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Comparison between W_2 distance and $\dot{H}^{-1}$ norm, and Localization of Wasserstein distance

Rémi Peyre*

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Abstract

It is well known that the quadratic Wasserstein distance $W_2(\cdot, \cdot)$ is formally equivalent, for infinitesimally small perturbations, to some weighted H^{-1} homogeneous Sobolev norm. In this article I show that this equivalence can be integrated to get non-asymptotic comparison results between these distances. Then I give an application of these results to prove that the W_2 distance exhibits some localization phenomenon: if μ and ν are measures on $\mathbf{R}^n$ and $\varphi: \mathbf{R}^n \rightarrow \mathbf{R}_+$ is some bump function with compact support, then under mild hypotheses, you can bound above the Wasserstein distance between $\varphi \cdot \mu$ and $\varphi \cdot \nu$ by an explicit multiple of $W_2(\mu, \nu)$.

Keywords: Wasserstein distance; homogeneous Sobolev norm; localization

Foreword

This article is divided into two sections, each of which having its own introduction. § 1 deals with general results of comparison between Wasserstein distance and homogeneous Sobolev norm, while § 2 handles an application to localization of W_2 distance.

1 Non-asymptotic equivalence between W_2 distance and $\dot{H}^{-1}$ norm

1.1 Introduction

In all this section, M denotes a connected Riemannian manifold endowed with its distance $dist(\cdot, \cdot)$ and its standard measure λ provided by the volume form (so, in the case $M = \mathbf{R}^n$, λ is the Lebesgue measure). Let us give a few standard definitions which will be at the core of our work:

- For μ, ν two positive measures on M , denoting by $\Pi(\mu, \nu)$ the set of (positive) measures on $M \times M$ whose respective marginals are μ and ν , for $\pi \in \Pi(\mu, \nu)$ one defines

$$I(\pi) := \int_{M \times M} dist(x, y)^2 \pi(dx, dy) \quad (1)$$

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and then

$$W_2(\mu, \nu) := \inf\{I(\pi) \mid \pi \in \Pi(\mu, \nu)\}^{1/2}. \quad (2)$$

W_2 is a (possibly infinite) distance, called the *quadratic Wasserstein distance* [13, § 7.1]. Note that this distance is finite only between measures having the same total mass.

- On the other hand, for μ a (positive) measure on M , if f is a $\mathcal{C}^1$ real function on M , one denotes

$$\|f\|_{\dot{H}^1(\mu)} := \left(\int_M |\nabla f(x)|^2 \mu(dx) \right)^{1/2}, \quad (3)$$

which defines a semi-norm; for ν a signed measure on M , one then denotes

$$\|\nu\|_{\dot{H}^{-1}(\mu)} := \sup\{|\langle f, \nu \rangle| \mid \|f\|_{\dot{H}^1(\mu)} \leq 1\}, \quad (4)$$

where the duality product $\langle f, \nu \rangle$ denotes the integral of the function f against the measure ν .^[*] We observe that $\|\cdot\|_{\dot{H}^{-1}(\mu)}$ defines a (possibly infinite) norm, which we will call the $\dot{H}^{-1}(\mu)$ *weighted homogeneous Sobolev norm*. Note that this norm is finite only for measures having zero total mass. In the case μ is the standard measure, we will merely write “ $\dot{H}^{-1}$ ” for “ $\dot{H}^{-1}(\lambda)$ ”.

The W_2 Wasserstein distance is an important object in analysis; but it is non-linear, which makes it harder to study. For infinitesimal perturbations however, the linearized behaviour of W_2 is well known: if μ is a positive measure on M and $d\mu$ is an infinitesimally small perturbation of this measure,^[†] one has formally (see [13, § 7.6] or [9, § 7])

$$W_2(\mu, \mu + d\mu) = \|d\mu\|_{\dot{H}^{-1}(\mu)} + o(d\mu). \quad (5)$$

More precisely, one has the following equality, known as the *Benamou–Brenier formula* ([2, Prop. 1.1] for $M = \mathbf{R}^n$; see [10] for general Riemannian manifolds): for two positive measures μ, ν on M ,

$$W_2(\mu, \nu) = \inf \left\{ \int_0^1 \|d\mu_t\|_{\dot{H}^{-1}(\mu_t)} \mid \mu_0 = \mu, \mu_1 = \nu \right\}. \quad (6)$$

Then, a natural question is the following: are there *non-asymptotic* comparisons between the W_2 distance and the $\dot{H}^{-1}$ norm? Concretely, we are looking for inequalities like

$$C_a \|\mu - \nu\|_{\dot{H}^{-1}(\mu)} \leq W_2(\mu, \nu) \leq C_b \|\mu - \nu\|_{\dot{H}^{-1}(\mu)} \quad (7)$$

for constants $0 < C_a \leq C_b < \infty$, under mild assumptions on μ and ν .

^[*]The rationale behind the use of duality notation in this article is that we cannot use the notation “ $d\mu$ ” to refer to the measure of a small volume: see indeed Footnote [†] below.

^[†]Beware that here $d\mu$ denotes a small measure on M , not the value of μ on a small volume.

^[‡]This formula has to be understood in the sense that, for every measure ν , one has $W_2(\mu, \mu + \varepsilon\nu) \stackrel{\varepsilon \rightarrow 0}{\sim} \|\varepsilon\nu\|_{\dot{H}^{-1}(\mu)} + o(\varepsilon)$. As explained in the references cited, some regularity assumptions on ν shall be required for that property to hold rigorously: in particular, one must have $\nu \ll \mu$ with a bounded and smooth enough density.

1.2 Controlling W_2 by $\dot{H}^{-1}$

Theorem 1. *For any positive measures μ, ν on M ,*

$$W_2(\mu, \nu) \leq 2\|\mu - \nu\|_{\dot{H}^{-1}(\mu)}. \quad (8)$$

Proof. We suppose that $\|\mu - \nu\|_{\dot{H}^{-1}(\mu)} < \infty$, otherwise there is nothing to prove. For $t \in [0, 1]$, let

$$\mu_t := (1 - t)\mu + t\nu, \quad (9)$$

so that $\mu_0 = \mu$, $\mu_1 = \nu$ and $d\mu_t = (\mu - \nu)dt$. Then, by the Benamou–Brenier formula (6):

$$W_2(\mu, \nu) \leq \int_0^1 \|\mu - \nu\|_{\dot{H}^{-1}(\mu_t)} dt. \quad (10)$$

Now, we use the following key lemma, whose proof is postponed:

Lemma 2. *If μ, μ' are two measures such that $\mu' \geq \rho\mu$ for some $\rho > 0$, then $\|\cdot\|_{\dot{H}^{-1}(\mu')} \leq \rho^{-1/2} \|\cdot\|_{\dot{H}^{-1}(\mu)}$.^[§]*

Here obviously $\mu_t \geq (1 - t)\mu$, so

$$W_2(\mu, \nu) \leq \int_0^1 (1 - t)^{-1/2} \|\mu - \nu\|_{\dot{H}^{-1}(\mu)} dt = 2\|\mu - \nu\|_{\dot{H}^{-1}(\mu)}, \quad (11)$$

