. Since S is consistent, $\langle(\delta'; \phi?)\Delta(\delta''; \psi?)\rangle\neg\chi \notin S$. Since S is closed under the rule of proof (**FOR**), there exists a propositional variable p such that $\langle\delta'; \phi?\rangle((\neg\chi \wedge p) \triangleleft \top) \vee \langle\delta''; \psi?\rangle(\top \triangleright (\neg\chi \wedge \neg p)) \notin S$. Thus, $\langle\delta'; \phi?\rangle((\neg\chi \wedge p) \triangleleft \top) \notin S$ and $\langle\delta''; \psi?\rangle(\top \triangleright (\neg\chi \wedge \neg p)) \notin S$. Since S is maximal, $[\delta'; \phi?]\neg((\neg\chi \wedge p) \triangleleft \top) \in S$ and $[\delta''; \psi?]\neg(\top \triangleright (\neg\chi \wedge \neg p)) \in S$. Therefore, $[\delta'](\phi \to \neg((\neg\chi \wedge p) \triangleleft \top)) \in S$ and $[\delta''](\psi \to \neg(\top \triangleright (\neg\chi \wedge \neg p))) \in S$. Consequently, $\phi \to \neg((\neg\chi \wedge p) \triangleleft \top) \in [\delta']S$ and $\psi \to \neg(\top \triangleright (\neg\chi \wedge \neg p)) \in [\delta'']S$. Since $\delta' \in \ker(A')$, $[\delta']S \subseteq U$. Since $\phi \to \neg((\neg\chi \wedge p) \triangleleft \top) \in [\delta']S$, $\phi \to \neg((\neg\chi \wedge p) \triangleleft \top) \in U$. Since $\phi \in U$, $\neg((\neg\chi \wedge p) \triangleleft \top) \in U$. Since $\delta'' \in \ker(A'')$, $[\delta'']S \subseteq V$. Since $\psi \to \neg(\top \triangleright (\neg\chi \wedge \neg p)) \in [\delta'']S$, $\psi \to \neg(\top \triangleright (\neg\chi \wedge \neg p)) \in V$. Since $\psi \in V$, $\neg(\top \triangleright (\neg\chi \wedge \neg p)) \in V$. Since $T \in U \star_c V$ and $\neg((\neg\chi \wedge p) \triangleleft \top) \in U$, $\neg((\neg\chi \wedge p) \triangleleft \top) \circ \neg(\top \triangleright (\neg\chi \wedge \neg p)) \in T$. Hence, $\chi \in T$. Thus, for all programs δ, if $\delta \in \ker((A'; \bar{U})\Delta(A''; \bar{V}))$, $[\delta]S \subseteq T$. Since $(f(\beta); \top?)\Delta(f(\gamma); \top?) \in \ker((A'; \bar{U})\Delta(A''; \bar{V}))$, it suffices to take $A = (A'; \bar{U})\Delta(A''; \bar{V})$.
- Suppose there exists a maximal program A such that $(f(\beta); \top?)\Delta(f(\gamma); \top?) \in \ker(A)$ and for all programs δ, if $\delta \in \ker(A)$, $[\delta]S \subseteq T$. We demonstrate $SR_{\mathcal{M}_c}(\beta\Delta\gamma)T$. Since $(f(\beta); \top?)\Delta(f(\gamma); \top?) \in \ker(A)$, there exists a maximal program A', there exists a maximal consistent theory U, there exists a maximal program A'' and there exists a maximal consistent theory V such that $f(\beta) \in \ker(A')$, $f(\gamma) \in \ker(A'')$ and $A = (A'; \bar{U})\Delta(A''; \bar{V})$. Now, let $\delta' \in \ker(A')$ and $\phi \in [\delta']S$. Hence, $[\delta']\phi \in S$. Thus, $[\delta'](\neg\phi \to \bot) \in S$. Therefore, $[\delta'; \neg\phi?]\bot \in S$. Let $\delta'' \in \ker(A'')$. Since $[\delta'; \neg\phi?]\bot \in S$, $[(\delta'; \neg\phi?)\Delta(\delta''; \top?)]\bot \in S$. Consequently, $\bot \in [(\delta'; \neg\phi?)\Delta(\delta''; \top?)]S$. Since T is consistent, $\bot \notin T$. Since for all programs δ, if $\delta \in \ker(A)$, $[\delta]S \subseteq T$ and $\bot \in [(\delta'; \neg\phi?)\Delta(\delta''; \top?)]S$, $(\delta'; \neg\phi?)\Delta(\delta''; \top?) \notin \ker(A)$. Since $A = (A'; \bar{U})\Delta(A''; \bar{V})$, $\delta' \in \ker(A')$ and $\delta'' \in \ker(A'')$, $\neg\phi \notin U$. Since U is maximal, $\phi \in U$. Hence, for all $\delta' \in \ker(A')$, $[\delta']S \subseteq U$. Since $f(\beta) \in \ker(A')$ and $P(\beta)$, $SR_{\mathcal{M}_c}(\beta)U$. The proof that $SR_{\mathcal{M}_c}(\gamma)V$ is similar. Now, let $\phi \in U$ and $\psi \in V$. Let $\delta' \in \ker(A')$ and $\delta'' \in \ker(A'')$. Since $A = (A'; \bar{U})\Delta(A''; \bar{V})$, $\phi \in U$ and $\psi \in V$, $(\delta'; \phi?)\Delta(\delta''; \psi?) \in \ker(A)$. Since for all programs δ, if $\delta \in \ker(A)$, $[\delta]S \subseteq T$, $[(\delta'; \phi?)\Delta(\delta''; \psi?)]S \subseteq T$. Thus, $[(\delta'; \phi?)\Delta(\delta''; \psi?)](\phi \circ \psi) \in S$. Therefore, $\phi \circ \psi \in [(\delta'; \phi?)\Delta(\delta''; \psi?)]S$. Since $[(\delta'; \phi?)\Delta(\delta''; \psi?)]S \subseteq T$, $\phi \circ \psi \in T$. Consequently, for all $\phi \in U$ and for all $\psi \in V$, $\phi \circ \psi \in T$. Hence, $T \in U \star_c V$. Since $SR_{\mathcal{M}_c}(\beta)U$ and $SR_{\mathcal{M}_c}(\gamma)V$, $SR_{\mathcal{M}_c}(\beta\Delta\gamma)T$.

Case $\alpha = \psi?$. Let S, T be maximal consistent theories.

- Suppose $SR_{\mathcal{M}_c}(\psi?)T$. We demonstrate there exists a maximal program A such that $\psi? \in \ker(A)$ and for all programs β, if $\beta \in \ker(A)$, $[\beta]S \subseteq T$. Since $SR_{\mathcal{M}_c}(\psi?)T$, $S = T$ and $T \in V_{\mathcal{M}_c}(\psi)$. Since $P(\psi)$, $\psi \in T$. Since $S = T$, $\psi \in S$. Hence, $\psi? \in \ker(\bar{S})$. Now, let $\chi? \in \ker(\bar{S})$. Thus, $\chi \in S$. By Lemma 3, $[\chi?]S = S$. Since $S = T$, $[\chi?]S \subseteq T$. Therefore, for all programs β, if $\beta \in \ker(\bar{S})$, $[\beta]S \subseteq T$. Since $\psi? \in \ker(\bar{S})$, it suffices to take $A = \bar{S}$.
- Suppose there exists a maximal program A such that $\psi? \in \ker(A)$ and for all programs β, if $\beta \in \ker(A)$, $[\beta]S \subseteq T$. We demonstrate $SR_{\mathcal{M}_c}(\psi?)T$. Since $\psi? \in \ker(A)$, there exists a maximal consistent theory U such that $\psi \in U$ and $A = \bar{U}$. Since for all programs β, if $\beta \in \ker(A)$, $[\beta]S \subseteq T$, for all formulas χ, if $\chi \in U$, $[\chi?]S \subseteq T$. Since $\psi \in U$ and $\top \in U$, $[\psi?]S \subseteq T$ and $[\top?]S \subseteq T$. Since $\top \in S$, by Lemma 3, $[\top?]S = S$. Since $[\top?]S \subseteq T$, $S \subseteq T$. Since S is maximal and T is consistent, $S = T$. Since $[\psi?]S \subseteq T$ and $[\psi?]\psi \in S$, $\psi \in T$. Since $P(\psi)$, $T \in V_{\mathcal{M}_c}(\psi)$. Since $S = T$, $SR_{\mathcal{M}_c}(\psi?)T$.

Hypothesis. Let ϕ be a formula such that for all expressions *exp* (either a program, or a formula), if *exp* is an expression strictly occurring in ϕ, $P(exp)$ holds.

Step. We demonstrate $P(\phi)$ holds.

Case $\phi = p$. Left to the reader.

Cases $\phi = \bot$, $\phi = \neg\psi$ and $\phi = \psi \vee \chi$. Left to the reader.

Case $\phi = [\beta]\psi$. Let S be a maximal consistent theory.

- Suppose $S \in V_{\mathcal{M}_c}([\beta]\psi)$. We demonstrate $[\beta]\psi \in S$. If not, by Lemma 10, there exists a maximal program A and there exists a maximal consistent theory T such that $f(\beta) \in \ker(A)$, for all programs γ, if $\gamma \in \ker(A)$, $[\gamma]S \subseteq T$ and $\psi \notin T$. Since $P(\beta)$ and $P(\psi)$, $SR_{\mathcal{M}_c}(\beta)T$ and $T \notin V_{\mathcal{M}_c}(\psi)$. Hence, $S \notin V_{\mathcal{M}_c}([\beta]\psi)$: a contradiction.
- Suppose $[\beta]\psi \in S$. We demonstrate $S \in V_{\mathcal{M}_c}([\beta]\psi)$. If not, there exists a maximal consistent theory T such that $SR_{\mathcal{M}_c}(\beta)T$ and $T \notin V_{\mathcal{M}_c}(\psi)$. Since $P(\beta)$ and $P(\psi)$, there exists a maximal program A such that $f(\beta) \in \ker(A)$ and for all programs γ, if $\gamma \in \ker(A)$, $[\gamma]S \subseteq T$ and $\psi \notin T$. Hence, $[f(\beta)]S \subseteq T$. Since $[\beta]\psi \in S$, $[f(\beta)]\psi \in S$. Thus, $\psi \in [f(\beta)]S$. Since $[f(\beta)]S \subseteq T$, $\psi \in T$: a contradiction.

Case $\phi = \psi \circ \chi$. Let S be a maximal consistent theory.