QED. □

Corollary 3. *If $\mu \geq \rho\lambda$ for some $\rho > 0$, then*

$$W_2(\mu, \nu) \leq 2\rho^{-1/2} \|\mu - \nu\|_{\dot{H}^{-1}}. \quad (12)$$

Proof. Just use that $\|\cdot\|_{\dot{H}^{-1}(\mu)} \leq \rho^{-1/2} \|\cdot\|_{\dot{H}^{-1}}$ by Lemma 2. □

Proof of Lemma 2. Take $\mu' \geq \rho\mu$ and let ν be a signed measure on M such that $\mu + \nu$ is positive; then $\mu' + \rho\nu$ is also positive. For m a measure on M , we denote by $diag(m)$ the measure on $M \times M$ supported by the diagonal whose marginals (which are equal) are m , i.e.:

$$(diag(m))(A \times B) := m(A \cap B); \quad (13)$$

with that notation,

$$\pi \in \Pi(\mu, \mu + \nu) \Rightarrow \rho\pi + diag(\mu' - \rho\mu) \in \Pi(\mu', \mu' + \rho\nu), \quad (14)$$

and

$$I(\rho\pi + diag(\mu' - \rho\mu)) = \rho I(\pi). \quad (15)$$

Therefore, taking infima,

$$\begin{aligned} W_2(\mu', \mu' + \rho\nu)^2 &= \inf\{I(\pi') \mid \pi' \in \Gamma(\mu', \mu' + \rho\nu)\} \\ &\leq \inf\{I(\rho\pi + diag(\mu' - \rho\mu)) \mid \pi \in \Gamma(\mu, \mu + \nu)\} \\ &= \rho \inf\{I(\pi) \mid \pi \in \Gamma(\mu, \mu + \nu)\} = \rho W_2(\mu, \mu + \nu)^2. \end{aligned} \quad (16)$$

^[§]Beware that here ‘ $\cdot$ ’ stands for a *measure*, not for a function: otherwise the formula would be false.—When f is a function, $\|f\|_{\dot{H}^{-1}(\mu)}$ stands for the $\dot{H}^{-1}(\mu)$ norm of the measure having density f w.r.t. μ .

For infinitesimally small ν ,^[¶] it follows by Equation (5) that $\|\rho\nu\|_{\dot{H}^{-1}(\mu')}^2 \leq \rho\|\nu\|_{\dot{H}^{-1}(\mu)}^2$, hence $\|\nu\|_{\dot{H}^{-1}(\mu')} \leq \rho^{-1/2}\|\nu\|_{\dot{H}^{-1}(\mu)}$. This relation remains true even for non-infinitesimal ν by linearity, which ends the proof. $\square$

Remark 4. Lemma 2 could also be proved very quickly by using the definition (3)-(4) of the $\dot{H}^{-1}(\mu)$ norm. The proof above, however, has the advantage that it does not need the precise expression of $\|\cdot\|_{\dot{H}^{-1}(\mu)}$, but only the fact that it is the linearized W_2 distance.

1.3 Controlling $\dot{H}^{-1}$ by W_2

Theorem 5. *Assume M has nonnegative Ricci curvature. Then for any positive measures μ, ν on M such that $\mu \leq \rho_0\lambda$ and $\nu \leq \rho_1\lambda$,*

$$\|\mu - \nu\|_{\dot{H}^{-1}} \leq \frac{2(\rho_0^{1/2} - \rho_1^{1/2})}{\ln(\rho_0 / \rho_1)} W_2(\mu, \nu). \quad (17)$$

(For $\rho_1 = \rho_0$, the right-hand side of (17) is to be taken as $\rho_0^{1/2} W_2(\mu, \nu)$ by continuity).

Remark 6. For $M = \mathbf{R}^n$ a similar result was already stated in [7, Proposition 2.8], with a different proof.

Proof. Assume that $W_2(\mu, \nu) < \infty$, otherwise there is nothing to prove. Let $(\mu_t)_{0 \leq t \leq 1}$ be the displacement interpolation between μ and ν (cf. [14, chap. 7]), which is such that $\mu_0 = \mu$, $\mu_1 = \nu$ and the infimum in (6) is attained with $\|\mathrm{d}\mu_t\|_{\dot{H}^{-1}(\mu_t)} = W_2(\mu, \nu) \mathrm{d}t \forall t$. Since Ricci curvature is nonnegative, the Lott–Sturm–Villani theory tells us that, denoting by $\|\mu\|_\infty$ the essential supremum of the density of μ w.r.t. λ , one has $\|\mu_t\|_\infty \leq \|\mu_0\|_\infty^{1-t} \|\mu_1\|_\infty^t \leq \rho_0^{1-t} \rho_1^t$ (see [14, Corollary 17.19] or [5, Lemma 6.1]); so that $\|\cdot\|_{\dot{H}^{-1}} \leq \rho_0^{(1-t)/2} \rho_1^{t/2} \|\cdot\|_{\dot{H}^{-1}(\mu_t)}$ by Lemma 2.

Then, by the integral triangle inequality for normed vector spaces,

$$\begin{aligned} \|\mu - \nu\|_{\dot{H}^{-1}} &= \left\| \int_0^1 \mathrm{d}\mu_t \right\|_{\dot{H}^{-1}} \leq \int_0^1 \|\mathrm{d}\mu_t\|_{\dot{H}^{-1}} \\ &\leq \int_0^1 \rho_0^{(1-t)/2} \rho_1^{t/2} \|\mathrm{d}\mu_t\|_{\dot{H}^{-1}(\mu_t)} = \left(\int_0^1 \rho_0^{(1-t)/2} \rho_1^{t/2} \mathrm{d}t \right) W_2(\mu, \nu) \\ &= \frac{2(\rho_0^{1/2} - \rho_1^{1/2})}{\ln(\rho_0 / \rho_1)} W_2(\mu, \nu), \end{aligned} \quad (18)$$

QED. $\square$

Remark 7. Taking into account the dimension n of the manifold M , the bound on $\|\mu_t\|_\infty$ could be refined into

$$\|\mu_t\|_\infty \leq ((1-t)\|\mu_0\|_\infty^{-1/n} + t\|\mu_1\|_\infty^{-1/n})^{-n} \quad (19)$$

^[¶]To make rigorous the formal argument of taking an infinitesimally small ν , according to Footnote [‡] above, one would have to replace ν by $\varepsilon\nu_1$, where ν_1 is a regular enough measure, and to let ε tend to 0; then the regularity assumption on ν_1 would be relaxed by a classical approximation argument. Anyway, Lemma 2 can also be proved easily and rigorously without referring to optimal transportation at all, cf. Remark 4 below.

(cf. [8, Theorem 2.3]), which would yield a slightly sharper bound in Equation (17), namely:

$$\begin{aligned} \|\mu - \nu\|_{\dot{H}^{-1}} &\leq \left(\int_0^1 ((1-t)\rho_0^{-1/n} + t\rho_1^{-1/n})^{-n/2} dt \right) W_2(\mu, \nu) \\ &= \begin{cases} \frac{\rho_0^{1/2-1/n} - \rho_1^{1/2-1/n}}{(n/2-1)(\rho_1^{-1/n} - \rho_0^{-1/n})} W_2(\mu, \nu) & n \geq 2; \\ \frac{\log(\rho_1/\rho_0)}{2(\rho_0^{-1/2} - \rho_1^{-1/2})} W_2(\mu, \nu) & n = 2. \end{cases} \end{aligned} \quad (20)$$

For $n = 1$ it turns out that one can let tend $\rho_1 \rightarrow \infty$ in (20) without making the integral diverge; which leads to a much more powerful result:

Theorem 8. When M is an interval of $\mathbf{R}$, then under the sole assumption that $\mu \leq \rho_0 \lambda$, one has for all positive measures ν on M :

$$\|\mu - \nu\|_{\dot{H}^{-1}} \leq 2\rho_0^{1/2} W_2(\mu, \nu). \quad (21)$$

Remark 9. For $n \geq 2$ there is no hope to get a bound valid for all ν , because then it can occur that $W_2(\mu, \nu) < \infty$ but $\|\mu - \nu\|_{\dot{H}^{-1}} = \infty$: for instance, take μ to be the uniform measure on the 2-dimensional sphere and ν a Dirac mass.

2 Application to localization of Wasserstein distance

2.1 Introduction

In all this section, we work in the Euclidian space $\mathbf{R}^n$, whose norm is denoted by $|\cdot|$. $\text{dist}(x, A) := \inf\{|x - y| \mid y \in A\}$ denotes the distance between a point x and a set A ; A^c denotes the complement of A ; λ denotes the Lebesgue measure. We will use the following notation to handle measures:

- For μ a measure on $\mathbf{R}^n$ and $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$ a measurable map, $f_* \mu$ denotes the pushforward of μ by f , that is, $(f_* \mu)(A) := \mu(f^{-1}(A))$.
- For μ a measure on $\mathbf{R}^n$ and $\varphi: \mathbf{R}^n \rightarrow \mathbf{R}_+$ a nonnegative measurable function, $\varphi \cdot \mu$ denotes the measure such that $(\varphi \cdot \mu)(dx) := \varphi(x)\mu(dx)$.

We will also use the following norms on measures:

- $\|\mu\|_{\dot{H}^{-1}(\nu)}$ has the same definition as in § 1;
- $\|\mu\|_1 := \int_{\mathbf{R}^n} |\mu(dx)|$ is the total variation norm of μ ; [||]
- For ν a positive measure with $\mu \ll \nu$, we define

$$\|\mu\|_{L^2(\nu)} := \left(\int_{\text{supp } \nu} \left(\frac{\mu(dx)}{\nu(dx)} \right)^2 \nu(dx) \right)^{1/2}. [**] \quad (22)$$

For $A \subset \mathbf{R}^n$, we also denote $\|\cdot\|_{L^2(A)}$ for $\|\cdot\|_{L^2(\mathbf{1}_A \cdot \lambda)}$.

[||]Note that in the case μ is a positive measure on $\mathbf{R}^n$, then $\|\mu\|_1$ is nothing but $\mu(\mathbf{R}^n)$.

[**]What we denote here by $\mu(dx)/\nu(dx)$ here is what is commonly called $(d\mu/d\nu)(x)$: indeed, as we already told, in this article we reserve the use of “ $d\mu$ ” to denote a mass distribution of infinitesimally small magnitude, rather than for the mass of an infinitely small volume.

The goal of this section is to give an application of Theorem 1 to the problem of *localization* of the quadratic Wasserstein distance. Morally, the question is the following: take two measures μ, ν on $\mathbf{R}^n$ being close to each other in the sense of W_2 distance; is it true that μ and ν remain close when you consider their restrictions to a subset of $\mathbf{R}^n$? Concretely, if φ is a non-negative real function on $\mathbf{R}^n$ with compact support (plus some technical assumptions to be specified later), we want to bound above $W_2(a\varphi \cdot \mu, \varphi \cdot \nu)$ by some multiple of $W_2(\mu, \nu)$ —where, in the former expression, a is a constant factor ensuring that $a\varphi \cdot \mu$ and $\varphi \cdot \nu$ have the same mass (for otherwise the distance between $\varphi \cdot \mu$ and $\varphi \cdot \nu$ is generically infinite).

This question, which was my initial motivation for the results of § 1, was asked to me by Xavier TOLSA, who needed such a result for his paper [12] on characterizing uniform rectifiability in terms of mass transport. Actually Xavier managed to devise a proof of his own [12, Theorem 1.1], but it was quite long (about thirty pages) and involved arguments of multi-scale analysis. With Theorem 1 at hand, however, the reasoning becomes far more direct; moreover we will be able to relax some of the assumptions of Xavier’s theorem.

2.2 Statement of the theorem

Theorem 10. *Let μ, ν be (positive) measures on $\mathbf{R}^n$ having the same total mass; let B be a ball of $\mathbf{R}^n$ (whose radius will be denoted by R when needed). Assume that on B , the density of μ w.r.t. the Lebesgue measure is bounded above and below:*

$$\exists 0 < m_1 \leq m_2 < \infty \quad \forall x \in B \quad m_1 \lambda(dx) \leq \mu(dx) \leq m_2 \lambda(dx). \quad (23)$$

Let $\varphi: \mathbf{R}^n \rightarrow \mathbf{R}_+$ be a function such that:

- (i) φ is zero outside B ;
- (ii) There exist $0 < c_1 \leq c_2 < \infty$ such that for all $x \in B$, $c_1 \text{dist}(x, B^c)^2 \leq \varphi(x) \leq c_2 \text{dist}(x, B^c)^2$.
- (iii) φ is k -Lipschitz for some $k < \infty$.

Then, denoting $a := \|\varphi \cdot \nu\|_1 / \|\varphi \cdot \mu\|_1$,

$$W_2(a\varphi \cdot \mu, \varphi \cdot \nu) \leq C(n) \frac{c_2^{3/2} m_2^{3/2}}{c_1^{3/2} m_1^{3/2}} k c_1^{-1/2} W_2(\mu, \nu), \quad (24)$$

for $C(n) < \infty$ some absolute constant only depending on n . Moreover, one can bound explicitly $C(n)$ in such a way that $C(n) = O(n^{1/2})$ when $n \rightarrow \infty$.^[††]

Remark 11. Theorem 10 relaxes the assumptions of Theorem 1.1 of [12] on the following points: first, Tolsa’s theorem required that $|\nabla \varphi|$ was bounded by a multiple of $\text{dist}(\cdot, B^c)$, while ours does not impose any specific control on $|\nabla \varphi|$ near the boundary of B ; second, Tolsa’s theorem worked only for radially symmetric φ . Also, contrary to [12], our conclusions state explicitly how the bound on $W_2(a\varphi \cdot \mu, \varphi \cdot \nu)$ depends on the constants k, c_1, c_2, m_1, m_2 and on the dimension n .