- Suppose $S \in V_{\mathcal{M}_c}(\psi \circ \chi)$. We demonstrate $\psi \circ \chi \in S$. Since $S \in V_{\mathcal{M}_c}(\psi \circ \chi)$, there exist maximal consistent theories T, U such that $S \in T \star_c U$, $T \in V_{\mathcal{M}_c}(\psi)$ and $U \in V_{\mathcal{M}_c}(\chi)$. Since $P(\psi)$ and $P(\chi)$, $\psi \in T$ and $\chi \in U$. Since $S \in T \star_c U$, $\psi \circ \chi \in S$.

$$\frac{d\colon \phi}{d\colon {\sim}\phi} \qquad \frac{d\colon \phi \vee \psi}{d\colon \phi \quad d\colon \psi}$$

$$\frac{d\colon \langle\alpha\rangle\phi}{d\colon \alpha \quad d+\mathrm{size}(\alpha)\colon \phi}$$

$$\frac{d\colon \phi?}{d\colon \phi} \qquad \frac{d\colon \alpha;\beta}{d\colon \alpha \quad d+\mathrm{size}(\alpha)\colon \beta} \qquad \frac{d\colon \alpha\Delta\beta}{d\colon \alpha \quad d\colon \beta}$$

$$\frac{d\colon \phi\circ\psi}{d+1\colon \phi \quad d+1\colon \psi} \qquad \frac{d\colon \phi\rhd\psi}{d+1\colon \phi \quad d+1\colon \psi} \qquad \frac{d\colon \phi\lhd\psi}{d+1\colon \phi \quad d+1\colon \psi}$$

Fig. 1. Rules for the decomposition of localized programs and formulas.

- Suppose $\psi \circ \chi \in S$. We demonstrate $S \in V_{\mathcal{M}_c}(\psi \circ \chi)$. Since $\psi \circ \chi \in S$, by Item (1) of Lemma 7, there exist maximal consistent theories T, U such that $S \in T \star_c U$, $\psi \in T$ and $\chi \in U$. Since $P(\psi)$ and $P(\chi)$, $T \in V_{\mathcal{M}_c}(\psi)$ and $U \in V_{\mathcal{M}_c}(\chi)$. Since $S \in T \star_c U$, $S \in V_{\mathcal{M}_c}(\psi \circ \chi)$.

Cases $\phi = \psi \rhd \chi$ and $\phi = \psi \lhd \chi$. Similar to the case $\phi = \psi \circ \chi$. $\dashv$

Now, we are ready for the completeness for $\mathbf{PDL}_0^{\Delta}$.

Proposition 17 *(Completeness for* $\mathbf{PDL}_0^{\Delta}$*). Let* ϕ *be a formula. If* ϕ *is valid in the class of all separated frames,* $\phi \in \mathbf{PDL}_0^{\Delta}$*.*

7. Decidability

In this section, we prove that the logic axiomatized in the previous sections has a strong finite model property, hence is decidable. The proof is by a selection procedure. We use the notation ${\sim}\phi$ which is defined by ${\sim}\phi = \psi$ if $\phi = \neg\psi$ for some ψ, otherwise ${\sim}\phi = \neg\phi$.

Definition. We use ν to denote an expression which may be either a program or a formula and $|\nu|$ to denote the number of occurrences of symbols in ν. To provide a more semantical measure on programs, the size function is defined inductively by:

$$\begin{aligned}
\mathrm{size}(\phi?) &= 0 \\
\mathrm{size}(a) &= 1 \\
\mathrm{size}(\alpha;\beta) &= \mathrm{size}(\alpha) + \mathrm{size}(\beta) \\
\mathrm{size}(\alpha\Delta\beta) &= \min\left(\mathrm{size}(\alpha), \mathrm{size}(\beta)\right) + 1
\end{aligned}$$

Obviously, if $xR_{\mathcal{M}}(\alpha)y$ and $\mathrm{size}(\alpha) = 0$ then $x = y$. Now we decompose expressions into subexpressions, associating a *depth* to each subformula.

Definition. A *localized expression* is a tuple $d\colon \nu$ where ν is an expression and $d \in \mathbb{N}$ is called the depth. Given any localized expression $d\colon \nu$, the decomposition $\mathrm{Cl}(d\colon \nu)$ of $d\colon \nu$ is the least set of localized expressions containing $d\colon \nu$ and closed by the application of the rules from Fig. 1. We write $\mathrm{Cl}(\phi)$ for $\mathrm{Cl}(0\colon \phi)$.

The following lemma is standard.

Lemma 12. *The cardinality of* $\mathrm{Cl}(\phi)$ *is linear in* $|\phi|$*.*

Proof. We first replace the rule for negation by the rule producing $d\colon \phi$ from $d\colon \neg\phi$, obtaining the closure $\mathrm{Cl}^+(\phi)$. Then it can be easily checked that the sum of the number of occurrences of symbols in the conclusions of the rules is strictly inferior to the number of occurrences of symbols in the premise. Finally, we observe that $\mathrm{Cl}(\phi) \subseteq \mathrm{Cl}^+(\phi) \cup \left\{d\colon \neg\psi : d\colon \psi \in \mathrm{Cl}^+(\phi)\right\}$. $\dashv$

More importantly, we have the following lemma.

Lemma 13. $\max\{d : \exists\nu, d\colon \phi \in \mathrm{Cl}(\phi)\}$ *is linear in* $|\phi|$*.*

Input: A formula ϕ_0, a model $\mathcal{M}_o = (W_o, R_o, \star_o, V_o)$ and an initial state $w_0 \in W_o$ such that $w_0 \in V_{\mathcal{M}_o}(\phi_0)$.

initialisation
  $n = 0$;
  $W_s = \{(0, 0, w_0)\}$;
  $R_s(a) = \emptyset$ for all $a \in \Pi_0$;
  $(0, 0, w_0) \star_s (0, 0, w_0) = \emptyset$;
  $K = \emptyset$;
end
while $K \neq W_s$ **do**
  choose an unmarked state $(k, d, w) \in W_s \setminus K$;
  while $(k, d, w) \notin K$ **do**
    let $V_s(p) = \{(k_x, d_x, x) \in W_s : x \in V_o(p)\}$ for all $p \in \Phi_0$;
    if *there exists* $d' : \langle\alpha\rangle\phi \in \mathrm{Cl}(\phi_0)$ *such that* $\mathrm{size}(\alpha) > 0$, $d' \geq d$, $w \in V_{\mathcal{M}_o}(\langle\alpha\rangle\phi)$ *and* $(k, d, w) \notin V_{\mathcal{M}_s}(\langle\alpha\rangle\phi)$ **then**
      choose y **s.t.** $w R_{\mathcal{M}_o}(\alpha) y$ and $y \in V_{\mathcal{M}_o}(\phi)$;
      let $d_y = d + \mathrm{size}(\alpha)$;
      let $n = n + 1$;
      add (n, d_y, y) **to** W_s ;
      call LINK$(\mathcal{M}_0, \mathcal{M}_s, n, (k, d, w), (n, d_y, y), \alpha)$;
    else if *there exists* $d' : \phi \circ \psi \in \mathrm{Cl}(\phi_0)$ *such that* $d' \geq d$, $w \in V_{\mathcal{M}_o}(\phi \circ \psi)$ *and there is no* $(k_x, d_x, x), (k_y, d_y, y) \in W_s$ *such that* $(k, d, w) \in (k_x, d_x, x) \star_s (k_y, d_y, y)$ **then**
      choose x *and* y **s.t.** $w \in x \star_s y$, $x \in V_{\mathcal{M}_o}(\phi)$ and $y \in V_{\mathcal{M}_o}(\psi)$;
      add $(n+1, d+1, x)$ and $(n+2, d+1, y)$ **to** W_s ;
      add (k, d, w) **to** $(n+1, d+1, x) \star_s (n+2, d+1, y)$;
      let $n = n + 2$;
    else if *there exists* $d' : \phi \rhd \psi \in \mathrm{Cl}(\phi_0)$ *such that* $d' \geq d$, $w \in V_{\mathcal{M}_o}(\phi \rhd \psi)$ *and* $(k, d, w) \notin V_{\mathcal{M}_s}(\phi \rhd \psi)$ **then**
      choose x *and* y **s.t.** $y \in x \star_s w$, $x \in V_{\mathcal{M}_o}(\phi)$ and $y \in V_{\mathcal{M}_o}(\psi)$;
      add $(n+1, d+1, x)$ and $(n+2, d+1, y)$ **to** W_s ;
      add $(n+2, d+1, y)$ **to** $(n+1, d+1, x) \star_s (k, d, w)$;
      let $n = n + 2$;
    else if *there exists* $d' : \phi \lhd \psi \in \mathrm{Cl}(\phi_0)$ *such that* $d' \geq d$, $w \in V_{\mathcal{M}_o}(\phi \lhd \psi)$ *and* $\mathcal{M}_s, (d, w) \lhd \notin V_\phi(\psi)$ **then**
      choose x *and* y **s.t.** $x \in w \star_s y$, $x \in V_{\mathcal{M}_o}(\phi)$ and $y \in V_{\mathcal{M}_o}(\psi)$;
      add $(n+1, d+1, x)$ and $(n+2, d+1, y)$ **to** W_s ;
      add $(n+1, d+1, x)$ **to** $(k, d, w) \star_s (n+2, d+1, y)$;
      let $n = n + 2$;
    else
      add (k, d, w) **to** K ;
    end
  end
end

Procedure 1: SELECTION.

Proof. We first prove by induction on n that for all $n > 0$, if for some $d \in \mathbb{N}$ and program α, $d : \alpha \in \mathrm{Cl}(\phi)$ and $\mathrm{size}(\alpha) = n$, then there exists ν such that $d + n - 1 : \nu \in \mathrm{Cl}(\phi)$. Then it can be easily proved that if $d : \nu \in \mathrm{Cl}(\phi)$ for some $d > 0$ and ν, then there exists ν' such that $d - 1 : \nu' \in \mathrm{Cl}(\phi)$. $\dashv$

We now prove the strong finite model property for $\mathbf{PDL}_0^{\triangle}$ interpreted over the class of separated frames. Given a formula φ_0, a model $\mathcal{M}_o$ and a state w_0 such that $w_0 \in V_{\mathcal{M}_o}(\varphi_0)$, a model $\mathcal{M}_s$ is constructed such that $\mathcal{M}_s$ satisfies φ_0 and the number of states in $\mathcal{M}_o$ is bounded by an exponential in $|\varphi_0|$. The construction of $\mathcal{M}_s$ is described by the procedure SELECTION on this page. SELECTION uses the recursive procedure LINK described on the next page. Whereas SELECTION ensures that the satisfiability of all subformulas is preserved, LINK ensures that subprograms can be executed between states in $\mathcal{M}_s$. The following lemmas are used to prove the strong finite model property.

Lemma 14. *The procedure* SELECTION *terminates and the cardinality of* W_s *is exponential in* $|\phi_0|$.

Proof. We consider the tree (V, E) such that $V = W_s$ and there is a edge from (k_x, d_x, x) to (k_y, d_y, y) iff (k_y, d_y, y) has been added to W_s while (k_x, d_x, x) was chosen in SELECTION. It can be easily proved by induction on $|\alpha|$ that during any call to LINK with program argument α, the number of states added to W_s is inferior or equal to $2|\alpha|$. Hence, by Lemma 12, the branching factor of (V, E) is bounded by a quadratic function in $|\phi_0|$. To prove that the depth of (V, E) is bounded by a linear function in $|\phi_0|$, we use Lemma 13 and prove that d is strictly increasing along the branches of (V, E). For that matter, it suffices to verify that whenever LINK$(\mathcal{M}_0, \mathcal{M}_s, n, (k_x, d_x, x), (k_y, d_y, y), \alpha)$ is called while (k_w, d_w, w) is chosen in SELECTION, then $d_x \geq d_w$ and $d_y > d_w$. $\dashv$

Lemma 15. *Whenever* LINK *is called,* $d_y \leq d_x + \mathrm{size}(\alpha)$.

The following lemma is essential for the forthcoming Lemma 19 as it makes the proof for the dual modalities $\bar{\circ}$, $\bar{\lhd}$ and $\bar{\rhd}$ trivial.

Input: Two models $\mathcal{M}_o = (W_o, R_o, \star_o, V_o)$ and $\mathcal{M}_s = (W_s, R_s, \star_s, V_s)$, an integer n, two states $(k_x, d_x, x), (k_y, d_y, y) \in W_s$ and a program α such that $xR_{\mathcal{M}_o}(\alpha)y$.

1 **if** *α is of the form $a \in \Pi_0$* **then**
2   **add** $\big((k_x, d_x, x), (k_y, d_y, y)\big)$ **to** $R_s(a)$;
3 **else if** *α is of the form $(\beta; \gamma)$* **then**
4   **if** $\mathrm{size}(\beta) = 0$ **then**
5   **call** LINK $\big(\mathcal{M}_o, \mathcal{M}_s, n, (k_x, d_x, x), (k_y, d_y, y), \gamma\big)$;
6   **else if** $\mathrm{size}(\gamma) = 0$ **then**
7   **call** LINK $\big(\mathcal{M}_o, \mathcal{M}_s, n, (k_x, d_x, x), (k_y, d_y, y), \beta\big)$;
8   **else**
9   **choose** z **s.t.** $xR_{\mathcal{M}_o}(\beta)z$ and $zR_{\mathcal{M}_o}(\gamma)y$;
10   **let** $n = n + 1$;
11   **let** $d_z = d_x + \mathrm{size}(\alpha)$;
12   **add** (n, d_z, z) **to** W_s ;
13   **call** LINK $(\mathcal{M}_o, \mathcal{M}_s, n, (k_x, d_x, x), (n, d_z, z), \beta)$;
14   **call** LINK $\big(\mathcal{M}_o, \mathcal{M}_s, n, (n, d_z, z), (k_y, d_y, y), \gamma\big)$;
15   **end**
16 **else if** *α is of the form $(\beta \Delta \gamma)$* **then**
17   **if** $\mathrm{size}(\beta) = 0$ *and* $\mathrm{size}(\gamma) = 0$ **then**
18   **add** (k_y, d_y, y) **to** $(k_x, d_x, x) \star_s (k_x, d_x, x)$;
19   **else if** $\mathrm{size}(\beta) = 0$ **then**
20   **choose** z **s.t.** $xR_{\mathcal{M}_o}(\gamma)z$ and $y \in x \star_o z$;
21   **let** $n = n + 1$;
22   **let** $d_z = \min\big(d_y + 1, d_x + \mathrm{size}(\gamma)\big)$;
23   **add** (n, d_z, z) **to** W_s ;
24   **add** (k_y, d_y, y) **to** $(k_x, d_x, x) \star_s (n, d_z, z)$;
25   **call** LINK $(\mathcal{M}_o, \mathcal{M}_s, n, (k_x, d_x, x), (n, d_z, z), \gamma)$;
26   **else if** $\mathrm{size}(\gamma) = 0$ **then**
27   **choose** w **s.t.** $xR_{\mathcal{M}_o}(\beta)w$ and $y \in w \star_o x$;
28   **let** $n = n + 1$;
29   **let** $d_w = \min\big(d_y + 1, d_x + \mathrm{size}(\beta)\big)$;
30   **add** (n, d_w, w) **to** W_s ;
31   **add** (k_y, d_y, y) **to** $(n, d_w, w) \star_s (k_x, d_x, x)$;
32   **call** LINK $(\mathcal{M}_o, \mathcal{M}_s, n, (k_x, d_x, x), (n, d_w, w), \beta)$;
33   **else**
34   **choose** *w and z* **s.t.** $xR_{\mathcal{M}_o}(\beta)w$, $xR_{\mathcal{M}_o}(\gamma)z$ and $y \in w \star_o z$;
35   **let** $n = n + 2$;
36   **let** $d_w = \min\big(d_y + 1, d_x + \mathrm{size}(\beta), d_x + \mathrm{size}(\gamma) + 1\big)$;
37   **let** $d_z = \min\big(d_y + 1, d_x + \mathrm{size}(\gamma), d_x + \mathrm{size}(\beta) + 1\big)$;
38   **add** $(n - 1, d_w, w)$ and (n, d_z, z) **to** W_s ;
39   **add** (k_y, d_y, y) **to** $(n - 1, d_w, w) \star_s (n, d_z, z)$;
40   **call** LINK $(\mathcal{M}_o, \mathcal{M}_s, n, (k_x, d_x, x), (n - 1, d_w, w), \beta)$;
41   **call** LINK $(\mathcal{M}_o, \mathcal{M}_s, n, (k_x, d_x, x), (n, d_z, z), \gamma)$;
42   **end**
43 **end**

Procedure 2: LINK.

Lemma 16. *For all $(k_y, d_y, y), (k_w, d_w, w), (k_z, d_z, z) \in W_s$, such that $(k_y, d_y, y) \in (k_w, d_w, w) \star_s (k_z, d_z, z)$, we have $y \in w \star_o z$, $|d_y - d_w| \leq 1$, $|d_y - d_z| \leq 1$ and $|d_w - d_z| \leq 1$.*

Proof. We prove that at line 39 of LINK, $|d_y - d_w| \leq 1$ and $|d_w - d_z| \leq 1$, the other cases and properties being either similar or straightforward.

Suppose first that $d_w = d_y + 1$. Then obviously $|d_y - d_w| \leq 1$. If $d_z = d_y + 1$, then $|d_w - d_z| \leq 1$ is trivial too. If $d_z = d_x + \mathrm{size}(\beta) + 1$, by minimality, $d_z \leq d_y + 1$ and $\mathrm{size}(\beta) < \mathrm{size}(\gamma)$. By Lemma 15, $d_y \leq d_z$. Therefore, $d_w - 1 \leq d_z \leq d_w$. If $d_z = d_x + \mathrm{size}(\gamma)$, by minimality, $d_z \leq d_y + 1$ and $\mathrm{size}(\gamma) \leq \mathrm{size}(\beta)$. By Lemma 15, $dy < d_z$. Therefore, $d_w - 1 < d_z \leq d_w$.

Suppose now that $d_w = d_x + \mathrm{size}(\gamma) + 1$. By minimality, $d_y > d_x + \mathrm{size}(\gamma)$ and $\mathrm{size}(\beta) > \mathrm{size}(\gamma)$. By Lemma 15, $d_y \leq d_x + \mathrm{size}(\gamma) + 1$. Therefore, $d_y = d_w$. If $d_z = d_y + 1$ or $d_z = d_x + \mathrm{size}(\gamma)$, then obviously $d_w - 1 \leq d_z \leq d_w + 1$. If $d_z = d_x + \mathrm{size}(\beta) + 1$, by minimality $\mathrm{size}(\beta) < \mathrm{size}(\gamma)$, which is impossible.

Suppose finally that $d_w = d_x + \mathrm{size}(\beta)$. By minimality, $d_y \geq d_x + \mathrm{size}(\beta)$ and $\mathrm{size}(\gamma) \geq \mathrm{size}(\beta)$. By Lemma 15, $d_y \leq d_x + \mathrm{size}(\beta) + 1$. Therefore, $d_w \leq d_y \leq d_w + 1$. If $d_z = d_x + \mathrm{size}(\beta) + 1$, then obviously $d_z = d_w + 1$. If $d_z = d_x + \mathrm{size}(\gamma) + 1$, by minimality, $\mathrm{size}(\gamma) \leq \mathrm{size}(\beta)$. Hence $\mathrm{size}(\gamma) = \mathrm{size}(\beta)$ and $d_z = d_w$. If $d_z = d_y + 1$, by minimality, $d_y < d_x + \mathrm{size}(\beta)$, which is impossible. $\dashv$

Lemma 17. *For all $(k_x, d_x, x), (k_y, d_y, y) \in W_s$ and all α, if $(k_x, d_x, x)R_{\mathcal{M}_s}(\alpha)(k_y, d_y, y)$, then $d_y \leq d_x + \mathrm{size}(\alpha)$.*

Lemma 18. *If $\mathcal{M}_o$ is separated, then $\mathcal{M}_s$ is separated too.*

Proof. Suppose (k_w, d_w, w) has been added to $(k_{w_1}, d_{w_1}, w_1) \star_s (k_{w_2}, d_{w_2}, w_2)$ while $(k_w, d_w, w) \in (k'_{w_1}, d'_{w_1}, w_1{}') \star_s (k'_{w_2}, d'_{w_2}, w_2{}')$. Then (k_w, d_w, w) cannot have been added to $(k_{w_1}, d_{w_1}, w_1) \star_s (k_{w_2}, d_{w_2}, w_2)$ in Selection because at lines 26 and 31, (k_w, d_w, w) is fresh and at line 21 the condition ensures that $(k_w, d_w, w) \notin (k'_{w_1}, d'_{w_1}, w_1{}') \star_s (k'_{w_2}, d'_{w_2}, w_2{}')$. Finally, to prove that (k_w, d_w, w) cannot have been added to $(k_{w_1}, d_{w_1}, w_1) \star_s (k_{w_2}, d_{w_2}, w_2)$ in Link, it suffices to verify that whenever Link is called there is no $(k_{y_1}, d_{y_1}, y_1), (k_{y_2}, d_{y_2}, y_2) \in W_s$ such that $(k_y, d_y, y) \in (k_{y_1}, d_{y_1}, y_1) \star_s (k_{y_2}, d_{y_2}, y_2)$, which is straightforward. $\dashv$

We can now prove the following lemma.

Lemma 19. *If $\mathcal{M}_o$ is separated, then $(0, 0, w_0) \in V_{\mathcal{M}_s}(\phi_0)$.*

Proof. The following properties are proved for all $n \in \mathbb{N}$, by induction on n:

1. for all $d : \phi \in \mathrm{Cl}(\phi_0)$ and all $(k_w, d_w, w) \in W_s$, if $|\phi| = n$, $d \geq d_w$ and $w \in V_{\mathcal{M}_o}(\phi)$ then $(k_w, d_w, w) \in V_{\mathcal{M}_s}(\phi)$
2. if Link has been called with last arguments (k_x, d_x, x), (k_y, d_y, y) and α with $|\alpha| = n$, then $(k_x, d_x, x) R_{\mathcal{M}_s}(\alpha)(k_y, d_y, y)$
3. for all α and all $(k_x, d_x, x), (k_y, d_y, y) \in W_s$, if $|\alpha| = n$ and $(k_x, d_x, x) R_{\mathcal{M}_s}(\alpha)(k_y, d_y, y)$ then $x R_{\mathcal{M}_o}(\alpha) y$

We give details of only the following cases, the other ones being either similar or straightforward.