^[††]For instance, with the estimates of this article, one finds that $C(n) := 47n^{1/2}$ fits—though this may be strongly suboptimal.

Remark 12. Actually the constraint that the support of φ is a ball is of little importance: we could assume as well that it would be a cube, a simplex, or many other shapes, as the corollary below shows:

Corollary 13. *Make the same assumptions as in Theorem 10, except that B need not be a ball: instead, we only assume that, denoting by $B_\circ$ the (true) ball having the same volume as B , there exists a bijection $\Phi: B \leftrightarrow B_\circ$ mapping the uniform measure on B onto the uniform measure on $B_\circ$ (i.e. such that $\Phi_*(\mathbf{1}_B \cdot \lambda) = \mathbf{1}_{B_\circ} \cdot \lambda$) such that Φ is bi-Lipschitz (i.e. such that both Φ and Φ^{-1} are Lipschitz). Denote by $\|\Phi\|_{Lip}$ and $\|\Phi^{-1}\|_{Lip}$ the optimal Lipschitz constants for resp. Φ and Φ^{-1} . Then, the conclusion of Theorem 10 remains true, except that now you have to replace the factor $C(n)$ by*

$$(\|\Phi\|_{Lip} \|\Phi^{-1}\|_{Lip})^5 C(n). \quad (25)$$

Proof. Consider the measures $\mu_\circ := \Phi_* \mu$ and $\nu_\circ := \Phi_* \nu$, and the bump function $\varphi_\circ := \varphi \circ \Phi^{-1}$; then, $\mu_\circ$, $\nu_\circ$ and $\varphi_\circ$ satisfy the original assumptions of Theorem 10, the roles of ‘ m_1 ’ and ‘ m_2 ’ (in the ball situation) being held by m_1 and m_2 (in the general situation) themselves, the role of ‘ k ’ being held by $\|\Phi^{-1}\|_{Lip} k$, and the roles of ‘ c_1 ’ and ‘ c_2 ’ being held by $c_1 / \|\Phi\|_{Lip}^2$ and $c_2 \|\Phi^{-1}\|_{Lip}^2$. Therefore, applying (24):

$$W_2(a\varphi_\circ \cdot \mu_\circ, \varphi_\circ \cdot \nu_\circ) \leq C(n) \|\Phi\|_{Lip}^4 \|\Phi^{-1}\|_{Lip}^4 \frac{c_2^{3/2} m_2^{3/2}}{c_1^{3/2} m_1^{3/2}} W_2(\mu_\circ, \nu_\circ). \quad (26)$$

But the optimal transportation plan from μ to ν , with cost $W_2(\mu, \nu)^2$, can be pushed forward by Φ into a (not optimal in general) transportation plan from $\mu_\circ$ to $\nu_\circ$, whose cost will then be $\leq \|\Phi\|_{Lip}^2 W_2(\mu, \nu)^2$; so $W_2(\mu_\circ, \nu_\circ) \leq \|\Phi\|_{Lip} W_2(\mu, \nu)$. Similarly $W_2(a\varphi \cdot \mu, \varphi \cdot \nu) \leq \|\Phi^{-1}\|_{Lip} W_2(a\varphi_\circ \cdot \mu_\circ, \varphi_\circ \cdot \nu_\circ)$. The announced result follows. $\square$

2.3 Proof of the main theorem

In the sequel we will shorthand $W_2(\mu, \nu) =: w$, and also $\varphi \cdot \mu =: \hat{\mu}$, resp. $\varphi \cdot \nu =: \hat{\nu}$. Let $g =: Id + S$ be a map achieving optimal transportation from ν to μ , i.e. such that $\mu = g_* \nu$ with $\int_{\mathbf{R}^n} |S(y)|^2 \nu(dy) = w^2$.^[††]

Our strategy will consist in transforming $\hat{\nu}$ into $a\hat{\mu}$ according to the following procedure:

- ① We apply the transportation plan g to $\hat{\nu}$; this transforms $\hat{\nu}$ into some measure $\hat{\mu}^*$. The measure $\hat{\mu}^*$ is not supported by B *a priori*, so we split it into $\hat{\mu}_B^* + \hat{\mu}_c^* := \mathbf{1}_B \cdot \hat{\mu}^* + \mathbf{1}_{B^c} \cdot \hat{\mu}^*$.
- ② Denoting $a_c := \|\hat{\mu}_c^*\|_1 / \|\hat{\mu}\|_1$, we then transform $\hat{\mu}_c^*$ into $a_c \hat{\mu}$ according to an arbitrary transference plan.
- ③ Finally, denoting $a_B := \|\hat{\mu}_B^*\|_1 / \|\hat{\mu}\|_1$,^[*] we transform $\hat{\mu}_B^*$ into $a_B \hat{\mu}$ according to the optimal transference plan: the cost of this operation is $W_2(\hat{\mu}_B^*, a_B \hat{\mu})$, which we bound above by $2\|\hat{\mu}_B^* - a_B \hat{\mu}\|_{\dot{H}^{-1}(a_B \hat{\mu})}$ thanks to Theorem 1.

^[††] Actually such an g does not always exist, as it can occur that the optimal transportation plan from ν to μ “splits points” if ν is not regular enough. However it would suffice to use the general formalism of transportation plans to handle that case: we do not do it here to keep notation light, but this is straightforward. Also note that it is not obvious that the infimum in (2) is attained: again, that is not a real problem as our proof still works by considering a sequence of transportation plans approaching optimality.

^[*] Observe that $a_B + a_c = a$.

Then, denoting by $W_2(\textcircled{1}), W_2(\textcircled{2}), W_2(\textcircled{3})$ the respective Wasserstein distances of these steps, we shall have $W_2(\hat{\nu}, a\hat{\mu}) \leq W_2(\textcircled{1}) + (W_2(\textcircled{2})^2 + W_2(\textcircled{3})^2)^{1/2}$.

Let us begin with bounding the cost of Step $\textcircled{1}$. The squared cost of this step is

$$\begin{aligned} W_2(\textcircled{1})^2 &= \int |S(y)|^2 \hat{\nu}(dy) = \int |S(y)|^2 \varphi(y) \nu(dy) \\ &\leq \sup \varphi \times \int |S(y)|^2 \nu(dy) = \sup \varphi \times w^2 \leq c_2 R^2 w^2, \end{aligned} \quad (27)$$

whence $W_2(\textcircled{1}) \leq c_2^{1/2} R w$.