Hypothesis 1 when $\phi = \langle\alpha\rangle\psi$. There exists $y \in W_o$ such that $w R_{\mathcal{M}_o}(\alpha) y$ and $y \in V_{\mathcal{M}_o}(\psi)$. Moreover, $d + \mathrm{size}(\alpha) : \psi \in \mathrm{Cl}(\phi_0)$. If $(k_w, d_w, w) \notin V_{\mathcal{M}_s}(\langle\alpha\rangle\psi)$, conditions at line 12 are satisfied. Therefore, a state $(k_y, d_w + \mathrm{size}(\alpha), y)$ is added to W_s and Link is called with last arguments (k_w, d_w, w), $(k_y, d_w + \mathrm{size}(\alpha), y)$ and α. Since $|\alpha| < |\phi|$, by induction hypothesis 2, $(k_w, d_w, w) R_{\mathcal{M}_s}(\alpha)(k_y, d_w + \mathrm{size}(\alpha), y)$. And since $d_w + \mathrm{size}(\alpha) \leq d + \mathrm{size}(\alpha)$ and $|\psi| < |\phi|$, by induction hypothesis 1, $(k_y, d_w + \mathrm{size}(\alpha), y) \in V_{\mathcal{M}_s}(\psi)$.

Hypothesis 1 when $\phi = [\alpha]\psi$. Suppose $(k_w, d_w, w) R_{\mathcal{M}_s}(\alpha)(k_y, d_y, y)$. Since $|\alpha| < |\phi|$, by induction hypothesis 3, $w R_{\mathcal{M}_o}(\alpha) y$. Therefore $y \in V_{\mathcal{M}_o}(\psi)$. By Lemma 17, $d_y \leq d_w + \mathrm{size}(\alpha)$. Since $d + \mathrm{size}(\alpha) : \psi \in \mathrm{Cl}(\phi_0)$, by induction hypothesis 1, $(k_y, d_y, y) \in V_{\mathcal{M}_s}(\psi)$.

Hypothesis 1 when $\phi = \psi \circ \chi$. By the condition at line 18, there exists (k_x, d_x, x) and (k_y, d_y, y) such that $(k_w, d_w, w) \in (k_x, d_x, x) \star_s (k_y, d_y, y)$. By Lemma 16, $w \in x \star_o y$, $d_x \leq d_w + 1$ and $d_y \leq d_w + 1$. Since $\mathcal{M}_o$ is separated, $x \in V_{\mathcal{M}_o}(\psi)$ and $y \in V_{\mathcal{M}_o}(\chi)$. Since $d + 1 : \psi \in \mathrm{Cl}(\phi_0)$ and $d + 1 : \chi \in \mathrm{Cl}(\phi_0)$, by induction hypothesis 1, $(k_x, d_x, x) \in V_{\mathcal{M}_s}(\psi)$ and $(k_y, d_y, y) \in V_{\mathcal{M}_s}(\chi)$.

Hypothesis 2 when $\alpha = \beta; \gamma$ and $\mathrm{size}(\beta) = 0$. First, it has to be verified that whenever Link is called, there exists $d \in \mathbb{N}$ such that $d \geq d_x$ and $d : \alpha \in \mathrm{Cl}(\phi_0)$, which is straightforward. Let d_α be such that $d_\alpha \geq d_x$ and $d_\alpha : \alpha \in \mathrm{Cl}(\phi_0)$. Since $\mathrm{size}(\alpha) = 0$, there is a list $\phi_1, \ldots, \phi_m$ such that $\beta = \phi_1?; \ldots; \phi_m?$. Moreover, for all $\ell \in 1..m$, $|\phi_\ell| < |\alpha|$ and $d_\alpha : \phi_\ell \in \mathrm{Cl}(\phi_0)$. Hence, by induction hypothesis 1, $(k_x, d_x, x) R_{\mathcal{M}_s}(\beta)(k_x, d_x, x)$. And since $|\gamma| < |\alpha|$, by induction hypothesis 2, $(k_x, d_x, x) R_{\mathcal{M}_s}(\gamma)(k_y, d_y, y)$.

Hypothesis 3 when $\alpha = a$. First, it has to be verified that whenever Link is called, $x R_{\mathcal{M}_o}(\alpha) y$, which is straightforward when $\mathcal{M}_o$ is separated. Finally, it suffices to remark that since $(k_x, d_x, x) R_{\mathcal{M}_s}(a)(k_y, d_y, y)$, line 2 of Link have been called. $\dashv$

We have proved the following proposition:

Proposition 18. *Any $\mathbf{PDL}_0^\triangle$ formula ϕ satisfiable in a separated model is satisfiable in a separated finite model with a number of states bounded by an exponential in $|\phi|$.*

The method from [17] can easily be adapted to prove that the model-checking problem for $\mathbf{PDL}_0^\triangle$ can be solved in polynomial time in the size of the model. Hence we have the following proposition.

Proposition 19. *The satisfiability problem for $\mathbf{PDL}_0^\triangle$ in the class of separated frames is decidable in non-deterministic exponential time.*

8. Conclusion

In modal logic, standard proofs of completeness for a given logic are usually based on the canonical frame construction consisting of the set of all maximal consistent sets of the logic equipped with standard definitions for the canonical accessibility relations. Since the program operation of fork considered in [11, Chapter 1] is not modally definable in the ordinary language of **PDL**, this method cannot work in our case. As a result, we have given an axiomatization of $\mathbf{PDL}_0^\triangle$, our variant of iteration-free **PDL** with fork, using an unorthodox rule of proof and we have proved its completeness using large programs. So, we have extended the canonical frame construction introducing new tools and techniques connected with an unorthodox rule of proof and large programs.

We anticipate a number of further investigations. First, there is the following general question: is it possible to eliminate the rule of proof (**FOR**) and to replace it with a finite set of additional axiom schemes? Second, more details on decidability/complexity issues would be relevant. Third, there is the question of the complete axiomatization of validity with respect to other classes of frames like the class of frames considered in [11, Chapter 1], i.e. the class of all separated, deterministic and serial frames. Fourth, is the validity problem with respect to the class of all separated, deterministic and serial frames decidable? If it is, what is its complexity? Fifth, it remains to see whether our approach can be extended to the full language of **PDL** with fork, this time with iteration.

A novelty in the paper is the proof that fork is modally definable in a language with propositional quantifiers and that the rule (**FOR**) in a sense simulates the quantifier rule for universal quantification in the context of the definition of fork. This is a new look on the nature of some context dependent rules of proof like (**FOR**). In some ways, (**FOR**) is similar to the rule for intersection from [3,4]. See also [1] for ideas about its elimination from the axiomatization of $\mathbf{PDL}_0^{\Delta}$ we have given. We expect that our variant of the canonical frame construction can be applied to other logics, for instance **PRSPDL**, the variant of **PDL** with fork given rise by the binary operation of fork ∇ considered in Benevides et al. [5, Section 2] and whose axiomatization is still open.

Declaration of Competing Interest

None declared.

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Annex

This Annex contains the proofs of some of our results.

Proof of Proposition 1. Left to the reader.

Proof of Proposition 2. By induction on the formation of parametrized actions and admissible forms. Left to the reader.

Proof of Proposition 3. Let $\mathcal{M} = (W, R, \star, V)$ be a model.

(**A**1)–(**A**15). Left to the reader.

(**A**16). It suffices to demonstrate by induction on the formation of α that for all $x, y \in W$, if $xR_{\mathcal{M}}(\alpha(\phi?))y$, either $xR_{\mathcal{M}}(\alpha((\phi \wedge \chi)?))y$, or $xR_{\mathcal{M}}(\alpha((\phi \wedge \neg\chi)?))y$. Left to the reader.

(**A**17). It suffices to demonstrate by induction on the formation of α that for all $x, y \in W$, $xR_{\mathcal{M}}(f(\alpha))y$ iff $xR_{\mathcal{M}}(\alpha)y$. Left to the reader.

Proof of Proposition 4. Left to the reader.

Proof of Proposition 5. (**MP**). Left to the reader.

(**N**). Left to the reader.

Proof of Proposition 7. Let $\mathcal{F} = (W, R, \star)$ be a frame.

(1). Suppose $\mathcal{F} \models \forall x\ \exists y\ \ y \in x \star x$. Suppose $\mathcal{F} \not\models \langle\top?\Delta\top?\rangle\top$. Hence, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ and there exists $x \in W$ such that $x \notin V_{\mathcal{M}}(\langle\top?\Delta\top?\rangle\top)$. Thus, for all $y \in W$, not $xR_{\mathcal{M}}(\top?\Delta\top?)y$. Since for all $z \in W$, $xR_{\mathcal{M}}(\top?\Delta\top?)z$ iff $z \in x \star x$, for all $y \in W$, $y \notin x \star x$. Therefore, $\mathcal{F} \not\models \forall x\ \exists y\ \ y \in x \star x$: a contradiction.

Suppose $\mathcal{F} \models \langle\top?\Delta\top?\rangle\top$. Suppose $\mathcal{F} \not\models \forall x\ \exists y\ \ y \in x \star x$. Hence, there exists $x \in W$ such that for all $y \in W$, $y \notin x \star x$. Let $\mathcal{M} = (W, R, \star, V)$ be a model on $\mathcal{F}$ (it suffices to take, for V, any valuation on $\mathcal{F}$). Since for all $y \in W$, $y \notin x \star x$ and for all $z \in W$, $z \in x \star x$ iff $xR_{\mathcal{M}}(\top?\Delta\top?)z$, for all $y \in W$, not $xR_{\mathcal{M}}(\top?\Delta\top?)y$. Thus, $x \notin V_{\mathcal{M}}(\langle\top?\Delta\top?\rangle\top)$. Therefore, $\mathcal{F} \not\models \langle\top?\Delta\top?\rangle\top$: a contradiction.

(2). Suppose $\mathcal{F} \models \forall x\ \forall y\ \forall z\ (y \in x \star x \wedge z \in x \star x \rightarrow y = z)$. Suppose $\mathcal{F} \not\models \langle\top?\Delta\top?\rangle p \rightarrow [\top?\Delta\top?]p$. Hence, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ and there exists $x \in W$ such that $x \notin V_{\mathcal{M}}(\langle\top?\Delta\top?\rangle p \rightarrow [\top?\Delta\top?]p)$. Thus, $x \in V_{\mathcal{M}}(\langle\top?\Delta\top?\rangle p)$ and $x \notin V_{\mathcal{M}}([\top?\Delta\top?]p)$. Therefore, there exists $y \in W$ such that $xR_{\mathcal{M}}(\top?\Delta\top?)y$ and $y \in V(p)$ and there exists $z \in W$ such that $xR_{\mathcal{M}}(\top?\Delta\top?)z$ and $z \notin V(p)$. Since for all $t \in W$, $xR_{\mathcal{M}}(\top?\Delta\top?)t$ iff $t \in x \star x$, and $y \in x \star x$ and $z \in x \star x$. Since $y \in V(p)$ and $z \notin V(p)$, $y \neq z$. Since $y \in x \star x$ and $z \in x \star x$, $\mathcal{F} \not\models \forall x\ \forall y\ \forall z\ (y \in x \star x \wedge z \in x \star x \to y = z)$: a contradiction.

Suppose $\mathcal{F} \models \langle\top?\Delta\top?\rangle p \to [\top?\Delta\top?]p$. Suppose $\mathcal{F} \not\models \forall x\ \forall y\ \forall z\ (y \in x \star x \wedge z \in x \star x \to y = z)$. Hence, there exists $x, y, z \in W$ such that $y \in x \star x$, $z \in x \star x$ and $y \neq z$. Thus, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ such that $y \in V(p)$ and $z \notin V(p)$ (it suffices to take, for V, any valuation on $\mathcal{F}$ such that $V(p) = \{y\}$). Since $y \in x \star x$, $z \in x \star x$ and for all $t \in W$, $xR_{\mathcal{M}}(\top?\Delta\top?)t$ iff $t \in x \star x$, $xR_{\mathcal{M}}(\top?\Delta\top?)y$ and $xR_{\mathcal{M}}(\top?\Delta\top?)z$. Since $y \in V(p)$ and $z \notin V(p)$, $x \in V_{\mathcal{M}}(\langle\top?\Delta\top?\rangle p)$ and $x \notin V_{\mathcal{M}}([\top?\Delta\top?]p)$. Therefore, $x \notin V_{\mathcal{M}}(\langle\top?\Delta\top?\rangle p \to [\top?\Delta\top?]p)$. Consequently, $\mathcal{F} \not\models \langle\top?\Delta\top?\rangle p \to [\top?\Delta\top?]p$: a contradiction.

(3). Suppose $\mathcal{F} \models \forall x\ \forall y\ (y \in x \star x \to x \in x \star y)$. Suppose $\mathcal{F} \not\models p \to [\top?\Delta\top?](p \triangleright p)$. Hence, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ and there exists $x \in W$ such that $x \notin V_{\mathcal{M}}(p \to [\top?\Delta\top?](p \triangleright p))$. Thus, $x \in V(p)$ and $x \notin V_{\mathcal{M}}([\top?\Delta\top?](p \triangleright p))$. Therefore, there exists $y \in W$ such that $xR_{\mathcal{M}}(\top?\Delta\top?)y$ and $y \notin V_{\mathcal{M}}(p \triangleright p)$. Since for all $z \in W$, $xR_{\mathcal{M}}(\top?\Delta\top?)z$ iff $z \in x \star x$, $y \in x \star x$. Since $x \in V(p)$ and $y \notin V_{\mathcal{M}}(p \triangleright p)$, $x \notin x \star y$. Since $y \in x \star x$, $\mathcal{F} \not\models \forall x\ \forall y\ (y \in x \star x \to x \in x \star y)$: a contradiction.

Suppose $\mathcal{F} \models p \to [\top?\Delta\top?](p \triangleright p)$. Suppose $\mathcal{F} \not\models \forall x\ \forall y\ (y \in x \star x \to x \in x \star y)$. Hence, there exists $x, y \in W$ such that $y \in x \star x$ and $x \notin x \star y$. Thus, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ such that $x \in V(p)$ and $y \notin V_{\mathcal{M}}(p \triangleright p)$ (it suffices to take, for V, any valuation on $\mathcal{F}$ such that $V(p) = \{x\}$). Since $y \in x \star x$ and for all $z \in W$, $xR_{\mathcal{M}}(\top?\Delta\top?)z$ iff $z \in x \star x$, $xR_{\mathcal{M}}(\top?\Delta\top?)y$. Since $y \notin V_{\mathcal{M}}(p \triangleright p)$, $x \notin V_{\mathcal{M}}([\top?\Delta\top?](p \triangleright p))$. Since $x \in V(p)$, $x \notin V_{\mathcal{M}}(p \to [\top?\Delta\top?](p \triangleright p))$. Therefore, $\mathcal{F} \not\models p \to [\top?\Delta\top?](p \triangleright p)$: a contradiction.

(4). Similar to (3).

(5). Suppose $\mathcal{F} \models \forall x\ \forall y\ \forall z\ (z \in x \star y \leftrightarrow z \in y \star x)$. Suppose $\mathcal{F} \not\models p \circ q \leftrightarrow q \circ p$. Hence, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ and there exists $x \in W$ such that $x \notin V_{\mathcal{M}}(p \circ q \leftrightarrow q \circ p)$. Thus, either $x \notin V_{\mathcal{M}}(p \circ q \to q \circ p)$, or $x \notin V_{\mathcal{M}}(q \circ p \to p \circ q)$. Without loss of generality, suppose $x \notin V_{\mathcal{M}}(p \circ q \to q \circ p)$. Therefore, $x \in V_{\mathcal{M}}(p \circ q)$ and $x \notin V_{\mathcal{M}}(q \circ p)$. Consequently, there exists $y, z \in W$ such that $x \in y \star z$, $y \in V(p)$ and $z \in V(q)$. Since $\mathcal{F} \models \forall x\ \forall y\ \forall z\ (z \in x \star y \leftrightarrow z \in y \star x)$, $x \in z \star y$. Since $z \in V(q)$ and $y \in V(p)$, $x \in V_{\mathcal{M}}(q \circ p)$: a contradiction.

Suppose $\mathcal{F} \models p \circ q \leftrightarrow q \circ p$. Suppose $\mathcal{F} \not\models \forall x\ \forall y\ \forall z\ (z \in x \star y \leftrightarrow z \in y \star x)$. Hence, there exists $x, y, z \in W$ such that either $z \in x \star y$ and $z \notin y \star x$, or $z \in y \star x$ and $z \notin x \star y$. Without loss of generality, suppose $z \in x \star y$ and $z \notin y \star x$. Thus, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ such that $z \in V_{\mathcal{M}}(p \circ q)$ (it suffices to take, for V, any valuation on $\mathcal{F}$ such that $V(p) = \{x\}$ and $V(q) = \{y\}$). Since $\mathcal{F} \models p \circ q \leftrightarrow q \circ p$, $z \in V_{\mathcal{M}}(q \circ p)$. Therefore $z \in y \star x$: a contradiction.

(6). Suppose $\mathcal{F} \models \forall x\ \exists y\ \exists z\ x \in y \star z$. Suppose $\mathcal{F} \not\models \top \circ \top$. Hence, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ and there exists $x \in W$ such that $x \notin V_{\mathcal{M}}(\top \circ \top)$. Thus, for all $y, z \in W$, $x \notin y \star z$. Therefore, $\mathcal{F} \not\models \forall x\ \exists y\ \exists z\ x \in y \star z$: a contradiction.

Suppose $\mathcal{F} \models \top \circ \top$. Suppose $\mathcal{F} \not\models \forall x\ \exists y\ \exists z\ x \in y \star z$. Hence, there exists $x \in W$ such that for all $y, z \in W$, $x \notin y \star z$. Let $\mathcal{M} = (W, R, \star, V)$ be a model on $\mathcal{F}$ (it suffices to take, for V, any valuation on $\mathcal{F}$). Since for all $y, z \in W$, $x \notin y \star z$, $x \notin V_{\mathcal{M}}(\top \circ \top)$. Therefore, $\mathcal{F} \not\models \top \circ \top$: a contradiction.

(7) and (8). Similar to (6).

(9). Suppose $\mathcal{F} \models \forall x\ \forall y\ \forall z\ \forall t\ (t \in (x \star y) \star z \leftrightarrow t \in x \star (y \star z))$. Suppose $\mathcal{F} \not\models (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$. Hence, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ and there exists $x \in W$ such that $x \notin V_{\mathcal{M}}((p \circ q) \circ r \leftrightarrow p \circ (q \circ r))$. Thus, either $x \notin V_{\mathcal{M}}((p \circ q) \circ r \to p \circ (q \circ r))$, or $x \notin V_{\mathcal{M}}(p \circ (q \circ r) \to (p \circ q) \circ r)$. Without loss of generality, suppose $x \notin V_{\mathcal{M}}((p \circ q) \circ r \to p \circ (q \circ r))$. Therefore, $x \in V_{\mathcal{M}}((p \circ q) \circ r)$ and $x \notin V_{\mathcal{M}}(p \circ (q \circ r))$. Consequently, there exists $y, z \in W$ such that $x \in y \star z$, $y \in V_{\mathcal{M}}(p \circ q)$ and $z \in V(r)$. Hence, there exists $t, u \in W$ such that $y \in t \star u$, $t \in V(p)$ and $u \in V(q)$. Since $x \in y \star z$, $x \in (t \star u) \star z$. Since $\mathcal{F} \models \forall x\ \forall y\ \forall z\ \forall t\ (t \in (x \star y) \star z \leftrightarrow t \in x \star (y \star z))$, $x \in t \star (u \star z)$. Thus, there exists $v \in W$ such that $v \in u \star z$ and $x \in t \star v$. Since $u \in V(q)$ and $z \in V(r)$, $v \in V_{\mathcal{M}}(q \circ r)$. Since $x \in t \star v$ and $t \in V(p)$, $x \in V_{\mathcal{M}}(p \circ (q \circ r))$: a contradiction.

Suppose $\mathcal{F} \models (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$. Suppose $\mathcal{F} \not\models \forall x\ \forall y\ \forall z\ \forall t\ (t \in (x \star y) \star z \leftrightarrow t \in x \star (y \star z))$. Hence, there exists $x, y, z, t \in W$ such that either $t \in (x \star y) \star z$ and $t \notin x \star (y \star z)$, or $t \in x \star (y \star z)$ and $t \notin (x \star y) \star z$. Without loss of generality, suppose $t \in (x \star y) \star z$ and $t \notin x \star (y \star z)$. Thus, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ such that $t \in V_{\mathcal{M}}((p \circ q) \circ r)$ (it suffices to take, for V, any valuation on $\mathcal{F}$ such that $V(p) = \{x\}$, $V(q) = \{y\}$ and $V(r) = \{z\}$). Since $\mathcal{F} \models (p \circ q) \circ r \leftrightarrow p \circ (q \circ r)$, $t \in V_{\mathcal{M}}(p \circ (q \circ r))$. Therefore $t \in x \star (y \star z)$: a contradiction.

(10). Suppose $\mathcal{F} \models \forall x\ \forall y\ \forall z\ x \notin y \star z$. Suppose $\mathcal{F} \not\models \bot \bar{\circ} \bot$. Hence, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ and there exists $x \in W$ such that $x \notin V_{\mathcal{M}}(\bot \bar{\circ} \bot)$. Thus, there exists $y, z \in W$ such that $x \in y \star z$. Therefore, $\mathcal{F} \not\models \forall x\ \forall y\ \forall z\ x \notin y \star z$: a contradiction.