Now consider Step $\textcircled{2}$. As $a_c \hat{\mu}$ is supported by B , one has obviously

$$W_2(\textcircled{2})^2 \leq \int (\text{dist}(x, B) + 2R)^2 \hat{\mu}_c^*(dx) = \int_{B^c} (\text{dist}(x, B) + 2R)^2 \hat{\mu}^*(dx). \quad (28)$$

From that we deduce that $W_2(\textcircled{2}) \leq 2c_2^{1/2} R w$ by the following computation:

$$\begin{aligned} \int_{B^c} (\text{dist}(x, B) + 2R)^2 \hat{\mu}^*(dx) &= \int_{g(y) \notin B} (\text{dist}(g(y), B) + 2R)^2 \varphi(y) \nu(dy) \\ &\leq c_2 \int_{\substack{y \in B \\ g(y) \notin B}} (\text{dist}(g(y), B) + 2R)^2 \text{dist}(y, B^c)^2 \nu(dy) \\ &\leq c_2 \int_{\substack{y \in B \\ g(y) \notin B}} (R \text{dist}(g(y), B) + 2R \text{dist}(y, B^c))^2 \nu(dy) \\ &\leq 4c_2 R^2 \int_{\substack{y \in B \\ g(y) \notin B}} (\text{dist}(g(y), B) + \text{dist}(y, B^c))^2 \nu(dy) \\ &\leq 4c_2 R^2 \int |y - g(y)|^2 \nu(dy) = 4c_2 R^2 w^2. \end{aligned} \quad (29)$$

Step $\textcircled{3}$ is the difficult one. We begin with observing that it is easy to bound the $L^2(B)$ distance between $\hat{\mu}_B^*$ and $\hat{\mu}$: indeed, denoting by $f =: Id + T$ the inverse map of $g^{[\dagger]}$,

$$\begin{aligned} \|\hat{\mu}_B^* - \hat{\mu}\|_{L^2(\mathbf{1}_B \cdot \mu)}^2 &= \int_B \left(\frac{\hat{\mu}^*(dx) - \varphi(x) \mu(dx)}{\mu(dx)} \right)^2 \mu(dx) \\ &= \int_B (\varphi(f(x)) - \varphi(x))^2 \mu(dx) \\ &\leq k^2 \int_{\mathbf{R}^n} |x - f(x)|^2 \mu(dx) = k^2 \int |T(x)|^2 \mu(dx) = k^2 w^2, \end{aligned} \quad (30)$$

(where we used that $\hat{\mu}^*(dx) = \hat{\nu}(d(f(x))) = \varphi(f(x)) \nu(d(f(x))) = \varphi(f(x)) \mu(dx)$), so that

$$\|\hat{\mu}_B^* - \hat{\mu}\|_{L^2(B)}^2 \leq k^2 m_2 w^2. \quad (31)$$

^[\dagger]For f to exist, g should be bijective, which is not always true *stricto sensu*; but we can safely carry out the reasoning with pretending so, by the same argument as in Footnote ^[\ddagger] on page 7.

^[\ddagger]Remember that when ν stands for a *measure*, $\|\nu\|_{L^2(\mu)}$ means what is more commonly denoted by $\|\nu\|_{L^2(\mu)}$, so that the relation $\mu \leq m\lambda$ implies that $\|\nu\|_{L^2(\lambda)}^2 \leq m \|\nu\|_{L^2(\mu)}^2$ —while on the other hand, when f stands for a *function*, one has $\|f\|_{L^2(\mu)}^2 \leq m \|f\|_{L^2(\lambda)}^2$.

Now we have to link $\|\cdot\|_{L^2(B)}$ with $\|\cdot\|_{\dot{H}^{-1}(\mu)}$. This is achieved by the following lemma, whose proof is postponed:

Lemma 14. *Define $\hat{\lambda}$ to be the measure on B such that $\hat{\lambda}(dx) := \text{dist}(x, B^c)^2 \lambda(dx)$. Then, for any signed measure m on B having total mass zero:*

$$\|m\|_{\dot{H}^{-1}(\hat{\lambda})} \leq C_1(n)^{1/2} \|m\|_{L^2(B)}, \quad (32)$$

where $C_1(n)$ is some absolute constant only depending on n . Moreover, taking $C_1(n) := ((2e+1)n-1) \vee 8e$ fits.

Thanks to Theorem 1 and Lemma 14, we have that

$$\begin{aligned} W_2(\textcircled{3}) &\leq 2\|a_B \hat{\mu} - \hat{\mu}_B^*\|_{\dot{H}^{-1}(a_B \hat{\mu})} \leq 2(a_B c_1 m_1)^{-1/2} \|a_B \hat{\mu} - \hat{\mu}_B^*\|_{\dot{H}^{-1}(\hat{\lambda})} \\ &\leq 2C_1(n)^{1/2} (a_B c_1 m_1)^{-1/2} \|a_B \hat{\mu} - \hat{\mu}_B^*\|_{L^2(B)}. \end{aligned} \quad (33)$$

Next, we compute

$$\begin{aligned} \|a_B \hat{\mu} - \hat{\mu}_B^*\|_{L^2(B)} &= \left\| \frac{\|\hat{\mu}_B^*\|_1}{\|\hat{\mu}\|_1} \hat{\mu} - \hat{\mu}_B^* \right\|_{L^2(B)} \leq \frac{\|\hat{\mu}_B^*\|_1 - \|\hat{\mu}\|_1}{\|\hat{\mu}\|_1} \|\hat{\mu}\|_{L^2(B)} + \|\hat{\mu}_B^* - \hat{\mu}\|_{L^2(B)} \\ &\leq \frac{\|\hat{\mu}\|_{L^2(B)}}{\|\hat{\mu}\|_1} \|\hat{\mu}_B^* - \hat{\mu}\|_1 + \|\hat{\mu}_B^* - \hat{\mu}\|_{L^2(B)} \leq \left(\frac{\|\hat{\mu}\|_{L^2(B)}}{\|\hat{\mu}\|_1} \lambda(B)^{1/2} + 1 \right) \|\hat{\mu}_B^* - \hat{\mu}\|_{L^2(B)} \\ &\leq \left(\frac{c_2 m_2}{c_1 m_1} \frac{\lambda(B)^{1/2} \|\hat{\lambda}\|_{L^2(B)}}{\|\hat{\lambda}\|_1} + 1 \right) \|\hat{\mu}_B^* - \hat{\mu}\|_{L^2(B)} \stackrel{[8]}{\leq} (\sqrt{6} \frac{c_2 m_2}{c_1 m_1} + 1) \|\hat{\mu}_B^* - \hat{\mu}\|_{L^2(B)} \\ &\stackrel{(31)}{\leq} (\sqrt{6} + 1) \frac{c_2 m_2}{c_1 m_1} k m_2^{1/2} w, \end{aligned} \quad (34)$$

so that, combining (33) and (34), we have got:

$$W_2(\textcircled{3}) \leq (2\sqrt{6} + 2) C_1(n)^{1/2} a_B^{-1/2} \frac{c_2 m_2^{3/2}}{c_1 m_1^{3/2}} \frac{k}{c_1^{1/2}} w. \quad (35)$$

Equation (35) is the kind of bound we were looking for, *provided* $a_B \lesssim 1$. Though this will be the case in practice (since we are mainly interested in cases where ν is close to μ and thus $\hat{\mu}^*$ is close to $\hat{\mu}$), this is not quite satisfactory yet. So, what can we do when $a_B \ll 1$, that is, when $\|\hat{\mu}_B^*\|_1 \ll \|\hat{\mu}\|_1$? In fact that case is easier, because transportation between small measures has low cost, while w has to be large to make $\hat{\mu}_B^*$ very different from $\hat{\mu}$.

The computations are the following. First, it is obvious that

$$W_2(\textcircled{3}) = W_2(\hat{\mu}_B^*, a_B \hat{\mu}) \leq 2R \|\hat{\mu}_B^*\|_1^{1/2}. \quad (36)$$

^[8]This step comes from the computation $\lambda(B)^{1/2} \|\hat{\lambda}\|_{L^2(B)} / \|\hat{\lambda}\|_1 = (\int_0^1 r^{n-1} dr)^{1/2} \times (\int_0^1 (1-r)^4 r^{n-1} dr)^{1/2} / (\int_0^1 (1-r)^2 r^{n-1} dr) = (6(1+n)(2+n) / (3+n)(4+n))^{1/2} \leq \sqrt{6} \forall n$.

Next, observing that $\varphi(f(x)) \geq \frac{c_1}{c_2} \varphi(x) - 2c_1 \operatorname{dist}(x, B^c) |T(x)|$,^[¶] we compute that

$$\begin{aligned} \|\hat{\mu}_B^*\|_1 &= \int_B \varphi(f(x)) \mu(dx) \geq \int_B \left(\frac{c_1}{c_2} \varphi(x) - 2c_1 \operatorname{dist}(x, B^c) |T(x)| \right) \mu(dx) \\ &\geq \frac{c_1}{c_2} \|\hat{\mu}\|_1 - 2c_1 \left(\int_B \operatorname{dist}(x, B^c)^2 \mu(dx) \right)^{1/2} \left(\int_B |T(x)|^2 \mu(dx) \right)^{1/2} \\ &= \frac{c_1}{c_2} \|\hat{\mu}\|_1 - 2c_1 \|\operatorname{dist}(\cdot, B^c)^2 \cdot \mu\|_1^{1/2} w \geq \frac{c_1}{c_2} \|\hat{\mu}\|_1 - 2c_1 m_2^{1/2} \|\hat{\lambda}\|_1^{1/2} w, \end{aligned} \quad (38)$$

whence

$$w \geq \frac{\left(\frac{c_1}{c_2} \|\hat{\mu}\|_1 - \|\hat{\mu}_B^*\|_1 \right)_+}{2c_1 m_2^{1/2} \|\hat{\lambda}\|_1^{1/2}} = \frac{\left(\frac{c_1}{c_2} - a_B \right)_+ \|\hat{\mu}\|_1}{2c_1 m_2^{1/2} \|\hat{\lambda}\|_1^{1/2}} \geq \frac{m_1^{1/2}}{2c_1^{1/2} m_2^{1/2}} \left(\frac{c_1}{c_2} - a_B \right)_+ \|\hat{\mu}\|_1^{1/2}. \quad (39)$$

So,

$$W_2(\textcircled{3}) \leq 2R \|\hat{\mu}_B^*\|_1^{1/2} = 2R a_B^{1/2} \|\hat{\mu}\|_1^{1/2} \leq 4R c_1^{1/2} \frac{m_2^{1/2}}{m_1^{1/2}} \frac{a_B^{1/2}}{\left(\frac{c_1}{c_2} - a_B \right)_+} w. \quad (40)$$

In the end, choosing either (35) if $a_B \geq c_1 / 2c_2$ or (40) if $c_1 / 2c_2$, and observing that $c_1 \leq kR^{-1}$, one has always:

$$W_2(\textcircled{3}) \leq ((4\sqrt{3} + 2\sqrt{2})C_1(n)^{1/2} \vee 4\sqrt{2}) \frac{c_2^{3/2} m_2^{3/2}}{c_1^{3/2} m_1^{3/2}} \frac{k}{c_1^{1/2}} w. \quad \square \quad (41)$$

Remark 15. To bound $W_2(\textcircled{3})$ in the situation where $a_B \ll 1$, we could also have started from “ $\varphi(f(x)) \geq \varphi(x) - k|T(x)|$ ” (instead of “ $\varphi(f(x)) \geq \frac{c_1}{c_2} \varphi(x) - 2c_1 \operatorname{dist}(x, B^c) |T(x)|$ ”) to get another bound analogous to (38). Following such an approach, the factor $(c_2 / c_1)^{3/2}$ in (40) would be improved into (c_2 / c_1) in the analogous formula; however the dimensional factor would behave in $O(n)$ rather than in $O(n^{1/2})$.

2.4 Proof of Lemma 14

It still remains to prove Lemma 14, whose statement we recall to be:

Lemma. Denoting $\hat{\lambda} := \operatorname{dist}(\cdot, B^c)^2 \cdot \lambda$, one has, for any signed measure m on B having total mass zero:

$$\|m\|_{\dot{H}^{-1}(\hat{\lambda})} \leq (((2e+1)n-1) \vee 8e)^{1/2} \|m\|_{L^2(B)}. \quad (42)$$

—In the sequel, “ $((2e+1)n-1) \vee 8e$ ” will be shorthanded into “ $C_1(n)$ ”.

Remark 16. The bound (42) is within a constant factor of being optimal, uniformly in n , as one sees by taking a linear function f in (45).

^[¶]This follows from the computation:

$$\begin{aligned} \varphi(f(x)) &\geq c_1 \operatorname{dist}(f(x), B^c)^2 \geq c_1 (\operatorname{dist}(x, B^c) - |T(x)|)^2_+ \\ &\geq c_1 \operatorname{dist}(x, B^c)^2 - 2c_1 \operatorname{dist}(x, B^c) |T(x)| \geq \frac{c_1}{c_2} \varphi(x) - 2c_1 \operatorname{dist}(x, B^c) |T(x)|. \end{aligned} \quad (37)$$

Proof of the lemma. We begin with translating the lemma into a functional analysis statement by a duality argument. Recall the duality definition of $\|m\|_{\dot{H}^{-1}(\hat{\lambda})}$ from § 1:

$$\|m\|_{\dot{H}^{-1}(\hat{\lambda})} := \sup\{|\langle f, m \rangle| \mid \|f\|_{\dot{H}^1(\hat{\lambda})} \leq 1\}. \quad (43)$$

There is a similar duality formula for $\|m\|_{L^2(B)}$:

$$\|m\|_{L^2(B)} = \sup\{|\langle f, m \rangle| \mid \|f\|_{L^2(B)} \leq 1\}, \quad (44)$$

where, for f a function, $\|f\|_{L^2(B)}$ has its usual meaning, namely $\|f\|_{L^2(B)} := (\int_B f(x)^2 \lambda(dx))^{1/2}$. Since m is assumed to have total mass zero, $|\langle f, m \rangle|$ does not change when one adds a constant to f . On the other hand, when f describes the set $\{\|f_0 + a\| \mid a \in \mathbf{R}\}$, $\|f\|_{L^2(B)}$ is minimal when a is such that f has zero mean on B , while the value of $\|f\|_{\dot{H}^1(\hat{\lambda})}$ remains constant.^[lll] As a consequence, we can restrict the supremum in (43) and (44) to those f having zero mean on B . Thus, the lemma will be implied^[**] by proving that

$$\langle f, \mathbf{1}_B \cdot \lambda \rangle = 0 \quad \Rightarrow \quad \|f\|_{L^2(B)} \leq C_1(n)^{1/2} \|f\|_{\dot{H}^1(\hat{\lambda})}. \quad (45)$$