Suppose $\mathcal{F} \models \bot \bar{\circ} \bot$. Suppose $\mathcal{F} \not\models \forall x\ \forall y\ \forall z\ x \notin y \star z$. Hence, there exists $x, y, z \in W$ such that $x \in y \star z$. Let $\mathcal{M} = (W, R, \star, V)$ be a model on $\mathcal{F}$ (it suffices to take, for V, any valuation on $\mathcal{F}$). Since $x \in y \star z$, $x \notin V_{\mathcal{M}}(\bot \bar{\circ} \bot)$. Therefore, $\mathcal{F} \not\models \bot \bar{\circ} \bot$: a contradiction.

Proof of Proposition 8. Let $\mathcal{F} = (W, R, \star)$ be a frame.

Suppose $\mathcal{F}$ is separated. Suppose $\mathcal{F} \not\models p \circ q \to (p \bar{\circ} \bot) \wedge (\bot \bar{\circ} q)$. Hence, there exists a model $\mathcal{M} = (W, R, \star, V)$ on $\mathcal{F}$ and there exists $x \in W$ such that $x \notin V_{\mathcal{M}}(p \circ q \to (p \bar{\circ} \bot) \wedge (\bot \bar{\circ} q))$. Thus, $x \in V_{\mathcal{M}}(p \circ q)$ and $x \notin V_{\mathcal{M}}((p \bar{\circ} \bot) \wedge (\bot \bar{\circ} q))$. Therefore, there exists $y, z \in W$ such that $x \in y \star z$, $y \in V(p)$ and $z \in V(q)$. Moreover, either $x \notin V_{\mathcal{M}}(p \bar{\circ} \bot)$, or $x \notin V_{\mathcal{M}}(\bot \bar{\circ} q)$.

Case $x \notin V_{\mathcal{M}}(p \bar{\circ} \bot)$. Hence, there exists $t, u \in W$ such that $x \in t \star u$ and $t \notin V(p)$. Since $y \in V(p)$, $y \neq t$. Since $\mathcal{F}$ is separated, $x \in y \star z$ and $x \in t \star u$, $y = t$: a contradiction.

Case $x \notin V_{\mathcal{M}}(\perp \bar{\circ} q)$. Similar to the case $x \notin V_{\mathcal{M}}(p \bar{\circ} \perp)$.

Suppose $\mathcal{F} \models p \circ q \rightarrow (p \bar{\circ} \perp) \wedge (\perp \bar{\circ} q)$. Suppose $\mathcal{F}$ is not separated. Hence, there exists $x, y, z, t, u \in W$ such that $u \in x \star y$, $u \in z \star t$ and either $x \neq z$, or $y \neq t$.

Case $x \neq z$. Let $\mathcal{M} = (W, R, \star, V)$ be a model on $\mathcal{F}$ such that $x \in V(p)$, $y \in V(q)$ and $z \notin V(p)$ (it suffices to take, for V, any valuation on $\mathcal{F}$ such that $V(p) = \{x\}$ and $V(q) = \{y\}$). Since $u \in x \star y$ and $u \in z \star t$, $u \in V_{\mathcal{M}}(p \circ q)$ and $u \notin V_{\mathcal{M}}(p \bar{\circ} \perp)$. Hence, $u \notin V_{\mathcal{M}}(p \circ q \rightarrow (p \bar{\circ} \perp) \wedge (\perp \bar{\circ} q))$. Thus, $\mathcal{F} \not\models p \circ q \rightarrow (p \bar{\circ} \perp) \wedge (\perp \bar{\circ} q))$: a contradiction.

Case $y \neq t$. Similar to the case $x \neq z$.

Proof of Proposition 9. Suppose the class of all deterministic frames is modally defined by the formula ϕ. Let $\mathcal{F} = (W, R, \star)$, $\mathcal{F}' = (W', R', \star')$ be the frames defined by:

- $W = \{(x, i) : i \in \mathbb{Z}\} \cup \{(y, j) : j \in \mathbb{Z}\} \cup \{(u, i, j) : i, j \in \mathbb{Z}\}$;
- R is the empty function;
- $(x, i) \star (y, j) = \{(u, i, j)\}$, otherwise $\star$ is the empty function;
- $W' = \{x', y', z', t'\}$;
- R' is the empty function;
- $x' \star' y' = \{z', t'\}$, otherwise $\star'$ is the empty function.

Obviously, $\mathcal{F}$ is deterministic and $\mathcal{F}'$ is not deterministic. Since the class of all deterministic frames is modally defined by the formula ϕ, $\mathcal{F} \models \phi$ and $\mathcal{F}' \not\models \phi$. Hence, there exists a model $\mathcal{M}' = (W', R', \star', V')$ on $\mathcal{F}'$ such that $V_{\mathcal{M}'}(\phi) \neq W'$. Let $f : v \in W \mapsto f(v) \in W'$ be such that for all $i \in \mathbb{Z}$, $f(x, i) = x'$, for all $j \in \mathbb{Z}$, $f(y, j) = y'$, for all $i, j \in \mathbb{Z}$, if $i < j$, $f(u, i, j) = z'$, else, $f(u, i, j) = t'$. Let $\mathcal{M} = (W, R, \star, V)$ be the model on $\mathcal{F}$ defined by $V(p) = f^{-1}(V'(p))$. The reader may easily prove by induction on the formation of the formula ψ that for all $v \in W$, $v \in V_{\mathcal{M}}(\psi)$ iff $f(v) \in V_{\mathcal{M}'}(\psi)$. Since $V_{\mathcal{M}'}(\phi) \neq W'$, $V_{\mathcal{M}}(\phi) \neq W$. Thus, $\mathcal{F} \not\models \phi$: a contradiction.

Proof of Proposition 10. Define the disjoint union $\mathcal{M}_1 \uplus \mathcal{M}_2 = (W_+, R_+, \star_+, V_+)$ of any two models $\mathcal{M}_1 = (W_1, R_1, \star_1, V_1)$ and $\mathcal{M}_2 = (W_2, R_2, \star_2, V_2)$ by

- $W_+ = \{(i, w) : i \in \{1, 2\}$ and $w \in W_i\}$,
- $R_+(a) = \{((i, w), (j, x)) : i = j$ and $w R_i(a) x\}$, for all a,
- $(i, w) \star_+ (j, x) = \{(k, y) : i = j = k$ and $y \in w \star_i x\}$, for all $(i, w), (j, x)$,
- $V_+(p) = \{(i, w) : w \in V_i(p)\}$, for all p.

Clearly, disjoint union of models does not preserve seriality, because even if $\mathcal{M}_1$ and $\mathcal{M}_2$ are serial, for all $w_1 \in W_1$ and all $w_2 \in W_2$, $(1, w_1) \star_+ (2, w_2)$ is empty. Moreover, it can easily be proved by induction on ϕ that for all $i \in \{1, 2\}$ and all $w \in W_i$, $(i, w) \in V_{\mathcal{M}_1 \uplus \mathcal{M}_2}(\phi)$ iff $w \in V_{\mathcal{M}_i}(\phi)$.

Suppose now that the class of serial frames is modally defined by the formula ϕ. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are serial then $V_{\mathcal{M}_1}(\phi) = W_1$ and $V_{\mathcal{M}_2}(\phi) = W_2$, hence $V_{\mathcal{M}_1 \uplus \mathcal{M}_2}(\phi) = W_+$. But we already proved that $\mathcal{M}_1 \uplus \mathcal{M}_2$ is not serial: a contradiction.

Proof of Proposition 11. Suppose there exists a test-free formula ϕ such that $\langle \top ? \Delta \top ? \rangle \top \leftrightarrow \phi$ is valid in the class of all separated deterministic frames. Let $\mathcal{M} = (W, R, \star, V)$, $\mathcal{M}' = (W', R', \star', V')$ be the models defined by:

- $W = \{x, y\}$;
- R is the empty function;
- $x \star x = \{y\}$, otherwise $\star$ is the empty function;
- V is the empty function;
- $W' = \{x'_1, x'_2, y'_1, y'_2\}$;
- R' is the empty function;
- $x'_1 \star' x'_2 = \{y'_1\}$ and $x'_2 \star' x'_1 = \{y'_2\}$, otherwise $\star'$ is the empty function;
- V' is the empty function.

Obviously, $\mathcal{M}, \mathcal{M}'$ are separated and deterministic, $x \in V_{\mathcal{M}}(\langle \top ? \Delta \top ? \rangle \top)$, $x'_1 \notin V_{\mathcal{M}'}(\langle \top ? \Delta \top ? \rangle \top)$ and $x'_2 \notin V_{\mathcal{M}'}(\langle \top ? \Delta \top ? \rangle \top)$. Since $\langle \top ? \Delta \top ? \rangle \top \leftrightarrow \phi$ is valid in the class of all separated deterministic frames, $x \in V_{\mathcal{M}}(\phi)$, $x'_1 \notin V_{\mathcal{M}'}(\phi)$ and $x'_2 \notin V_{\mathcal{M}'}(\phi)$. Let $Z = \{(x, x'_1), (x, x'_2), (y, y'_1), (y, y'_2)\}$. The reader may easily prove by induction on the formation of the test-free formula ψ that for all $u \in W$, for all $u' \in W'$, if $u Z u'$, $u \in V_{\mathcal{M}}(\psi)$ iff $u' \in V_{\mathcal{M}'}(\psi)$. Since $x Z x'_1$, $x Z x'_2$ and $x \in V_{\mathcal{M}}(\phi)$, $x'_1 \in V_{\mathcal{M}'}(\phi)$ and $x'_2 \in V_{\mathcal{M}'}(\phi)$: a contradiction.

Proof of Proposition 12. Suppose there exists a fork-free formula ϕ such that $\langle a \Delta a \rangle \top \leftrightarrow \phi$ is valid in the class of all separated deterministic frames. Let $\mathcal{M} = (W, R, \star, V)$, $\mathcal{M}' = (W', R', \star', V')$ be the models defined by:

- $W = \{x, y, z, t\}$;
- $R(a) = \{(x, y), (x, z)\}$, otherwise R is the empty function;
- $y \star z = \{t\}$, otherwise $\star$ is the empty function;
- V is the empty function;
- $W' = \{x', y'_1, y'_2, z'_1, z'_2, t'_1, t'_2\}$;
- $R'(a) = \{(x', y'_1), (x', z'_2)\}$, otherwise R' is the empty function;
- $y'_1 \star' z'_1 = \{t'_1\}$, $y'_2 \star' z'_2 = \{t'_2\}$, otherwise $\star'$ is the empty function;
- V' is the empty function.