Going back to the definitions of $\|\cdot\|_{\dot{H}^{-1}(\hat{\lambda})}$ and $\|\cdot\|_{L^2(B)}$, relaxing the condition on f to be centred by projecting it orthogonally in $L^2(B)$ onto the subspace of centred functions, and denoting by P the uniform probability measure on B , Equation (45) turns into:

$$\forall f \quad \text{Var}_P(f) \leq C_1(n) \int \text{dist}(x, B^c)^2 |\nabla f(x)|^2 P(dx), \quad (46)$$

which we recognize to be a so-called “improved Poincaré inequality” [3, 6]. In general, Poincaré inequalities, bounding the variance of f by a quadratic integral of its first derivative, are linked with the exponential convergence of a certain diffusion Markov process towards equilibrium (*cf.* [1, Chap. 2]): that probabilistic vision initially guided me to tackle Equation (46), although this will not be apparent in the sequel.

To prove (46), the first key idea (inspired by [4]) is to separate radial and spherical coordinates. This is, considering the bijection

$$\begin{aligned} \varphi: (0, R) \times \mathbb{S}^{n-1} &\rightarrow B \setminus \{0\} \\ (r, \theta) &\mapsto r\theta \end{aligned} \quad (47)$$

(the origin of space being set at the center of B), we introduce the measure $\tilde{P} := \varphi^{-1} * P$, which is obviously the product measure $\tilde{P}_r \otimes \tilde{P}_\theta$, where $\tilde{P}_r$ is the probability measure on $(0, R)$ such that $\tilde{P}_r(dr) := nR^{-n}r^{n-1}dr$, resp. $\tilde{P}_\theta$ is the uniform measure on the sphere $\mathbb{S}^{n-1}$. With this notation, we perform can a change of variables to see that (46) is equivalent to proving that, for all $g \in L^2(\tilde{P})$:

$$C_1(n)^{-1} \text{Var}_{\tilde{P}}(g) \leq \int_0^R \int_{\mathbb{S}^{n-1}} (R-r)^2 (|\nabla_r g(r, \theta)|^2 + r^{-2} |\nabla_\theta g(r, \theta)|^2) \tilde{P}_r(dr) \tilde{P}_\theta(d\theta), \quad (48)$$

where ∇_r and ∇_θ denote the gradient along resp. the r coordinate and the θ coordinate.^[††] We will denote the right-hand side of (48) by $\mathcal{E}(g, g)$.

^[lll]Here we implicitly assume that $\int_B |f(x)| \lambda(dx) < \infty$, which is legitimate since an approximation argument allows to restrict the suprema in (43) and (44) to those f having a $\mathcal{C}^\infty$ continuation on $\text{cl}(B)$.

^[**]Actually there is even equivalence.

^[††]In the latter case, we have to use the Riemannian definition of the gradient on $\mathbb{S}^{n-1}$.

Because $\tilde{P} = \tilde{P}_r \otimes \tilde{P}_\theta$, we know that $L^2(\tilde{P})$ can be seen as (the closure of) the tensor product of $L^2(\tilde{P}_r)$ and $L^2(\tilde{P}_\theta)$:

$$L^2(\tilde{P}) = \text{cl}(L^2(\tilde{P}_r) \overset{\perp}{\otimes} L^2(\tilde{P}_\theta)), \quad (49)$$

where the symbol ' $\overset{\perp}{\otimes}$ ' means that the Hilbertian structure of $L^2(\tilde{P})$ is compatible with the Hilbertian structures of $L^2(\tilde{P}_r)$ and $L^2(\tilde{P}_\theta)$ —i.e., that $\langle h_a \otimes u_a, h_b \otimes u_b \rangle_{L^2(\tilde{P})} = \langle h_a, h_b \rangle_{L^2(\tilde{P}_r)} \times \langle u_a, u_b \rangle_{L^2(\tilde{P}_\theta)}$. Now consider the spherical harmonics $Y_0, Y_1, \dots$, which by definition are an orthonormal basis, in $L^2(\tilde{P}_\theta)$, of eigenfunctions of the Laplace–Beltrami operator Δ on $\mathbb{S}^{n-1}$; and call $\ell_0, \ell_1, \dots$ the associated eigenvalues, which are known to be such that (up to permuting indices) $Y_0 \equiv 1$ with $\ell_0 = 0$, and $\ell_i \leq -(n-1) \forall i \neq 0$ (see for instance [11]). By construction, $L^2(\tilde{P}_\theta) = \text{cl}(\bigoplus_{i \in \mathbf{N}} (\mathbf{R} \cdot Y_i))$; therefore, one has that

$$L^2(\tilde{P}) = \text{cl}\left(\bigoplus_{i \in \mathbf{N}} L^2(\tilde{P}_r) \cdot Y_i\right): \quad (50)$$

in other words, the functions of $L^2(\tilde{P})$ are those of the form

$$g(r, \theta) = \sum_{i \in \mathbf{N}} h_i(r) Y_i(\theta), \quad (51)$$

with $\sum_i \|h_i\|_{L^2(\tilde{P}_r)}^2 < \infty$, and the correspondence is bijective. An interesting point is that, then, one has:

$$\text{Var}_{\tilde{P}}(g) = \text{Var}_{\tilde{P}_r}(h_0) + \sum_{i \neq 0} \|h_i\|_{L^2(\tilde{P}_r)}^2. \quad (52)$$

On the other hand, one has

$$\mathcal{E}(g, g) = -\langle Lg, g \rangle_{L^2(\tilde{P})}, \quad (53)$$

where

$$(Lg)(r, \theta) := (R-r)^2 \Delta_r g + \left((n-1) \frac{(R-r)^2}{r} - 2(R-r) \right) \mathbf{e}_r \cdot \nabla_r g + \frac{(R-r)^2}{r^2} \Delta_\theta g. \quad (54)$$

From (54) we see that, since the Y_i are eigenfunctions of Δ_θ , all the $L^2(\tilde{P}_r) \cdot Y_i$ are invariant by L , and that one has:

$$\mathcal{E}(g, g) = \sum_{i \in \mathbf{N}} \int_0^R \left((R-r)^2 |\nabla h_i(r)|^2 - \ell_i \frac{(R-r)^2}{r^2} h_i(r)^2 \right) \tilde{P}_r(dr). \quad (55)$$

So, proving (48) becomes equivalent to proving that both following formulas hold for all $h \in L^2(\tilde{P}_r)$:

$$\text{Var}_{\tilde{P}_r}(h) \leq C_1(n) \int_0^R (R-r)^2 |\nabla h(r)|^2 \tilde{P}_r(dr); \quad (56)$$

$$\|h\|_{L^2(\tilde{P}_r)}^2 \leq C_1(n) \int_0^R \left((R-r)^2 |\nabla h(r)|^2 + (n-1) \frac{(R-r)^2}{r^2} h(r)^2 \right) \tilde{P}_r(dr). \quad (57)$$