Obviously, $\mathcal{M}, \mathcal{M}'$ are separated and deterministic, $x \in V_{\mathcal{M}}(\langle a\Delta a\rangle\top)$ and $x' \notin V_{\mathcal{M}'}(\langle a\Delta a\rangle\top)$. Since $\langle a\Delta a\rangle\top \leftrightarrow \phi$ is valid in the class of all separated deterministic frames, $x \in V_{\mathcal{M}}(\phi)$ and $x' \notin V_{\mathcal{M}'}(\phi)$. Let $Z = \{(x, x'), (y, y'_1), (y, y'_2), (z, z'_1), (z, z'_2), (t, t'_1), (t, t'_2)\}$. The reader may easily prove by induction on the formation of the fork-free formula ψ that for all $u \in W$, for all $u' \in W'$, if uZu', $u \in V_{\mathcal{M}}(\psi)$ iff $u' \in V_{\mathcal{M}'}(\psi)$. Since xZx' and $x \in V_{\mathcal{M}}(\phi)$, $x' \in V_{\mathcal{M}'}(\phi)$: a contradiction.

Proof of Proposition 16. By Propositions 3–6.

Proof of Lemma 1. Left to the reader.

Proof of Lemma 2. Left to the reader.

Proof of Lemma 3. (1). Left to the reader.

(2). Leaving to the reader the proof that $\mathbf{PDL}_0^\Delta \subseteq [\alpha]S$ and $[\alpha]S$ is closed under the rule of proof (**MP**), we prove that $[\alpha]S$ is closed under the rule of proof (**FOR**). Suppose $\{\breve{\phi}(\langle\beta\rangle((\psi \wedge p) \lhd \top) \vee \langle\gamma\rangle(\top \rhd (\psi \wedge \neg p)))$: p is a propositional variable$\} \subseteq [\alpha]S$. Hence, $\{[\alpha; \neg\breve{\phi}(\langle\beta\rangle((\psi \wedge p) \lhd \top) \vee \langle\gamma\rangle(\top \rhd (\psi \wedge \neg p)))?]\bot$: p is a propositional variable$\} \subseteq S$. Since S is closed under the rule of proof (**FOR**), $[\alpha; \neg\breve{\phi}(\langle\beta\Delta\gamma\rangle\psi)?]\bot \in S$. Thus, $\breve{\phi}(\langle\beta\Delta\gamma\rangle\psi) \in [\alpha]S$.

(3). By (1) and (2).

(4) and (5). Left to the reader.

Proof of Lemma 6. Suppose $\phi \otimes \psi \in S$.

(1). Suppose $(\phi \wedge \chi) \otimes \psi \notin S$. Hence, $(\phi \wedge \neg\chi) \otimes \psi \in S$. Obviously, there are finitely many, say $k \geq 0$, representations of χ in the form of a conclusion of the rule of proof (**FOR**): $\breve{\tau}_1(\langle\alpha_1\Delta\beta_1\rangle\mu_1), \ldots, \breve{\tau}_k(\langle\alpha_k\Delta\beta_k\rangle\mu_k)$. We define by induction a sequence $(\theta_0, \ldots, \theta_k)$ of formulas such that for all $l \in \mathbb{N}$, if $l \leq k$, the following conditions are satisfied: $(\phi \wedge \theta_l) \otimes \psi \in S$; $\theta_l \to \neg\chi \in \mathbf{PDL}_0^\Delta$; for all $m \in \mathbb{N}$, if $1 \leq m \leq l$, there exists a propositional variable p such that $\theta_l \to \neg\breve{\tau}_m(\langle\alpha_m\rangle((\mu_m \wedge p) \lhd \top) \vee \langle\beta_m\rangle(\top \rhd (\mu_m \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. First, let $\theta_0 = \neg\chi$. Obviously, the following conditions are satisfied: $(\phi \wedge \theta_0) \otimes \psi \in S$; $\theta_0 \to \neg\chi \in \mathbf{PDL}_0^\Delta$. Second, let $l \geq 1$ be such that $l \leq k$ and the formulas $\theta_0, \ldots, \theta_{l-1}$ have already been defined. Hence, $(\phi \wedge \theta_{l-1}) \otimes \psi \in S$; $\theta_{l-1} \to \neg\chi \in \mathbf{PDL}_0^\Delta$; for all $m \in \mathbb{N}$, if $1 \leq m \leq l-1$, there exists a propositional variable p such that $\theta_{l-1} \to \neg\breve{\tau}_m(\langle\alpha_m\rangle((\mu_m \wedge p) \lhd \top) \vee \langle\beta_m\rangle(\top \rhd (\mu_m \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. Third, since $(\phi \wedge \theta_{l-1}) \otimes \psi \in S$ and $\theta_{l-1} \to \neg\chi \in \mathbf{PDL}_0^\Delta$, $([(\phi \wedge \theta_{l-1})?; \neg\chi?]\bot)\bar{\otimes}\neg\psi \notin S$. Since S is closed under the rule of proof (**FOR**), there exists a propositional variable p such that $([(\phi \wedge \theta_{l-1})?; \neg\breve{\tau}_l(\langle\alpha_l\rangle((\mu_l \wedge p) \lhd \top) \vee \langle\beta_l\rangle(\top \rhd (\mu_l \wedge \neg p)))?]\bot)\bar{\otimes}\neg\psi \notin S$. Let $\theta_l = \theta_{l-1} \wedge \neg\breve{\tau}_l(\langle\alpha_l\rangle((\mu_l \wedge p) \lhd \top) \vee \langle\beta_l\rangle(\top \rhd (\mu_l \wedge \neg p)))$. Obviously, the following conditions are satisfied: $(\phi \wedge \theta_l) \otimes \psi \in S$; $\theta_l \to \neg\chi \in \mathbf{PDL}_0^\Delta$; for all $m \in \mathbb{N}$, if $1 \leq m \leq l$, there exists a propositional variable p such that $\theta_l \to \neg\breve{\tau}_m(\langle\alpha_m\rangle((\mu_m \wedge p) \lhd \top) \vee \langle\beta_m\rangle(\top \rhd (\mu_m \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. Finally, the reader may easily verify that the following conditions are satisfied: $(\phi \wedge \theta_k) \otimes \psi \in S$; $\theta_k \to \neg\chi \in \mathbf{PDL}_0^\Delta$; if χ is in the form $\breve{\tau}(\langle\alpha\Delta\beta\rangle\mu)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\theta_k \to \neg\breve{\tau}(\langle\alpha\rangle((\mu \wedge p) \lhd \top) \vee \langle\beta\rangle(\top \rhd (\mu \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$.

(2). Similar to (1).

Proof of Lemma 7. (1). Suppose $\phi \circ \psi \in S$. Since there are countably many formulas, there exists an enumeration $\chi_1, \chi_2, \ldots$ of the set of all formulas. Let $\theta_0, \theta_1, \ldots$ and $\tau_0, \tau_1, \ldots$ be the sequences of formulas inductively defined as follows such that for all $n \in \mathbb{N}$, $\theta_n \circ \tau_n \in S$. First, let $\theta_0 = \phi$ and $\tau_0 = \psi$. Obviously, $\theta_0 \circ \tau_0 \in S$. Second, let $n \geq 1$ be such that formulas $\theta_0, \ldots, \theta_{n-1}$ and $\tau_0, \ldots, \tau_{n-1}$ have already been defined. Hence, $\theta_{n-1} \circ \tau_{n-1} \in S$. Third, by Lemma 6, either $(\theta_{n-1} \wedge \chi_n) \circ \tau_{n-1} \in S$, or there exists a formula μ such that the following conditions are satisfied: $(\theta_{n-1} \wedge \mu) \circ \tau_{n-1} \in S$; $\mu \to \neg\chi_n \in \mathbf{PDL}_0^\Delta$; if χ_n is in the form $\breve{\nu}(\langle\alpha\Delta\beta\rangle\omega)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\mu \to \neg\breve{\nu}(\langle\alpha\rangle((\omega \wedge p) \lhd \top) \vee \langle\beta\rangle(\top \rhd (\omega \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. In the former case, let $\theta_n = \theta_{n-1} \wedge \chi_n$. In the latter case, let $\theta_n = \theta_{n-1} \wedge \mu$. Obviously, $\theta_n \circ \tau_{n-1} \in S$. By Lemma 6, either $\theta_n \circ (\tau_{n-1} \wedge \chi_n) \in S$, or there exists a formula μ such that the following conditions are satisfied: $\theta_n \circ (\tau_{n-1} \wedge \mu) \in S$; $\mu \to \neg\chi_n \in \mathbf{PDL}_0^\Delta$; if χ_n is in the form $\breve{\nu}(\langle\alpha\Delta\beta\rangle\omega)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\mu \to \neg\breve{\nu}(\langle\alpha\rangle((\omega \wedge p) \lhd \top) \vee \langle\beta\rangle(\top \rhd (\omega \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. In the former case, let $\tau_n = \tau_{n-1} \wedge \chi_n$. In the latter case, let $\tau_n = \tau_{n-1} \wedge \mu$. Obviously, $\theta_n \circ \tau_n \in S$. Finally, the reader may easily verify that $T = \bigcup\{\mathbf{PDL}_0^\Delta + \theta_n : n \in \mathbb{N}\}$ and $U = \bigcup\{\mathbf{PDL}_0^\Delta + \tau_n : n \in \mathbb{N}\}$ are maximal consistent theories such that $T \circ U \subseteq S$, $\phi \in T$ and $\psi \in U$.

(2) and (3). Similar to (1).

Proof of Lemma 8. Suppose $\mathcal{F}_c$ is not separated. Hence, there exists $X, Y, Z, T, U \in W_c$ such that $U \in X \star Y$, $U \in Z \star T$ and either $X \neq Z$, or $Y \neq T$.

Case $X \neq Z$. Hence, there exists a formula ϕ such that $\phi \in X$ and $\phi \notin Z$. Since $U \in X \star Y$, $\phi \circ \top \in U$. Thus, $\neg(\neg\phi \circ \top) \in U$. Since $U \in Z \star T$, $\phi \in Z$: a contradiction.

Case $Y \neq T$. Similar to the case $X \neq Z$.