Let us start with (56). In all the sequel of the proof, we introduce

$$b := 1 - n^{-1}. \quad (58)$$

By the Cauchy–Schwarz inequality, one has, for all $r \in (bR, R)$:

$$\begin{aligned} (h(r) - h(bR))^2 &= \left(\int_{bR}^r h'(s) ds \right)^2 \leq \left(\int_{bR}^r (R-s)^{-3/2} ds \right) \times \int_{bR}^r (R-s)^{3/2} |\nabla h(s)|^2 ds \\ &\leq 2((R-r)^{-1/2} - (R-bR)^{-1/2}) \int_{bR}^r (R-s)^{3/2} |\nabla h(s)|^2 ds \\ &\leq 2(R-r)^{-1/2} \int_{bR}^r (R-s)^{3/2} |\nabla h(s)|^2 ds. \end{aligned} \quad (59)$$

Integrating and using Fubini's formula, it follows that

$$\begin{aligned} \int_{bR}^R (h(r) - h(bR))^2 \tilde{P}_r(dr) &\leq \\ &2 \int_{s=bR}^R \left(\int_{r=s}^R nR^{-n} (R-r)^{-1/2} r^{n-1} dr \right) (R-s)^{3/2} |\nabla h(s)|^2 ds \\ &\leq 2 \int_{s=bR}^R \left(\int_{r=s}^R nR^{-n} (b^{-1}s)^{n-1} (R-r)^{-1/2} dr \right) (R-s)^{3/2} |\nabla h(s)|^2 ds \\ &= 2b^{-(n-1)} \int_{s=bR}^R \left(\int_{r=s}^R (R-r)^{-1/2} dr \right) (R-s)^{3/2} |\nabla h(s)|^2 \tilde{P}_r(ds) \\ &= 4b^{-(n-1)} \int_{s=bR}^R (R-s)^2 |\nabla h(s)|^2 ds. \end{aligned} \quad (60)$$

One can apply the same line of reasoning for $r \in (0, bR)$: the (unweighted this time) Cauchy–Schwarz inequality then yields $(h(r) - h(bR))^2 \leq (bR - r) \times \int_r^{bR} |\nabla h(s)|^2 ds$, whence:

$$\begin{aligned} \int_0^{bR} (h(r) - h(bR))^2 \tilde{P}_r(dr) &\leq \int_{s=0}^{bR} \left(\int_{r=0}^s nR^{-n} (bR-r)^{n-1} dr \right) |\nabla h(s)|^2 ds \\ &\leq bR^{-(n-1)} \int_{s=0}^{bR} \left(\int_{r=0}^s nr^{n-1} dr \right) |\nabla h(s)|^2 ds = bR \int_0^{bR} |\nabla h(s)|^2 s^n ds \\ &\leq bn^{-1} R^2 \int_0^{bR} |\nabla h(s)|^2 \tilde{P}_r(ds) \leq b(1-b)^{-2} n^{-1} \int_0^{bR} (R-s)^2 |\nabla h(s)|^2 \tilde{P}_r(ds). \end{aligned} \quad (61)$$

Summing (60) and (61), we get that

$$\int_0^R (h(r) - h(bR))^2 \tilde{P}_r(dr) \leq (4b^{-(n-1)} \vee b(1-b)^{-2} n^{-1}) \int_0^R (R-s)^2 |\nabla h(s)|^2 \tilde{P}_r(ds), \quad (62)$$

where $(4b^{-(n-1)} \vee b(1-b)^{-2} n^{-1})$ can itself be bounded by $((n-1) \vee 4e)$. The left-hand-side of (62) being an upper bound for $\text{Var}_{\tilde{P}_r}(h)$, this proves (56).

Now we turn to (57). For $r \in (bR, R)$ we have, similarly to (59), that

$$(h(r) - h(bR))^2 \leq 2(R-r)^{-1/2} \int_{bR}^r (R-s)^{3/2} |\nabla h(s)|^2 ds, \quad (63)$$

so that

$$h(r)^2 \leq 2h(br)^2 + 4(R-r)^{-1/2} \int_{br}^r (R-s)^{3/2} |\nabla h(s)|^2 ds. \quad (64)$$

Then, integrating and applying Fubini's formula:

$$\begin{aligned} \int_{bR}^R h(r)^2 \tilde{P}_r(dr) &\leq 2 \int_{bR}^R h(br)^2 \tilde{P}_r(dr) + \\ &4 \int_{s=b^2R}^R \left(\int_{r=s \vee bR}^{b^{-1}s \wedge R} nR^{-n} r^{n-1} (R-r)^{-1/2} dr \right) (R-s)^{3/2} |\nabla h(s)|^2 ds. \end{aligned} \quad (65)$$

By change of variables, the first term of the right-hand side of (65) is equal to $2b^{-n} \int_{b^2R}^{bR} h(s)^2 \tilde{P}_r(ds)$, which we can bound by

$$\begin{aligned} 2b^{-(n-2)} \frac{(1-b)^{-2}}{n-1} \int_{b^2R}^{bR} (n-1) \frac{(R-r)^2}{r^2} h(s)^2 \tilde{P}_r(ds) \\ \leq 2ne \int_0^R (n-1) \frac{(R-r)^2}{r^2} h(s)^2 \tilde{P}_r(ds). \end{aligned} \quad (66)$$

The second term of the right-hand side of (65) is itself bounded by

$$\begin{aligned} 4b^{-(n-1)} \int_{s=b^2R}^R \left(\int_{r=s}^R (R-r)^{-1/2} dr \right) (R-s)^{3/2} |\nabla h(s)|^2 \tilde{P}_r(ds) \\ \leq 8e \int_0^R (R-s)^2 |\nabla h(s)|^2 \tilde{P}_r(ds). \end{aligned} \quad (67)$$

This way, we have bounded $\int_{bR}^R h(r)^2 \tilde{P}_r(dr)$.

On the other hand, it is trivial that, for $r \leq bR$,

$$h(r)^2 \leq \frac{b^2}{(n-1)(1-b)^2} \times (n-1) \frac{(R-r)^2}{r^2} h(r)^2, \quad (68)$$

whence:

$$\int_0^{bR} h(r)^2 \tilde{P}_r(dr) \leq (n-1) \int_0^R (n-1) \frac{(R-r)^2}{r^2} h(r)^2 \tilde{P}_r(dr). \quad (69)$$

Combining (66), (67) and (69), we finally get the wanted bound (57). $\square$

Remark 17. At the time I wrote that proof I was not aware of the already existing results on improved Poincaré inequalities, in particular [6, Theorem 1.3], which Equation (46) is actually a particular case of; nor were the people whom I had asked about such inequalities. Compared to the result of [6] however, my Equation (46) states an explicit value for the constant in the inequality, which moreover is within a constant factor of being optimal, uniformly in the dimension n ; also, it uses a quite different proof, which may be interesting $\smile$

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