Proof of Lemma 9. Suppose $\langle\alpha(\phi?)\rangle\top \in S$. Suppose $\langle\alpha((\phi \wedge \psi)?)\rangle\top \notin S$. Hence, $\langle\alpha((\phi \wedge \neg\psi)?)\rangle\top \in S$. Obviously, there are finitely many, say $k \geq 0$, representations of ψ in the form of a conclusion of the rule of proof (**FOR**): $\breve{\tau}_1(\langle\beta_1\Delta\gamma_1\rangle\theta_1), \ldots, \breve{\tau}_k(\langle\beta_k\Delta\gamma_k\rangle\theta_k)$. We define by induction a sequence $(\chi_0, \ldots, \chi_k)$ of formulas such that for all $l \in \mathbb{N}$, if $l \leq k$, the following conditions are satisfied: $\langle\alpha((\phi \wedge \chi_l)?)\rangle\top \in S$; $\chi_l \to \neg\psi \in \mathbf{PDL}_0^\Delta$; for all $m \in \mathbb{N}$, if $1 \leq m \leq l$, there exists a propositional variable p such that $\chi_l \to \neg\breve{\tau}_m(\langle\beta_m\rangle((\theta_m \wedge p) \lhd \top) \vee \langle\gamma_m\rangle(\top \rhd (\theta_m \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. First, let $\chi_0 = \neg\psi$. Obviously, the following conditions are satisfied: $\langle\alpha((\phi \wedge \chi_0)?)\rangle\top \in S$; $\chi_0 \to \neg\psi \in \mathbf{PDL}_0^\Delta$. Second, let $l \geq 1$ be such that $l \leq k$ and the formulas $\chi_0, \ldots, \chi_{l-1}$ have already been defined. Hence, $\langle\alpha((\phi \wedge \chi_{l-1})?)\rangle\top \in S$; $\chi_{l-1} \to \neg\psi \in \mathbf{PDL}_0^\Delta$; for all $m \in \mathbb{N}$, if $1 \leq m \leq l-1$, there exists a propositional variable p such that $\chi_{l-1} \to \neg\breve{\tau}_m(\langle\beta_m\rangle((\theta_m \wedge p) \lhd \top) \vee \langle\gamma_m\rangle(\top \rhd (\theta_m \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. Third, since $\langle\alpha((\phi \wedge \chi_{l-1})?)\rangle\top \in S$ and $\chi_{l-1} \to \neg\psi \in \mathbf{PDL}_0^\Delta$, $[\alpha(\neg[(\phi \wedge \chi_{l-1})?; \neg\psi?]\bot?)]\bot \notin S$. Since S is closed under the rule of proof (**FOR**), there exists a propositional variable p such that $[\alpha(\neg[(\phi \wedge \chi_{l-1})?; \neg\breve{\tau}_l(\langle\beta_l\rangle((\theta_l \wedge p) \lhd \top) \vee \langle\gamma_l\rangle(\top \rhd (\theta_l \wedge \neg p)))?]\bot?)]\bot \notin S$. Let $\chi_l = \chi_{l-1} \wedge \neg\breve{\tau}_l(\langle\beta_l\rangle((\theta_l \wedge p) \lhd \top) \vee \langle\gamma_l\rangle(\top \rhd (\theta_l \wedge \neg p)))$. Obviously, the following conditions are satisfied: $\langle\alpha((\phi \wedge \chi_l)?)\rangle\top \in S$; $\chi_l \to \neg\psi \in \mathbf{PDL}_0^\Delta$; for all $m \in \mathbb{N}$, if $1 \leq m \leq l$, there exists a propositional variable p such that $\chi_l \to \neg\breve{\tau}_m(\langle\beta_m\rangle((\theta_m \wedge p) \lhd \top) \vee \langle\gamma_m\rangle(\top \rhd (\theta_m \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. Finally, the reader may easily verify that the following conditions are satisfied: $\langle\alpha((\phi \wedge \chi_k)?)\rangle\top \in S$; $\chi_k \to \neg\psi \in \mathbf{PDL}_0^\Delta$; if ψ is in the form $\breve{\tau}(\langle\beta\Delta\gamma\rangle\theta)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\chi_k \to \neg\breve{\tau}(\langle\beta\rangle((\theta \wedge p) \lhd \top) \vee \langle\gamma\rangle(\top \rhd (\theta \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$.

Proof of Lemma 10. Suppose $[\alpha]\phi \notin S$. Since S is maximal, $\langle\alpha\rangle\neg\phi \in S$. Hence, $\langle f(\alpha)\rangle\neg\phi \in S$. Without loss of generality, suppose $f(\alpha)$ contains exactly one test, say $\psi?$. Thus, $\langle f(\alpha)(\psi?); \neg\phi?\rangle\top \in S$. Since there are countably many formulas, there exists an enumeration $\chi_1, \chi_2, \ldots$ of the set of all formulas. Let $\theta^0, \theta^1, \ldots$ and $\tau^0, \tau^1, \ldots$ be the sequences of formulas inductively defined as follows such that for all $n \in \mathbb{N}$, $\langle f(\alpha)(\theta^n?); \tau^n?\rangle\top \in S$. First, let $\theta^0 = \psi$ and $\tau^0 = \neg\phi$. Obviously, $\langle f(\alpha)(\theta^0?); \tau^0?\rangle\top \in S$. Second, let $n \geq 1$ be such that formulas $\theta^0, \ldots, \theta^{n-1}$ and $\tau^0, \ldots, \tau^{n-1}$ have already been defined. Hence, $\langle f(\alpha)(\theta^{n-1}?); \tau^{n-1}?\rangle\top \in S$. Third, by Lemma 9, either $\langle f(\alpha)((\theta^{n-1} \wedge \chi_n)?); \tau^{n-1}?\rangle\top \in S$, or there exists a formula μ such that the following conditions are satisfied: $\langle f(\alpha)((\theta^{n-1} \wedge \mu)?); \tau^{n-1}?\rangle\top \in S$; $\mu \to \neg\chi_n \in \mathbf{PDL}_0^\Delta$; if χ_n is in the form $\breve{\omega}(\langle\beta\Delta\gamma\rangle\nu)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\mu \to \neg\breve{\omega}(\langle\beta\rangle((\nu \wedge p) \lhd \top) \vee \langle\gamma\rangle(\top \rhd (\nu \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. In the former case, let $\theta^n = \theta^{n-1} \wedge \chi_n$. In the latter case, let $\theta^n = \theta^{n-1} \wedge \mu$. Obviously, $\langle f(\alpha)(\theta^n?); \tau^{n-1}?\rangle\top \in S$. By Lemma 9, either $\langle f(\alpha)(\theta^n?); (\tau^{n-1} \wedge \chi_n)?\rangle\top \in S$, or there exists a formula μ such that the following conditions are satisfied: $\langle f(\alpha)(\theta^n?); (\tau^{n-1} \wedge \mu)?\rangle\top) \in S$; $\mu \to \neg\chi_n \in \mathbf{PDL}_0^\Delta$; if χ_n is in the form $\breve{\omega}(\langle\beta\Delta\gamma\rangle\nu)$ of a conclusion of the rule of proof (**FOR**), there exists a propositional variable p such that $\mu \to \neg\breve{\omega}(\langle\beta\rangle((\nu \wedge p) \lhd \top) \vee \langle\gamma\rangle(\top \rhd (\nu \wedge \neg p))) \in \mathbf{PDL}_0^\Delta$. In the former case, let $\tau^n = \tau^{n-1} \wedge \chi_n$. In the latter case, let $\tau^n = \tau^{n-1} \wedge \mu$. Obviously, $\langle f(\alpha)(\theta^n?); \tau^n?\rangle\top \in S$. Finally, the reader may easily verify that $T = \bigcup\{\mathbf{PDL}_0^\Delta + \theta^n : n \in \mathbb{N}\}$ and $U = \bigcup\{\mathbf{PDL}_0^\Delta + \tau^n : n \in \mathbb{N}\}$ are maximal consistent theories such that $f(\alpha) \in \ker(f(\alpha)(\bar{T}))$, for all programs β, if $\beta \in \ker(f(\alpha)(\bar{T}))$, $[\beta]S \subseteq U$ and $\phi \notin U$.

Proof of Proposition 17. Suppose ϕ is valid in the class of all separated frames. Suppose $\phi \notin \mathbf{PDL}_0^\Delta$. By Lemma 3, $\mathbf{PDL}_0^\Delta + \neg\phi$ is a consistent theory containing $\neg\phi$. By Lemma 5, there exists a maximal consistent theory S such that $\mathbf{PDL}_0^\Delta + \neg\phi \subseteq S$. Since $\mathbf{PDL}_0^\Delta + \neg\phi$ contains $\neg\phi$, $\phi \notin S$. By Lemma 11, $S \notin V_{\mathcal{M}_c}(\phi)$. By Lemma 8, ϕ is not valid in the class of all separated frames: a contradiction.

References

[1] P. Balbiani, Eliminating unorthodox derivation rules in an axiom system for iteration-free **PDL** with intersection, Fundam. Inform. 56 (2003) 211–242.
[2] P. Balbiani, J. Boudou, Iteration-free **PDL** with storing, recovering and parallel composition: a complete axiomatization, J. Log. Comput. 25 (2015).
[3] P. Balbiani, D. Vakarelov, Iteration-free **PDL** with intersection: a complete axiomatization, Fundam. Inform. 45 (2001) 173–194.
[4] P. Balbiani, D. Vakarelov, **PDL** with intersection of programs: a complete axiomatization, J. Appl. Non-Class. Log. 13 (2003) 231–276.
[5] M. Benevides, R. de Freitas, P. Viana, Propositional dynamic logic with storing, recovering and parallel composition, Electron. Notes Theor. Comput. Sci. 269 (2011) 95–107.
[6] P. Blackburn, M. de Rijke, Y. Venema, Modal Logic, Cambridge University Press, 2001.
[7] R. Danecki, Nondeterministic propositional dynamic logic with intersection is decidable, in: A. Skowron (Ed.), Computation Theory, Springer, 1985, pp. 34–53.
[8] K. Došen, A brief survey of frames for the Lambek calculus, Z. Math. Log. Grundl. Math. 38 (1992) 179–187.
[9] L. Fariñas del Cerro, E. Orłowska, **DAL** — a logic for data analysis, Theor. Comput. Sci. 36 (1985) 251–264.
[10] M. Fisher, R. Ladner, Propositional dynamic logic of regular programs, J. Comput. Syst. Sci. 18 (1979) 194–211.
[11] M. Frias, Fork Algebras in Algebra, Logic and Computer Science, World Scientific, 2002.

[12] M. Frias, G. Baum, A. Haeberer, Fork algebras in algebra, logic and computer science, Fundam. Inform. 32 (1997) 1–25.
[13] M. Frias, P. Veloso, G. Baum, Fork algebras: past, present and future, J. Relat. Methods Comput. Sci. 1 (2004) 181–216.
[14] G. Gargov, S. Passy, A note on Boolean modal logic, in: P. Petkov (Ed.), Mathematical Logic, Plenum Press, 1990, pp. 299–309.
[15] D. Harel, D. Kozen, J. Tiuryn, Dynamic Logic, MIT Press, 2000.
[16] D. Kozen, R. Parikh, An elementary proof of the completeness of **PDL**, Theor. Comput. Sci. 14 (1981) 113–118.
[17] M. Lange, Model checking propositional dynamic logic with all extras, J. Appl. Log. 4 (2006) 39–49.
[18] S. Mikulás, Complete calculus for conjugated arrow logic, in: M. Marx, L. Pólos, M. Masuch (Eds.), Arrow Logic and Multi-Modal Logic, CSLI, 1996, pp. 125–139.
[19] G. Mirkowska, **PAL** — propositional algorithmic logic, Fundam. Inform. 4 (1981) 675–760.
[20] A. Tarski, On the calculus of relations, J. Symb. Log. 6 (1941) 73–89.